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ON COUPLING CONSTANT THRESHOLDS IN TWO DIMENSIONS

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1. INTRODUCTION

Recently there has been an increased interest in low-energy phenomena and also in the study of two-dimensional systems due to applications to surfaces in solid state physics.

In this paper we study the low-energy behaviour of a two-dimensional two-body system.

Consider namely the Schrödinger operator

$$H_\lambda = -\triangle + \lambda V$$

on $L^2(\mathbf{R}^\nu)$ where $\triangle$ is the Laplacian and V is a suitable short-range potential. The spectrum of H_λ then consists of possibly some negative eigenvalues in addition to the continuous spectrum $[0, \infty)$.

Assume now that $E(\lambda)$ is a negative eigenvalue of H_λ such that

$$E(\lambda) \uparrow 0$$

as $\lambda \downarrow \lambda_0$, i.e. $E(\lambda)$ approaches the continuous spectrum.

While perturbation of discrete eigenvalues for self-adjoint operators yield analytic expansions for the eigenvalue in terms of the perturbation parameter, this is not necessarily the case when an eigenvalue approaches the continuous spectrum.

An interesting question is therefore whether one can find convergent expansions in singular quantities like $(\lambda - \lambda_0)^\alpha$ or $(\lambda - \lambda_0)^\alpha \ln(\lambda - \lambda_0)$.

This problem has recently been extensively studied by Klaus and Simon [8] in all dimensions ν, and they find convergent expansions for $E(\lambda)$ in various functions of λ like the two mentioned above and others. The results do very much depend the dimension ν. However there is one case in which Klaus and Simon [8] do not give a convergent expansion, namely in the two dimensional case which corresponds to s-waves when V is central and where $\lambda_0 \neq 0$.

In this paper we show that $E(\lambda)$ in this case has a convergent expansion given by

$$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m(\lambda-\lambda_0)^n$$

where

$$\sigma = \exp(-a(\lambda-\lambda_0)^{-1})$$

$$\tau = (\lambda-\lambda_0)^{-2}\exp(-a(\lambda-\lambda_0)^{-1})$$

and a and c_{200} are explicitly computed and given by (14) and (15) respectively provided $\int d^2x\, V(x) \neq 0$.

In one and two dimensions it is also possible to have $\lambda_0 = 0$. This case has been considered by Klaus [6] *in one dimension and* Simon [10] in both one and two dimensions giving only the leading order for the eigenvalue in two dimensions. The leading order in two dimensions was already computed nonrigorously in Landau and Lifshitz [9], p. 163.

We show that in this case when $\int d^2x\, V(x) < 0$

$$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m\lambda^n$$

where

$$\sigma = \exp(-a/\lambda)$$

$$\tau = \frac{a}{\lambda}\exp(-a/\lambda)$$

$$a = \left[-\frac{1}{2\pi}\int d^2x\, V(x)\right]^{-1}$$

and c_{200} is given by (8). When $\int d^2x\, V(x) = 0$ we have

$$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m\lambda^n$$

where

$$\sigma = \exp\left(-\frac{a}{\lambda^2} + \frac{b}{\lambda}\right)$$

$$\tau = \lambda^{-3}\exp\left(-\frac{a}{\lambda^2} + \frac{b}{\lambda}\right)$$

and where a, b and c_{200} are computed explicitly and are given by (10), (10) and (12) respectively.

We remark here that the case $\lambda_0 = 0$ is related to the so-called short-range expansion in connection with point interactions.

Namely one can show [5] that

$$H_\varepsilon = -\triangle + \frac{\mu(\varepsilon)}{\ln \varepsilon} V(\cdot/\varepsilon)$$

will converge as $\varepsilon \downarrow 0$ *to the Hamiltonian with a point interaction at zero, formally*

$$H = -\triangle - \nu\delta$$

when $\mu(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$. Thus we see that the coupling constant $\mu(\varepsilon)/\ln \varepsilon \to 0$ as $\varepsilon \downarrow 0$. See also [3], [2], [1] for related work in one and three dimensions.

The method we use is based on the Birman-Schwinger principle in which the eigenvalue equation $H_\lambda \psi_\lambda = E(\lambda)\psi_\lambda$ is replaced by the equivalent equation

$$(1 + \lambda K_{E(\lambda)})\varphi = 0 \quad \text{where } K_{E(\lambda)} = |V|^{1/2}(-\triangle - E(\lambda))^{-1}|V|^{1/2}\operatorname{sgn} V$$

and using the special form $K_{E(\lambda)}$ has in two dimensions we derive an implicit equation, using the so-called Weinstein-Aronszajn determinant, which gives the expansion.

The approach using determinants, also Fredholm determinants, instead of perturbation theory, has the advantage that it can be used to study resonances in addition to eigenvalues. See [3], [4].

2. THE EXPANSIONS

Recall the following two theorems, the first one treating the case when $\lambda_0 = 0$ and the second treating $\lambda_0 > 0$.

THEOREM 1. (Simon [10]). *Let* $\int d^2x|V(x)|^{1+\delta} < \infty$ *and* $\int d^2x|V(x)|(1 + |x|^\delta) < < \infty$ *for some* $\delta > 0$.

Then $H_\lambda = -\Delta + \lambda V$ *has for all sufficiently small* λ *a negative eigenvalue* $E(\lambda)$ *approaching zero as* $\lambda \downarrow 0$ *iff* $\int d^2x V(x) \leqslant 0$.

Moreover

$$\lim_{\lambda \downarrow 0} \lambda \ln(-E(\lambda)) = \left[-\frac{1}{4\pi} \int d^2x V(x) \right]^{-1} \tag{1}$$

if $\int d^2x V(x) < 0$ *and*

$$\lim_{\lambda \downarrow 0} \lambda^2 \ln(-E(\lambda)) < 0 \tag{2}$$

if $\int d^2x V(x) = 0$.

THEOREM 2. (Klaus and Simon [8]). *Let* $V \in C_0^\infty(\mathbf{R}^2)$ *and assume* $V \leqslant 0$. *If* $E(\lambda)$ *is a negative eigenvalue of* $H_\lambda = -\Delta + \lambda V$ *and* $E(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0 > 0$, *then one and only one of the following three situations holds:*

(i) $0 \in \sigma_p(H_{\lambda_0})$. *Then*

$$E(\lambda) = \sum_{\substack{m=0 \\ n=2}}^{\infty} c_{mn} (\lambda - \lambda_0)^{n/2} [(\lambda - \lambda_0)^{1/2} \ln(\lambda - \lambda_0)]^m \tag{3}$$

and $c_{20} < 0$.

(ii) $0 \notin \sigma_p(H_{\lambda_0})$ *and*

$$\begin{aligned} E(\lambda) = \sum_{\substack{m,k=0 \\ n=2}}^{\infty} c_{nmk} [-(\lambda - \lambda_0)/\ln(\lambda - \lambda_0)]^{n/2} [-1/\ln(\lambda - \lambda_0)]^m \cdot \\ \cdot \{-\ln[\ln(\lambda - \lambda_0)^{-1}]/\ln(\lambda - \lambda_0)\}^k \end{aligned} \tag{4}$$

and $c_{200} < 0$.

(iii) $0 \notin \sigma_p(H_{\lambda_0})$ *and*

$$\lim_{\lambda \downarrow \lambda_0} E(\lambda) \exp(1/c(\lambda - \lambda_0)) < 0 \quad \textit{for some } c > 0. \tag{5}$$

If several eigenvalues approach zero at once, at most one is in case (iii) and at most two in case (ii). For a central potential V we have that s-waves are in case (iii), p-waves in case (ii) and $l \geqslant 2$ in case (i).

REMARKS. 1. Both proofs use the Birman-Schwinger principle, i.e. to replace the equation $H_\lambda \psi_\lambda = E(\lambda)\psi_\lambda$ by the equivalent equation $(1 + \lambda K)\varphi_\lambda = 0$ where K_E is given by $K_E = |V|^{1/2}(-\triangle - E)^{-1}|V|^{1/2}\operatorname{sgn} V$. However the proof of Theorem 1 uses modified Fredholm determinants to derive an implicit equation determining the eigenvalue while in the proof of Theorem 2 one uses the so-called Weinstein--Aronszajn determinant to derive the same equation.

2. The very strong conditions on the potential in Theorem 2 reflects the fact that Klaus-Simon [8] consider all cases in all dimensions and therefore want to avoid too much technicalities.

We can now state the following theorem.

THEOREM 3. *Let* $\int d^2x |V(x)|^{1+\delta} < \infty$ *and* $\int d^2x |V(x)| e^{\eta|x|} < \infty$ *for some* $\delta, \eta > 0$.

Assume that $H_\lambda = -\triangle + \lambda V$ *has a negative eigenvalue* $E(\lambda) \uparrow 0$ *as* $\lambda \downarrow \lambda_0 \geqslant 0$. *If* $\lambda_0 > 0$, *we assume furthermore that* $E(\lambda)$ *is in case* (iii) *of Theorem* 2.

Then $E(\lambda)$ *has the following convergent expansion*

(a) $\lambda_0 = 0$.

$$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m\lambda^n \tag{6}$$

where

$$\sigma = \exp(-a/\lambda)$$

$$\tau = -\frac{a}{\lambda}\exp(-a/\lambda) \tag{7}$$

$$a = \left[-\frac{1}{2\pi}\int d^2x V(x)\right]^{-1}$$

and

$$c_{200} = 4\exp\left[-2\left(\int d^2x V(x)\right)^{-2}\iint d^2x\, d^2y V(x)\ln|x-y|V(y) + 2C\right] \tag{8}$$

(C is Euler's constant) when $\int d^2x V(x) < 0$.

If $\int d^2x V(x) = 0$, *we have*

$$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m\lambda^n \tag{9}$$

where

$$\sigma = \exp\left(-\frac{a}{\lambda^2} + \frac{b}{\lambda}\right)$$

$$\tau = \lambda^{-3}\exp\left(-\frac{a}{\lambda^2} + \frac{b}{\lambda}\right)$$

(10)

$$a = \left[(2\pi)^{-2}\iint d^2x d^2y V(x)\ln|x-y|V(y)\right]^{-1} > 0$$

$$b = -\frac{a^2}{2\pi}(u, M_0^2 v)$$

where

(11) $$M_0(x, y) = -\frac{1}{2\pi}v(x)\ln|x-y|u(y) + \frac{1}{2\pi}(\ln 2 + C)v(x)u(y)$$

and

(12) $$c_{200} = \exp 2\left[\frac{a^2}{2\pi}(u, M_0^3 v) - b^2 a^{-1}\right].$$

(b) $\lambda_0 > 0$. *In this case we have if* $\int d^2x V(x) \neq 0$

(13) $$E(\lambda) = -\sum_{\substack{m,n=0\\ l=2}}^{\infty} c_{lmn}\sigma^l\tau^m(\lambda-\lambda_0)^n$$

where σ *and* τ *are defined by the following expressions when there is only one eigenvalue approaching zero as* $\lambda \downarrow \lambda_0$

$$\sigma = \exp(-a(\lambda-\lambda_0)^{-1})$$

(14) $$\tau = (\lambda-\lambda_0)^{-2}\exp(-a(\lambda-\lambda_0)^{-1})$$

$$a = \left[-\frac{\lambda_0^2}{(2\pi)^2}\iint d^2x\, d^2y\, u(x)(1+\lambda_0 M_0)^{-1}v(x)\ln|x-y|u(y)(1+\lambda_0 M_0)^{-1}v(y)\right]^{-1}$$

(M_0 as above) and

(15) $$c_{200} = b^2$$

where b is given by (62) at the end of the paper. When there are several eigenvalues approaching zero as $\lambda \downarrow \lambda_0$, the resolvent $(1 + \lambda M_0)^{-1}$ has to be replaced by the reduced resolvent T given by (65) in the definitions of a and b.

Proof. We will use the Birman-Schwinger principle followed by the Weinstein-Aronszajn determinant.

According to the Birman-Schwinger principle $E(\lambda) = -\alpha^2(\lambda)$ $(\alpha(\lambda) > 0)$ is a negative eigenvalue of $H_\lambda = -\triangle + \lambda V$ iff -1 is an eigenvalue of $\lambda K_\alpha, \alpha = \alpha(\lambda)$, where K_α has integral kernel

(16) $$K_\alpha(x, y) = v(x) G_\alpha(x - y) u(y)$$

where

(17) $$v(x) = |V(x)|^{1/2}, \quad u(x) = v(x) \operatorname{sgn} V(x)$$

and $G_\alpha(x - y)$ is the integral kernel of $(-\triangle + \alpha^2)^{-1}$.

In two dimensions one has (see Lemma 3.1 in Simon [10]) that

(18) $$G_\alpha(x - y) = f(\alpha|x - y|) \ln \alpha|x - y| + g(\alpha|x - y|)$$

where f, g are entire functions with

(19) $$f(0) = -\frac{1}{2\pi}$$ $$g(0) = \frac{1}{2\pi}(\ln 2 + C)$$

(C is Euler's constant). The problem is that, due to the logarithmic singularity when $\alpha \to 0$, K_α has no limit as $\alpha \to 0$. Therefore we split off the divergent part

(20) $$K_\alpha = M_\alpha + L_\alpha$$

where

(21) $$M_\alpha(x, y) = v(x)[f(\alpha|x - y|) \ln |x - y| + g(\alpha|x - y|) + \alpha \ln \alpha \, h(\alpha|x - y|)] u(y)$$

and

(22) $$h(x) = \frac{1}{x}(f(x) - f(0))$$

is entire and

(23) $$L_\alpha(x, y) = -\frac{\ln \alpha}{2\pi} v(x) u(y).$$

M_α converges in Hilbert-Schmidt norm as $\alpha \to 0$ to M_0 where

(24) $$M_0(x, y) = -\frac{1}{2\pi} v(x) \ln |x - y| u(y) + \frac{1}{2\pi}(\ln 2 + C) v(x) u(y).$$

Observe that L_α equals, up to a constant, a non-orthogonal projection when $\int d^2x V(x) \neq 0$, while L_α is nilpotent when $\int d^2x V(x) = 0$.

Following e.g. Klaus-Simon [8] we have, first for $|z| \gg 1$, that

$$(z - K_\alpha)^{-1} = (z - M_\alpha - L_\alpha)^{-1} = (1 - (z - M_\alpha)^{-1}L_\alpha)^{-1}(z - M_\alpha)^{-1} =$$

$$= (z - M_\alpha)^{-1} + \sum_{n=0}^{\infty} (z - M_\alpha)^{-1}[L_\alpha(z - M_\alpha)^{-1}L_\alpha]^n(z - M_\alpha)^{-1} =$$

(25)

$$= (z - M_\alpha)^{-1} + \sum_{n=0}^{\infty} (z - M_\alpha)^{-1}\left[-\frac{\ln\alpha}{2\pi}(u, (z - M_\alpha)^{-1}v)\right]^n L_\alpha(z - M_\alpha)^{-1} =$$

$$= (z - M_\alpha)^{-1} + \omega_\alpha^{-1}(z)(z - M_\alpha)^{-1}L_\alpha(z - M_\alpha)^{-1}$$

where

$$\omega_\alpha(z) = 1 + \frac{\ln\alpha}{2\pi}(u, (z - M_\alpha)^{-1}v) \tag{26}$$

is the Weinstein-Aronszajn determinant.

By analytic continuation this identity extends to all z such that

$$z \notin \sigma(M_\alpha) \cup \{z \mid \omega_\alpha(z) = 0\}. \tag{27}$$

Let $z = -\lambda^{-1}$. Then

$$\omega_\alpha(-\lambda^{-1}) = 1 - \lambda\frac{\ln\alpha}{2\pi}(u, (1 + \lambda M_\alpha)^{-1}v). \tag{28}$$

Since $1 + \lambda M_\alpha$ always is invertible when λ is small, we see that eigenvalues are determined by $\omega_\alpha(-\lambda^{-1}) = 0$ when $\lambda_0 = 0$, while the situation is more involved when $\lambda_0 > 0$.

First we discuss $\lambda_0 = 0$: $\omega_\alpha(-\lambda^{-1}) = 0$ can be rewritten as

$$\frac{1}{\ln\alpha} = \frac{\lambda}{2\pi}(u, (1 + \lambda M_\alpha)^{-1}v). \tag{29}$$

We remark here that this is the same equation one obtains by manipulating the modified Fredholm determinant, see Simon [10].

While Klaus-Simon [8] first use perturbation theory to give an expansion for λ in terms of α (in the cases where they give expansions) and then an implicit function theorem to find α as a function of λ, we will use the implicit function theorem on (29) directly.

From Theorem 1 we know that the asymptotic behaviour depends on whether $\int d^2x V(x)$ equals zero or not.

a) $\int d^2x V(x) < 0$. We then write

$$\alpha(\lambda) = e^{-\frac{a}{\lambda}}(b + \beta(\lambda)) \tag{30}$$

where $b > 0$ and $\beta(\lambda)$ are to be determined and

$$a = -\left[\frac{1}{2\pi}\int d^2x V(x)\right]^{-1} > 0. \tag{31}$$

Consider now

$$\begin{aligned}\lambda^{-2}\left[1/\ln\alpha - \frac{\lambda}{2\pi}(u, (1 + \lambda M_\alpha)^{-1}v)\right] &= \lambda^{-1}\left[-a^{-1}\left(1 - \frac{\lambda}{a}\ln(b+\beta)\right)^{-1} - \right.\\ &\left. - \frac{1}{2\pi}(u, (1 - \lambda M_0 + \lambda(M_\alpha - M_0) + \lambda^2 M_\alpha^2(1 + \lambda M_\alpha)^{-1})v)\right] = \\ &= -\frac{1}{a^2}\ln(b+\beta) - \frac{\lambda}{a^2}\ln^2(b+\beta)(\lambda\ln(b+\beta) - a)^{-1} + \frac{1}{2\pi}(u, M_0 v) + \\ &+ (u, (M_\alpha - M_0)v) - \frac{\lambda}{2\pi}(u, M_\alpha^2(1 + \lambda M_\alpha)^{-1}v).\end{aligned} \tag{32}$$

By letting $\lambda \downarrow 0$, and hence $\alpha \to 0$ and $\beta \to 0$, we see that

$$\begin{aligned}\frac{\ln b}{a^2} &= \frac{1}{2\pi}(u, M_0 v) = \\ &= -\frac{1}{(2\pi)^2}\iint d^2x\, d^2y\, V(x)\ln|x-y|V(y) + \frac{1}{(2\pi)^2}(\ln 2 + C)\left[\int d^2x V(x)\right]^2,\end{aligned} \tag{33}$$

$$b = 2\exp\left[-\left(\int d^2x V(x)\right)^{-2}\iint d^2x\, d^2y\, V(x)\ln|x-y|V(y) + C\right]. \tag{34}$$

Introduce now the variables

$$\text{(35)} \qquad \begin{aligned} \sigma &= \exp(-a/\lambda) \\ \tau &= -\frac{a}{\lambda}\exp(-a/\lambda). \end{aligned}$$

Then

$$\text{(36)} \qquad \begin{aligned} \alpha &= \sigma(b+\beta) \\ \alpha\ln\alpha &= (\tau+\sigma\ln(b+\beta))(b+\beta) \end{aligned}$$

which implies that M_α can be considered as a function of β, σ, τ, viz. $M_\alpha = M_{\beta,\sigma,\tau}$, which is analytic in β, σ, τ.

We define ($M_0 = M_{0,0,0}$)

$$\text{(37)} \qquad \begin{aligned} F(\beta,\sigma,\tau,\lambda) &= -\frac{1}{a^2}\ln(b+\beta) - \frac{\lambda}{a^2}\ln^2(b+\beta)(\lambda\ln(b+\beta)-a)^{-1} + \\ &+ \frac{1}{2\pi}(u, M_0 v) + \frac{1}{2\pi}(u,(M_{\beta,\sigma,\tau}-M_0)v) - \\ &- \frac{\lambda}{2\pi}(u, M^2_{\beta,\sigma,\tau}(1+\lambda M_{\beta,\sigma,\tau})^{-1}v). \end{aligned}$$

Then

(38)
(i) F is analytic
(ii) $F(0,0,0,0)=0$
(iii) $\dfrac{\partial F}{\partial\beta}(0,0,0,0) = -\dfrac{1}{a^2 b} \neq 0$

and hence we can find an analytic function

$$\text{(39)} \qquad \beta = \beta(\sigma,\tau,\lambda) = \sum_{l,m,n=0}^{\infty} \tilde{b}_{lmn}\sigma^l\tau^m\lambda^n$$

such that

$$\text{(40)} \qquad F(\beta(\sigma,\tau,\lambda),\sigma,\tau,\lambda) = 0.$$

Thus

$$\text{(41)} \qquad \alpha(\lambda) = (b+\beta)\exp(-a/\lambda) = \sum_{\substack{m,n=0\\ l=1}}^{\infty} b_{lmn}\sigma^l\tau^m\lambda^n$$

with

(42) $$b_{10} = b.$$

(b) $\int d^2x V(x) = 0$. This case is more difficult as can also be seen the one--dimensional case ([6], [7], [10]).

Again we use Theorem 1 to write

(43) $$\alpha(\lambda) = \exp(-a\lambda^{-2} + b\lambda^{-1})(c + \beta(\lambda))$$

where a, b, c and β are to be determined, $a > 0$.

Let

(44) $$\sigma = \exp(-a\lambda^{-2} + b\lambda^{-1})$$
$$\tau = \lambda^{-3}\exp(-a\lambda^{-2} + b\lambda^{-1}).$$

Regarding M_α as a function of β, σ, τ, λ we see that

(45) $$M_{\beta,\sigma,\tau,\lambda} - M_0 = \lambda^3 A_{\beta,\sigma,\tau,\lambda}.$$

Similarly

(46) $$M^2_{\beta,\sigma,\tau,\lambda} - M^2_0 = \lambda^3 B_{\beta,\sigma,\tau,\lambda}$$
$$M^3_{\beta,\sigma,\tau,\lambda} - M^3_0 = \lambda^3 C_{\beta,\sigma,\tau,\lambda}$$

where

(47) $$\|A_{\beta,\sigma,\tau,\lambda}\|,\ \|B_{\beta,\sigma,\tau,\lambda}\|,\ \|C_{\beta,\sigma,\tau,\lambda}\| \to 0.$$

Consider now

$$F(\beta, \sigma, \tau, \lambda) \equiv \lambda^{-4}\left[1/\ln\alpha - \frac{\lambda}{2\pi}(u, (1 + \lambda M_\alpha)^{-1}v)\right] =$$

(48) $$= \lambda^{-2}\left[-a^{-1}\left(1 - \left(\frac{\lambda}{a}b + \frac{\lambda^2}{a}\ln(c+\beta)\right)\right)^{-1} + \frac{1}{2\pi}(u, M_\alpha v) - \frac{\lambda}{2\pi}(u, M^2_\alpha v) + \right.$$
$$\left. + \frac{\lambda^2}{2\pi}(u, M^3_\alpha v) - \frac{\lambda^3}{2\pi}(u, M^4_\alpha(1 + \lambda M_\alpha)^{-1}v)\right] -$$
$$-\frac{1}{a^2}\ln(c + \beta) + \left(\frac{b}{a} + \frac{\lambda}{a}\ln(c + \beta)\right)^2(\lambda^2\ln(c + \beta) + \lambda b - a)^{-1} +$$
$$+ \frac{\lambda}{2\pi}(u, A_{\beta,\sigma,\tau,\lambda}v) - \frac{\lambda^2}{2\pi}(u, B_{\beta,\sigma,\tau,\lambda}v) + \frac{\lambda^3}{2\pi}(u, C_{\beta,\sigma,\tau,\lambda}v) -$$
$$- \frac{\lambda}{2\pi}(u, M^4_{\beta,\sigma,\tau,\lambda}(1 + \lambda M_{\beta,\sigma,\tau,\lambda})^{-1}v)$$

when we choose

(49)
$$a=\left[\frac{1}{2\pi}(u,M_0v)\right]^{-1}=\left[(2\pi)^{-2}\iint d^2x\,d^2y\,V(x)\ln|x-y|V(y)\right]^{-1}>0,$$
$$b=-\frac{a^2}{2\pi}(u,M_0^2v)$$

and

(50)
$$\frac{\ln c}{a^2}=\frac{1}{2\pi}(u,M_0^3v)-b^2/a^3$$

or

(51)
$$c=\exp\left[\frac{a^2}{2\pi}(u,M_0^3v)-b^2/a\right].$$

Again we have

(52)
(i) F is analytic
(ii) $F(0,0,0,0)=0$
(iii) $\frac{\partial F}{\partial\beta}(0,0,0,0)=-\frac{1}{a^2c}\neq 0.$

Hence we find

(53)
$$\alpha(\lambda)=\sum_{\substack{m,n=0\\ l=1}}^{\infty}b_{lmn}\sigma^l\tau^m\lambda^n$$

where

(54)
$$b_{100}=c.$$

We now consider $\lambda_0>0$: As we already mentioned, $\lambda_0>0$ is more complicated in the sense that we now have to take into consideration whether $1+\lambda_0M_0$ is invertible or not.

However it follows from Klaus and Simon [8] that in case (iii) of Theorem 2, which is the case we are interested in here, is characterized by the existence of $(1+\lambda_0M_0)^{-1}$ and

(55)
$$\omega_\alpha(-\lambda^{-1})=0,$$

when there is only one eigenvalue approaching zero as $\lambda\downarrow\lambda_0$. In addition we have

(56)
$$(u,(1+\lambda_0M_0)^{-1}v)=0.$$

Thus we obtain the same equation as before, viz.

(57)
$$\frac{1}{\ln\alpha}=\frac{\lambda}{2\pi}(u,(1+\lambda M_\alpha)^{-1}v)$$

which however now has to be expanded around $\lambda=\lambda_0>0$.

We use the same strategy, i.e. we write

(58) $$\alpha(\lambda) = e^{(-a(\lambda-\lambda_0)^{-1})}(b + \beta(\lambda))$$

and define

(59) $$\sigma = \exp(-a(\lambda - \lambda_0)^{-1})$$
$$\tau = (\lambda - \lambda_0)^{-2}\exp(-a(\lambda - \lambda_0)^{-1}).$$

Let

$$F(\beta, \sigma, \tau, \lambda) \equiv (\lambda - \lambda_0)^{-2}\left[1/\ln\alpha - \frac{\lambda}{2\pi}(u, (1 + \lambda M_\alpha)^{-1}v)\right] =$$

$$= (\lambda - \lambda_0)^{-2}\left[-a^{-1}(\lambda - \lambda_0)\left(1 - \frac{\lambda - \lambda_0}{a}\ln(b + \beta)\right)^{-1} -\right.$$

(60) $$\left. - \frac{\lambda}{2\pi}(u, (1 + \lambda_0 M_0 + \lambda(M_\alpha - M_0) + (\lambda - \lambda_0)M_0)^{-1}v)\right] =$$

$$= -\frac{\ln(b + \beta)}{a^2} + \frac{1}{2\pi}(u, (1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}v) -$$

$$-\frac{\lambda_0}{2\pi}(u, (1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}v) + (\lambda - \lambda_0)C_{\beta,\sigma,\tau,\lambda,\lambda_0}$$

when we let

$$a = \left[\frac{\lambda_0^2}{2\pi}(u, (1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}v)\right]^{-1} =$$

(61) $$= \left[-\frac{\lambda_0^2}{(2\pi)^2}\iint d^2x\, d^2y\, u(x)(1 + \lambda_0 M_0)^{-1}v(x)\ln|x - y|u(y)\cdot\right.$$
$$\left.\cdot(1 + \lambda_0 M_0)^{-1}v(y)\right]^{-1} > 0$$

since $-\ln|x - y|$ is conditional strict positive definite and $(u, (1 + \lambda_0 M_0)^{-1}v) = 0$.

When we let

(62) $$\frac{\ln b}{a^2} = -\frac{1}{2\pi}(u, (1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}v) -$$
$$-\frac{\lambda_0}{2\pi}(u, (1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}M_0(1 + \lambda_0 M_0)^{-1}v)$$

we see as before that we can find a convergent expansion for β in terms of σ, τ, λ and hence

(63) $$\alpha(\lambda) = \sum_{\substack{m,n=0\\ l=1}}^{\infty} b_{lmn}\sigma^l\tau^m(\lambda - \lambda_0)^n$$

where

$$b_{100} = b. \tag{64}$$

When there are several eigenvalues approaching zero as $\lambda \downarrow \lambda_0$, the operator $1 + \lambda_0 M_0$ is no longer invertible. Let P denote the projection onto the eigenspace corresponding to the eigenvalue -1 of $\lambda_0 M_0$. Then the operator $1 + \lambda_0 M_0 + P$ is invertible with inverse $P + T$ where T is the reduced resolvent,

$$T = \lim_{z \to 0} (1 + z + \lambda_0 M_0)^{-1}(1 - P). \tag{65}$$

Writing $(1 + \lambda M_\alpha)^{-1} = (1 + \lambda M_\alpha + P - P)^{-1}$ and expanding one finds that a similar analysis as above still applies with the result that the resolvent $(1 + \lambda_0 M_0)^{-1}$ has to be replaced by the reduced resolvent T.

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