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THE TYPE OF THE REGULAR REPRESENTATION OF CERTAIN TRANSITIVE GROUPOIDS

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1. INTRODUCTION

Let G be a (not necessarily connected) Lie group. If $G\times G$ is provided with a groupoid structure in a trivial manner (i.e., for $(g_1, g_2), (g_1', g_2')\in G\times G$, $(g_1, g_2)^{-1} = (g_2, g_1)$, and (g_1, g_2) is composable with (g_1', g_2') if and only if $g_2 = g_1'$ with the resulting composition equal to (g_1, g_2')), we can use Haar measure to form a convolution algebra $C_c^\infty(G\times G)$ and its regular representation on $L^2(G\times G)$. Then the generated von Neumann algebra is a type I factor and acts on $L^2(G\times G)$ as a standard representation. Although these facts are all trivial, if we take a quotient of $G\times G$ by a suitable subgroup of $G\times G$, the situation becomes rather complicated and the analysis of its regular representation becomes an interesting problem.

In this paper, we will carry out the type analysis of the von Neumann algebra associated with a groupoid of this type. More precisely, let H be a closed subgroup of G and let $\underline{H}$ be a normal subgroup of H which contains the connected component of H. We form a closed subgroup D of $G\times G$:

$$D = \{(h_1, h_2)\in H\times H\ ;\ h_1^{-1}h_2\in \underline{H}\}. \tag{1}$$

Then $\Gamma = G\times G/D$ has the structure of groupoid induced from that of $G\times G$, which is transitive in the sense that the canonical equivalence relation of the groupoid is transitive (see [10], for example). We assume that $\underline{H}$ has a unitary character χ, which is invariant under the action of H, i.e. $\chi(h\underline{h}h^{-1}) = \chi(\underline{h})$ for $h\in H$, $\underline{h}\in \underline{H}$. We obtain a character χ_d of D defined by

$$\chi_d(a, b) = \chi(a^{-1}b) \quad \text{for } (a, b)\in D. \tag{2}$$

Let φ be a C^∞-function on $G\times G$ which satisfies

$$\varphi(g_1a_1, g_2a_2) = \Delta_{G,H}(a_1a_2)^{1/2}\chi_d(a_2, a_1)\varphi(g_1, g_2) \tag{3}$$

for $(g_1, g_2) \in G\times G$, $(a_1, a_2) \in D$, and has a compact support modulo D. Here $\Delta_{G,H}(a) = \det_{\mathfrak{G}/\mathfrak{H}} \mathrm{Ad}\, a$; ($\mathfrak{G}$ and $\mathfrak{H}$ denote Lie algebras of G and H respectively). We denote the set of all these φ's by $\mathfrak{A}$ and equip it with the following $*$-algebra structure: For $\varphi, \varphi_1, \varphi_2 \in \mathfrak{A}$, $\varphi_1 * \varphi_2 \in \mathfrak{A}$ and $\varphi^* \in \mathfrak{A}$ are defined by

(4) $$(\varphi_1 * \varphi_2)(g_1, g_2) = \oint_{G/\underline{H}} dg\, \varphi_1(g_1, g)\varphi_2(g, g_2),$$

(5) $$\varphi^*(g_1, g_2) = \overline{\varphi(g_2, g_1)}$$

$\left(\text{see [2], for example, for the meaning of } \oint_{G/\underline{H}}\right)$. Furthermore, the following inner product makes $\mathfrak{A}$ a (unimodular) Hilbert algebra:

(6) $$(\varphi \mid \varphi) = \oint_{G\times G/D} dg_1\, dg_2\, |\varphi(g_1, g_2)|^2.$$

We denote by $\mathfrak{A}''$ (resp. $\mathfrak{A}'$) the left (resp. right) von Neumann algebra of this Hilbert algebra which acts on the L^2-completion $L^2(\mathfrak{A})$ of $\mathfrak{A}$.

Our first result is a concrete realization of the center of $\mathfrak{A}''$ as a subalgebra of a convolution algebra over $H/\underline{H}$ (Theorem 1). Next, we work out the type analysis of $\mathfrak{A}''$ under the assumption that the commutator subgroup of H is contained in $\underline{H}$; $\mathfrak{A}''$ is of type I or type II according as the coset space H/S is finite or infinite (Theorem 2). Here S is a normal subgroup of H defined by $S = \{a \in H\,;\, \chi(aba^{-1}b^{-1}) = 1$ for all $b \in H\}$.

The shortest way to obtain the results is as follows: Take a nowhere-vanishing function s on $G\times G$ satisfying $s(g_1h_1, g_2h_2) = \chi(h_2^{-1}h_1)s(g_1, g_2)$, $(h_1, h_2) \in D$. Then $\sigma((g_1, g)D, (g, g_2)D) = s(g_1, g)s(g, g_2)s(g_1, g_2)^{-1}$ defines a 2-cocycle on Γ and the left von Neumann algebra $\mathfrak{A}''$ is spatially equivalent to the von Neumann algebra $M(\Gamma, \sigma)$ generated by the σ-left regular representation of Γ. Now define a 2-cocycle σ_1 of $H/\underline{H}$ by $\sigma_1(h_1\underline{H}, h_2\underline{H}) = \sigma((1, h_1)D, (1, h_2)D)$. Then by the inclusion $H/\underline{H} \ni h\underline{H} \mapsto (1, h)D \in \Gamma$, $(H/\underline{H}, \sigma_1)$ is similar to (Γ, σ) and then $M(H/\underline{H}, \sigma_1)$ is stably-isomorphic to $M(\Gamma, \sigma)$ ([4]). In particular the center of $M(H/\underline{H}, \sigma_1)$ is isomorphic to the center of $M(\Gamma, \sigma)$ (a version of Theorem 1 in this paper). Furthermore, the type analysis of $M(H/\underline{H}, \sigma_1)$ is given in [7] (see also [1]), so the above Theorem 2 is obtained. Although this may be sufficient in the abstract, a much more direct proof is presented in this paper, which has several advantages:

(i) concrete realization of the center of $\mathfrak{A}''$,

(ii) applicability to certain non-transitive groupoids (cf. [9]),

(iii) easy to generalize to the holomorphically constrained case.

The last point will be important when one applies (or extends) to complex polarizations (in geometric quantization). I would like to discuss these applications in a future paper.

2. DESCRIPTION OF THE CENTER OF $\mathfrak{A}''$

Main result of this section is a concrete realization of the center of $\mathfrak{A}''$. To do this, we need distributions (more precisely generalized sections of a line bundle) in order to express bounded linear operator on $L^2(\mathfrak{A})$ in kernel form. For this reason, we have restricted ourselves to Lie group, but the result itself may be valid for arbitrary locally compact groups. To begin with, we define some line bundles over $G/\underline{H}$: We regard G as a principal $\underline{H}$-bundle by the right translation, and form a line bundle $B = G\times_{\underline{H}} \mathbf{C}$ over $G/\underline{H}$, where the action of $\underline{H}$ on $\mathbf{C}$ is given by $h\cdot z = \Delta_{G,H}(h)^{-1/2}\chi(h)z$, $h \in H$, $z \in \mathbf{C}$. We denote by $\bar{B}$ the conjugate bundle of B ($\bar{B}$ is constructed as B if we replace the action of $\underline{H}$ on $\mathbf{C}$ by $h\cdot z = \Delta_{G,H}(h)^{-1/2}\overline{\chi(h)}z$). Since χ is H-invariant, for any $a \in H$, $B \ni g\cdot z \mapsto \Delta_{G,H}(a)^{1/2} ga\cdot z \in B$ gives rise to a bundle morphism of B and hence $H\times H$ acts on the exterior tensor product bundle $\bar{B}\boxtimes B$ (this is a line bundle over $G/\underline{H}\times G/\underline{H}$ with the fibre at (x, y) equal to $\bar{B}_x\otimes B_y$). In particular, restricting the action to the diagonal subgroup of $H\times H$, we get an action of H. Then the quotient $B_d = \bar{B}\boxtimes B/H$ is a line bundle over Γ, and there is a 1-1 correspondence between elements in $\mathfrak{A}$ and support-compact C^∞-sections of $B_d \to \Gamma$.

LEMMA 1. *For any $L \in \mathfrak{A}''$, there is a generalized section $l \in C^{-\infty}(G/\underline{H}\times G/\underline{H};$ $\bar{B}\boxtimes B)$ ($=$ dual space of $C_c^\infty(G/\underline{H}\times G/\underline{H};\ B\boxtimes\bar{B})$, see [5] for the information of generalized section) such that*

$$l(x_1 a, x_2 a)(a^{-1}, a^{-1}) = l(x_1, x_2) \tag{7}$$

for $a \in H$, and for $F \in \mathfrak{A} \subset C^\infty(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$;

$$(LF)(x_1, x_2) = \int_{x\in G/\underline{H}} l(x_1, x)F(x, x_2) \tag{8}$$

(both sides should be regarded as elements in $C^{-\infty}(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$).

COMMENTS. (i) In (7), the action of $a \in H$ on l is through the bundle morphism of B described above.

(ii) In (8), for fixed x_2, $x \mapsto F(x, x_2)$ has a compact support and therefore the pairing of $l(x_1, \cdot)$ with $F(\cdot, x_2)$ has meaning.

Proof. By Schwartz's kernel theorem, $\mathscr{B}(L^2(\mathfrak{A}))$ is imbedded into $C^{-\infty}(\Gamma\times\Gamma;\ B_d\boxtimes\bar{B}_d)$. On the other hand, $C^{-\infty}(\Gamma\times\Gamma;\ B_d\boxtimes\bar{B}_d)$ is identified with

$$(9)\quad \begin{aligned}\mathscr{K} &= \{\varphi\in C^{-\infty}(G/\underline{H}\times G/\underline{H}\times G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B\boxtimes B\boxtimes\bar{B});\\ &\varphi(x_1a_1, y_1a_1; x_2a_2, y_2a_2)\cdot(a_1^{-1}, a_1^{-1}, a_2^{-1}, a_2^{-1}) = \varphi(x_1, y_1; x_2, y_2)\ \text{for}\ a_1, a_2\in H\},\end{aligned}$$

so we have a continuous linear map $\varkappa$ of $\mathscr{B}(L^2(\mathfrak{A}))$ into $C^{-\infty}(G/\underline{H}\times G/\underline{H}\times G/\underline{H}\times \times G/\underline{H};\ \bar{B}\boxtimes B\boxtimes B\boxtimes\bar{B})$, where $\mathscr{B}(L^2(\mathfrak{A}))$ is equipped with the weak operator topology and the space of generalized sections is topologized by the weak* topology. Let δ be a generalized section of $B\boxtimes\bar{B}$ defined by

$$(10)\qquad \langle\delta, f\rangle = \int_{x\in G/\underline{H}} f(x, x)$$

for $f\in C_c^\infty(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$ and let ι be a linear mapping of $C^{-\infty}(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$ into $C^{-\infty}(G/\underline{H}\times G/\underline{H}\times G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B\boxtimes B\boxtimes\bar{B})$ defined by

$$(11)\quad \begin{aligned}&(\iota\varphi)(x_1, y_1; x_2, y_2) = \varphi(x_1, x_2)\otimes\delta(y_1, y_2)\in\\ &\in \bar{B}_{x_1}\otimes B_{x_2}\otimes B_{y_1}\otimes\bar{B}_{y_2} \cong \bar{B}_{x_1}\otimes B_{y_1}\otimes B_{x_2}\otimes\bar{B}_{y_2}.\end{aligned}$$

Then ι is a topological imbedding of $C^{-\infty}(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$ into a closed subspace $\mathscr{L}$ of $C^{-\infty}(G/\underline{H}\times G/\underline{H}\times G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B\boxtimes B\boxtimes\bar{B})$. Since $\varkappa(\mathfrak{A})\subset\mathscr{K}\cap\mathscr{L}$ by (3), (4), and since $\mathscr{K}\cap\mathscr{L}$ is a closed subspace, we have $\varkappa(\mathfrak{A}'')\subset\mathscr{K}\cap\mathscr{L}$. Now set $l=\iota^{-1}\circ\varkappa(L)$ for a given $L\in\mathfrak{A}''$. Then l satisfies (8) by construction and (7) follows from

$$(12)\qquad \delta(x_1a, x_2a)(a^{-1}, a^{-1}) = \delta(x_1, x_2),\quad a\in H. \qquad ▧$$

In the same way as in Lemma 1, given $R\in\mathfrak{A}'$, we can find a generalized section $r\in C^{-\infty}(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$ such that

$$(13)\qquad r(x_1a, x_2a)(a^{-1}, a^{-1}) = r(x_1, x_2)$$

and

$$(14)\qquad (RF)(x_1, x_2) = \int_{x\in G/\underline{H}} F(x_1, x)r(x, x_2).$$

Lemma 2. *If* $L\in\mathfrak{A}''\cap\mathfrak{A}'$, *then*

$$\operatorname{supp} l\subset\{(x, xa)\ ;\ x\in G/\underline{H},\ a\in H\}.$$

Proof. In (8) and (14), let $L = R$;

$$\int_{x\in G/\underline{H}} l(x_1, x)F(x, x_2) = \int_{x\in G/\underline{H}} F(x_1, x)r(x, x_2) \tag{15}$$

for $F \in \mathfrak{A} \subset C^\infty(G/\underline{H} \times G/\underline{H};\ \bar{B} \boxtimes B)$. Take $x_0 \in G/\underline{H}$ and an open neighborhood U of x_0 such that $Ua_1 \cap Ua_2 \neq \emptyset$ $(a_1, a_2 \in H)$ implies $a_1^{-1}a_2 \in \underline{H}$. If we choose F so that the support of F (considered as a section of $\bar{B} \boxtimes B$) is contained in $\bigcup_{a\in H}(Ua \times Ua)$, then the right hand side of (15) vanishes when $x_1 \notin \bigcup_{a\in H} \bar{U}a$, and

$$\int_{x\in G/\underline{H}} \int_{x_2\in G/\underline{H}} l(x_1, x)F(x, x_2)\varphi(x_2) =$$

$$= \int_{x\in U} l(x_1, x) \int_{x_2\in G/\underline{H}} F(x, x_2)\varphi(x_2)$$

for $\varphi \in C_c^\infty(U, \bar{B})$, because the support of $x \mapsto \int_{x_2\in G/\underline{H}} F(x, x_2)\varphi(x_2)$ is contained in U. In view of the density of $\left\{x \mapsto \int_{x_2\in G/\underline{H}} F(x, x_2)\varphi(x_2)\right\}$ in $C_c(U, \bar{B})$, we have, from these facts, that $l(x_1, x_2) = 0$ if $x_1 \notin \bigcup_{a\in H} \bar{U}a$ and $x_2 \in U$. Since U can be chosen arbitrarily small, we obtain the desired result. ▨

LEMMA 3. *Let X be a C^∞-manifold with a nowhere vanishing C^∞ measure dx and $k(x, x')$ be a distribution on $X \times X$ with its support contained in $\{(x, x) \in X \times X;\ x \in X\}$. Suppose that $C_c^\infty(X) \ni \xi \mapsto (K\xi)(x) = \int_X dx'\, k(x, x')\xi(x')$ yields a bounded linear operator in $L^2(X)$. Then there exists a bounded measurable function f on X such that $(K\xi)(x) = f(x)\xi(x)$.*

Proof. We will show that K commutes with multiplication operators in $L^2(X)$. Take an open ball $U \subset X$ and a sequence $\{\varphi_n\}_{n\geq 1}$ in $C_c^\infty(U)$ such that $\varphi_n \nearrow 1$ on U. Then for any $\xi \in C_c^\infty(X)$, the support of which does not intersect with the boundary of U, $\varphi_n\xi|U = \xi|U$ and therefore $K\varphi_n\xi = \varphi_n K\xi$ for sufficiently large n. Since K is bounded, taking limit $n \to \infty$, we get $Km_U = m_U K$, where m_U is the multiplication operator by characteristic function of U. As U is an arbitrary open ball, K commutes with every multiplication operator. Thus K itself is a multiplication operator. ▨

By applying this lemma to l in Lemma 2, we can find $l_a \in L^\infty(G/\underline{H})$ for every $a \in H$ which satisfies

$$l_{a\underline{h}} = \chi(\underline{h})^{-1} l_a \quad \text{for } \underline{h} \in \underline{H}, \tag{16}$$

$$\begin{aligned} &\int_{(x_1,x_2)\in G/\underline{H}\times G/\underline{H}} l(x_1, x_2) f_1(x_1)\overline{f_2(x_2)} = \\ &= \sum_{a\in H/\underline{H}} \int_{x\in G/\underline{H}} l_a(x) f_1(x)(\overline{f_2(xa)}a^{-1}) \end{aligned} \tag{17}$$

for $f_1, f_2 \in C_c^\infty(G/\underline{H}; B)$ ($\sum_{a\in H/\underline{H}}$ means that summation is taken over a representative of coset space $H/\underline{H}$, which does not depend on a special choice of representative by (16)). Symbolically, (17) is written as

$$\int_{x_2\in G/\underline{H}} l(x_1, x_2) f(x_2) = \sum_{a\in H/\underline{H}} l_a(x_1)(f(x_1 a)a^{-1}), \tag{18}$$

for $f \in C_c^\infty(G/\underline{H}; \bar{B})$. Similarly, there exists $r_a \in L^\infty(G/\underline{H})$ for each $a \in H$, which satisfies

$$r_{a\underline{h}} = \chi(\underline{h}) r_a, \quad \underline{h} \in \underline{H}, \tag{19}$$

$$\int_{x_1\in G/\underline{H}} f(x_1) r(x_1, x_2) = \sum_{a\in H/\underline{H}} r_a(x_2)(f(x_2 a)a^{-1}), \tag{20}$$

for $f \in C_c^\infty(G/\underline{H}; B)$, and the equation (15) is reduced to

$$\sum_{b\in H/\underline{H}} l_b(x_1) F(x_1 b, x_2)(b^{-1}, 1) = \sum_{b\in H/\underline{H}} F(x_1, x_2 b)(1, b^{-1}) r_b(x_2). \tag{21}$$

LEMMA 4. *l_a is a constant function on $G/\underline{H}$, and as a function of $a \in H$, it is a class function; $l_a = l_{bab^{-1}}$, for $a, b \in H$.*

Proof. Take $x_0 \in G/\underline{H}$, $a \in H$, and a neighborhood U of x_0 as in the proof of Lemma 2. If, in (21), we choose F such that supp $F \subset \bigcup_{b\in H/\underline{H}} (U\times U)(1, a)(b, b)$, then for $x_1 \in U$ and for $x_2 \in Uc$ (c is an arbitrary element in H)

(22) $$\text{r.h.s.} = r_{c^{-1}a}(x_2)F(x_1, x_2c^{-1}a)(1, a^{-1}c)$$

and

$$\text{l.h.s.} = l_{a^{-1}c}(x_1)F(x_1a^{-1}c, x_2)(c^{-1}a, 1) =$$

(23)

$$= l_{a^{-1}c}(x_1)F(x_1, x_2c^{-1}a)(1, a^{-1}c).$$

Thus comparing (22) and (23),

(24) $$r_{c^{-1}a}(x_2) = l_{a^{-1}c}(x_1)$$

for $x_1 \in U$, $x_2 \in Uc$. Since a, c are arbitrary, we have obtained the desired conclusion. ▨

At this place, we claim that the converse of Lemma 4 holds. To be precise, let $\{l_a\}_{a\in H}$ be a function on H which is a class function,

(25) $$l_{ab} = l_{ba}, \quad a, b \in H,$$

and is $\underline{H}$-covariant;

(26) $$l_{a\underline{h}} = \chi(\underline{h})^{-1}l_a, \quad \underline{h} \in \underline{H}.$$

Further suppose that

(27) $$F(x_1, x_2) \mapsto \sum_{a\in H/\underline{H}} l_a F(x_1a, x_2)(a^{-1}, 1)$$

for $F \in \mathfrak{A}$, gives rise to a bounded linear operator L in $L^2(\mathfrak{A})$. Then we have

Lemma 5. *L belongs to $\mathfrak{A}'' \cap \mathfrak{A}'$.*

Proof. Let $f, F \in \mathfrak{A}$. We must show that $L(f * F) = f * LF$ and $L(F * f) = = LF * f$. These equalities follow from a direct computation. ▨

Comment. Conditions (25) and (26) insure that (27) is a well-defined map of $\mathfrak{A}$ into $C^{-\infty}(\Gamma, B_d)$.

Lemma 6. *Retain the assumption before Lemma 5 and suppose that $l_a \neq 0$ for some $a \in H$. Then a satisfies the following conditions.*

(∗) *If* $b \in H$ *commutes with* a *modulo* $\underline{H}$, i.e., $a^{-1}b^{-1}ab \in \underline{H}$, *then* $\chi(a^{-1}b^{-1}ab) = 1$.

(∗∗) *The cardinality of the conjugate class of* a *in* $H/\underline{H}$ *is finite.*

Proof. Here we use the original definition of $\mathfrak{A}$; elements in $\mathfrak{A}$ are C^∞-functions on $G\times G$ (see (3)). Then the operation of L on $\mathfrak{A}$ is expressed as

$$(LF)(g_1, g_2) = \sum_{a\in H/\underline{H}} l_a F(g_1 a, g_2)\Delta_{G,H}(a)^{-1/2} \tag{28}$$

for $g_1, g_2 \in G$, $F \in \mathfrak{A}$. Take a section $\varphi \in C_c^\infty(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$ whose support is contained in a sufficiently small neighborhood of $([1],[1])$ and use the same letter φ to denote the corresponding function on $G\times G$. Then $\sum_{a\in H/\underline{H}} \Delta_{G,H}(a)^{-1}\varphi(g_1a, g_2a)$ defines an element in $\mathfrak{A}$, and if we use this function as F in (28), we have

$$|(LF)(g_1, g_2)|^2 =$$

$$= \sum_{a,b,a',b'} \Delta_{G,H}(bb')^{-1/2}\Delta_{G,H}(aa')^{-1}\bar{l}_b l_{b'}\overline{\varphi(g_1ba, g_2a)}\varphi(g_1b'a', g_2a') = \tag{29}$$

$$= \sum_{a,b} |l_b|^2\Delta_{G,H}(a)^{-2}|\varphi(g_1ba, g_2a)|^2\Delta_{G,H}(b)^{-1}$$

(the support of φ should be assumed to be so small that $\overline{\varphi(g_1ba, g_2a)}$ and $\varphi(g_1b'a', g_2a')$ have overlapping supports if and only if $a = a'$ and $b = b'$). Integrating (29) over $G\times G/D$, we obtain

$$(LF|LF) = \oint_{G/\underline{H}\times G/\underline{H}} dg_1\, dg_2 \sum_b |l_b|^2|\varphi(g_1b, g_2)|^2\Delta_{G,H}(b)^{-1} =$$

$$= \sum_b |l_b|^2 \oint_{G/\underline{H}\times G/\underline{H}} dg_1\, dg_2 |\varphi(g_1, g_2)|^2 = \sum_b |l_b|^2(F|F).$$

From this, we have

$$\sum_a |l_a|^2 \leqslant \|L\|^2. \tag{30}$$

Now, in view of (25), boundedness of L implies the condition (∗∗). To prove (∗), let $b \in H$ be such that $a^{-1}b^{-1}ab \in \underline{H}$. Then by (25), (26),

$$l_a = l_{b^{-1}ab} = l_{aa^{-1}b^{-1}ab} = \chi(a^{-1}b^{-1}ab)^{-1}l_a$$

and (∗) holds. ▨

To formulate the theorem in this section, we need some more definitions concerning the Hilbert algebra of $H/\underline{H}$. Consider a function l on H with the following properties

(31) $$l(h\underline{h}) = \chi(\underline{h})^{-1}l(h), \quad h \in H,\ \underline{h} \in \underline{H},$$

(32) support of l is compact modulo $\underline{H}$.

We denote by $\mathscr{B}$ the totality of such functions. $\mathscr{B}$ is furnished with $*$-algebra structure: for l_1, l_2, $l \in \mathscr{B}$,

(33) $$(l_1 * l_2)(a) = \sum_{b \in H/\underline{H}} l_1(b)l_2(b^{-1}a),$$

(34) $$l^*(a) = \overline{l(a^{-1})}.$$

For $l \in \mathscr{B}$, define a linear operator $\Phi(l)$ on $\mathfrak{A}$ by

(35) $$(\varphi(l)F)(g_1, g_2) = \sum_{a \in H/\underline{H}} l(a)F(g_1 a, g_2)\Delta_{G,H}(a)^{-1/2}$$

for $F \in \mathfrak{A}$. Since the relevant summation is finite, $\Phi(l)$ is bounded with respect to L^2-norm and can be extended to a bounded linear operator on $L^2(\mathfrak{A})$ by continuity, which is also denoted by $\Phi(l)$. A direct computation shows that the correspondence $l \mapsto \Phi(l)$ is a faithful $*$-representation of $\mathscr{B}$ on $L^2(\mathfrak{A})$, and its range is contained in $\mathfrak{A}''$. Let S be the set of $a \in H$ which satisfies $(*)$, $(**)$ in Lemma 6 and set $\mathscr{C} = \{l \in \mathscr{B};\ l(ab) = l(ba)$ for a, $b \in H$ and the support of l is contained in $S\}$. Since $\Phi(\mathscr{C}) = \Phi(\mathscr{B}) \cap \mathfrak{A}'' \cap \mathfrak{A}'$ by Lemma 5 and Lemma 6, $\mathscr{C}$ forms a $*$-subalgebra of $\mathscr{B}$ and is commutative.

THEOREM 1. *The center of the left von Neumann algebra of $\mathfrak{A}$ is generated by $\Phi(\mathscr{C})$.*

COROLLARY. *The left von Neumann algebra of $\mathfrak{A}$ is a factor if and only if $S = \underline{H}$*

Before the proof of Theorem 1, we will give some preparatory discussions. First we remark that the $*$-algebra $\mathscr{B}$, described in (31)—(34), becomes a Hilber algebra by the inner product

(36) $$(l \mid l) = \sum_{a \in H/\underline{H}} |l(a)|^2, \quad l \in \mathscr{B}.$$

We construct an isometry of $L^2(\mathscr{B})$ into $L^2(\mathfrak{A})$ which intertwines Φ. Take an approximate δ-function $\xi \in \mathfrak{A}$; $(\xi \mid \xi) = 1$ and ξ is supported by a sufficiently smal neighborhood of $(1, 1)D \in G \times G/D$. Define a linear map I of $\mathscr{B}$ into $L^2(\mathfrak{A})$ by

(37) $$I(l)(g_1, g_2) = \sum_{a \in H/\underline{H}} l(a)\xi(g_1 a, g_2)\Delta_{G,H}(a)^{-1/2}, \quad l \in \mathscr{B}.$$

Then, by a computation as in the proof of Lemma 6, we have

$$(I(l)\,|\,I(l)) = (l\,|\,l), \quad l \in \mathscr{B}. \tag{38}$$

Since $I(l) = \Phi(l)\xi$, I intertwines Φ. Now we will show that Φ can be extended to a normal homomorphism of $\mathscr{B}''$ into $\mathfrak{A}''$. Set

$$\mathscr{M} = \{L \in \mathscr{B}(L^2(\mathfrak{A}))\ ;\ \text{there is a function } l \text{ on } H \text{ satisfying (31) such that}$$

$$(LF)(g_1, g_2) = \sum_{a \in H/\underline{H}} \Delta_{G,H}(a)^{-1/2} l(a) F(g_1 a, g_2) \text{ for } F \in \mathfrak{A}\}. \tag{39}$$

Then $\mathscr{M}$ becomes a von Neumann subalgebra of $\mathfrak{A}''$ and makes $\overline{I(\mathscr{B})}$ invariant ($\overline{I(\mathscr{B})}$ denotes the L^2-closure of $I(\mathscr{B})$). Furthermore, the restriction of $\mathscr{M}$ to the subspace $\overline{I(\mathscr{B})}$ yields an isomorphism of $\mathscr{M}$ onto $\mathscr{M}_{\overline{I(\mathscr{B})}}$, and the induced algebra $\mathscr{M}_{\overline{I(\mathscr{B})}}$ is transfered into a subalgebra of $\mathscr{B}''$ by the isometry I. Thus we get a normal isomorphism Ψ of $\mathscr{M}$ into $\mathscr{B}''$. Since $\Psi(\Phi(\mathscr{B})) = \mathscr{B}$ and $\mathscr{B}$ is weakly dense in $\mathscr{B}''$, we conclude that Ψ is an isomorphism of $\mathscr{M}$ onto $\mathscr{B}''$ and the restriction of Ψ^{-1} to $\mathscr{B}$ coincides with Φ. In other words, we can extend Φ to $\mathscr{B}''$ as a normal homomorphism and $\Phi(\mathscr{B}'') = \mathscr{M}$.

Proof of Theorem 1. Since $\mathfrak{A}'' \cap \mathfrak{A}' \subset \mathscr{M}$, we need to prove that $\mathscr{C}'' = \Phi^{-1}(\mathfrak{A}'' \cap \mathfrak{A}')$. By Lemma 5, $\mathscr{C}'' \subset \Phi^{-1}(\mathfrak{A}'' \cap \mathfrak{A}')$. For the reverse inclusion, we will show that $\mathscr{C}' \subset \Phi^{-1}(\mathfrak{A}'' \cap \mathfrak{A}')'$. Let K be a bounded linear operator in $L^2(\mathscr{B})$. Then K can be expressed in a matrix form:

$$(Kf)(a) = \sum_{a' \in H/\underline{H}} k(a, a') f(a') \tag{40}$$

where k (call the matrix function of K) is a function of $H \times H$ satisfying

$$k(a\underline{h}, a'\underline{h}') = \chi(\underline{h}^{-1}\underline{h}') k(a, a'), \tag{41}$$

here $\underline{h}, \underline{h}' \in \underline{H}$. For $L \in \mathfrak{A}'' \cap \mathfrak{A}'$, we can find a function l on H which satisfies (25), (26) and represents L by the relation (28). Then a direct computation shows that the matrix functions of $K\Phi^{-1}(L)$ and $\Phi^{-1}(L)K$ are given by

$$\sum_{a \in H/\underline{H}} l(a) k(a_1, a a_2), \tag{42}$$

$$\sum_{a \in H/\underline{H}} l(a) k(a^{-1} a_1, a_2). \tag{43}$$

Thus $K \in \Phi^{-1}(\mathfrak{A}'' \cap \mathfrak{A}')'$ if and only if (42) = (43) for $a_1, a_2 \in H$. By Lemma 6, we can find a family $\{l_\alpha\}$ of elements in $\mathscr{C}$, with pairwise disjoint support, such that

$$(44) \qquad l(a) = \sum_\alpha l_\alpha(a).$$

Now suppose that K commutes with $\mathscr{C}$. Then K satisfies (42) = (43) with l replaced by l_α, and therefore it holds for $l = \sum_\alpha l_\alpha$. (Note that the summations in (42), (43) are absolutely convergent.) In other words, K commutes with $\Phi(\mathfrak{A}'' \cap \mathfrak{A}')$. This shows that $\mathscr{C}' \subset \Phi^{-1}(\mathfrak{A}'' \cap \mathfrak{A}')'$ and the proof of Theorem 1 is completed. ▨

3. TYPE ANALYSIS OF $\mathfrak{A}''$

In this section, we assume that

(C) the commutator subgroup of H is contained in $\underline{H}$.

Then the condition (**) is trivially satisfied and the set S (see before Theorem 1) is characterized as follows:

$$(45) \qquad S = \{a \in H\,;\, \chi(a^{-1}b^{-1}ab) = 1 \text{ for all } b \in H\}.$$

From this, we see that S is a normal subgroup of H. To investigate the type of $\mathfrak{A}''$, we first give an explicit central decomposition of $\mathfrak{A}''$. Set

$$(46) \quad \hat{S}_\chi = \{\eta\,;\, \eta \text{ is a unitary character of } S \text{ whose restriction to } \underline{H} \text{ is equal to } \chi\}.$$

By pointwise multiplication, $\hat{S}_\chi$ is an $(S/\underline{H})^\wedge$-principal homogeneous space. (The irreducible decomposition of $\mathrm{Ind}_{\underline{H}\uparrow S}\chi$ shows that $\hat{S}_\chi$ is non-empty.) For each $\eta \in \hat{S}_\chi$, we can construct a Hilbert algebra $\mathfrak{A}_\eta$ as in § 1, here we use S, η in place of $\underline{H}, \chi$. Take $f \in \mathfrak{A}$ and set

$$(47) \qquad f_\eta(g_1, g_2) = \sum_{a \in S/\underline{H}} \Delta_{G,H}(a)^{-1/2}\eta(a)^{-1} f(g_1 a, g_2).$$

(Note that $S/\underline{H} \ni [a] \mapsto [(a, 1)] \in D_S/D$ is an isomorphism, where D_S is the 'D' for S.) Then $f_\eta \in \mathfrak{A}_\eta$ and one sees that $\{\hat{f} = \{f_\eta\}_{\eta \in \hat{S}_\chi}\,;\, f \in \mathfrak{A}\}$ forms a continuous field $\{\mathfrak{A}_\eta\}_{\eta \in \hat{S}_\chi}$ of Hilbert algebras, by a routine calculation. Also a direct computation shows that

$$(F|F) = \int_{\hat{S}_\chi} d\eta\,(F_\eta|F_\eta)_\eta \quad \text{for } F \in \mathfrak{A}, \tag{48}$$

where $(\ |\ )_\eta$ is the inner product in $\mathfrak{A}_\eta$ and $d\eta$ is a Haar measure through identification $\hat{S}_\chi \cong (S/\underline{H})^\wedge$. Note that $d\eta$ is unique up to positive constant. Furthermore, by applying Theorem 1 to $\mathfrak{A}_\eta$, one sees that $\mathfrak{A}_\eta''$ is a factor and hence the above decomposition is central (see [3], p. 191 Corollaire). Summarizing these considerations, we have:

PROPOSITION. *Through* $\mathfrak{A} \ni f \mapsto \hat{f} = \{f_\eta\}_{\eta\in\hat{S}_\chi}$, *the Hilbert algebra* $\mathfrak{A}$ *is identified with the direct integral of Hilbert algebras,* $\int_{\hat{S}_\chi}^{\oplus} d\eta\ \mathfrak{A}_\eta$, *and the center of* $\mathfrak{A}''$ *is identified with the algebra of diagonalizable operators* $L^\infty(\hat{S}_\chi)$.

THEOREM 2. *Under the condition* (C), *the left von Neumann algebra* $\mathfrak{A}''$ *is of type* II *if* H/S *is infinite and it is of type* I *otherwise.*

Proof. By Proposition, it suffices to prove the theorem when $\mathfrak{A}''$ is a factor, i.e., $S = \underline{H}$. Then $\mathscr{B}''$ is a factor by Corollary to Theorem 1. Let $\tau_{\mathfrak{A}}$ (resp. $\tau_{\mathscr{B}}$) be the canonical trace of $\mathfrak{A}''$ (resp. $\mathscr{B}''$) associated with the Hilbert algebra structure. By (38),

$$\tau_{\mathfrak{A}}(\Phi(l)^* * \Phi(l)) = \tau_{\mathscr{B}}(l^* * l), \quad \text{for } l \in \mathscr{B}''. \tag{49}$$

From this, one sees that $\mathfrak{A}''$ is of type II if $\mathscr{B}''$ is so. As $\mathscr{B}''$ is a finite factor, it is of type II if $\dim L^2(\mathscr{B}) = |H/\underline{H}| = \infty$. Thus we have showed that $\mathfrak{A}''$ is of type II if $|H/\underline{H}| = \infty$. Now suppose that $|H/\underline{H}| < \infty$. In this case, we can regard $L^2(\mathfrak{A})$ as a subspace of $L^2(G/\underline{H}\times G/\underline{H})$ and the projection P to $L^2(\mathfrak{A})$ is given by

$$(PF)(g_1, g_2) = |H/\underline{H}|^{-1} \sum_{a\in H/\underline{H}} \Delta_{G,H}(a)^{-1} F(g_1 a, g_2 a), \tag{50}$$

for $F \in C_c^\infty(G/\underline{H}\times G/\underline{H};\ \bar{B}\boxtimes B)$. Through this identification, we see that $\mathfrak{A}'' \subset \mathscr{B}(L^2(G/\underline{H}))\otimes 1$ and $\tau_{\mathfrak{A}}$ coincides with the restriction to $\mathfrak{A}''$ of the canonical trace of $\mathscr{B}(L^2(G/\underline{H}))$. In particular, the image of $\tau_{\mathfrak{A}}$ is discrete and hence $\mathfrak{A}$ is of type I. This completes the proof of Theorem 2. ▨

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