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# A NOTE ON THE SPACE OF PSEUDODIFFERENTIAL PROJECTIONS WITH THE SAME PRINCIPAL SYMBOL

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## 1. STATEMENT OF THE RESULT

In this paper we study the topological structure of the following space

$$(1.1)\qquad \mathscr{P}_{P_0} = \left\{ \begin{array}{c} P \in \mathscr{B}(H) : P^2 = P, \quad \dim \operatorname{Ran} P = \infty = \dim \operatorname{Ran}(\mathrm{Id} - P) \\ \text{and } P - P_0 \in \mathscr{K}(H) \end{array} \right\};$$

here $\mathscr{B}(H)$ is the algebra of all bounded operators in a separable Hilbert space $H$ and $\mathscr{K}(H)$ is the ideal of compact operators.

Here is an important example. Let $M$ be a closed smooth manifold, $E$ a complex vector bundle over $M$ and let $\mathrm{Pdiff}_p$ be the space of pseudodifferential projections $P$, with the same principal symbol $p : \pi^*(E) \to \pi^*(E)$ where $\pi : SM \to M$ is the natural projection from the cotangent sphere bundle (we have fixed a Riemannian structure on $M$ and a Hermitian structure on $E$). In this case $p$ is the projection onto some subbundle of $\pi^*(E)$ (actually the dimension of the fibre can change over connected components of $SM$). If $p$ and $\mathrm{Id} - p$ are not identically equal to 0 then the ranges of the projection $P$ and $\mathrm{Id} - P$ are infinite dimensional. The reason (at least for classical pseudodifferential operators) is that if $p \equiv 0$ then $P$ is of order $-1$ hence it is a compact operator. The operator is a projection so its range is finite dimensional. On the other hand, if the range of $P$ is finite dimensional, then $P$ is compact, so the 0-th order term in its symbol will vanish. Pseudodifferential projections described above are in many cases spectral projections of elliptic operators. The projections onto the Cauchy data spaces of elliptic differential operators are of this type, too (see for instance [1], [2], [3], [4], [8]). Now the space of projections in a separable Hilbert space with infinite dimensional range and kernel is contractible ([7]). For the space $\mathscr{P}_{P_0}$ the situation is different.

THEOREM 1.1. *Let $H$ be a separable Hilbert space, $P_0$ a projection with infinite dimensional range and kernel. Let $\mathscr{P}_{P_0}$ be the space defined in* (1.1). *$\mathscr{P}_{P_0}$ is a classi-*

*fying space for the* K*-functor. This means that for any compact space* $X$, $\mathrm{K}^0(X) = [X, \mathscr{P}_{P_0}]$, *where* $[X, \mathscr{P}_{P_0}]$ *denotes the set of homotopy classes of continuous maps from* $X$ *to* $\mathscr{P}_{P_0}$. *In particular*

$$\pi_i(\mathscr{P}_{P_0}) = \begin{cases} \mathbf{Z} & \text{for } i = 2k \\ 0 & \text{for } i = 2k+1. \end{cases} \tag{1.2}$$

Let us now study the pseudodifferential case. The following facts are well--known (see for instance Section 2 of [6] and the references given there). Let $\mathrm{Pdiff}(E)$ denote the set of all 0-th order pseudodifferential operators acting on sections of $E$. $\overline{\mathrm{Pdiff}(E)}$, the norm closure of $\mathrm{Pdiff}(E)$ in the space $\mathscr{B}(L^2(E))$, is a $*$-subalgebra of $\mathscr{B}(L^2(E))$, containing the identity and the ideal $\mathscr{K}(L^2(E))$ of compact operators ([6], Theorems 2.4 and 2.7). The principal symbol of the element from $\overline{\mathrm{Pdiff}(E)}$ is well-defined and two operators from $\overline{\mathrm{Pdiff}(E)}$ have the same principal symbol if and only if their difference is a compact operator ([6], Theorem 2.7). This gives us the following corollary.

COROLLARY 1.2. *Let* $M$ *be a closed smooth Riemannian manifold,* $E$ *a Hermitian vector bundle over* $M$, $\pi: SM \to M$ *the natural projection of the cotangent sphere bundle and* $p: \pi^*(E) \to \pi^*(E)$ *a bundle morphism such that*

$$p^2 = p, \quad p \text{ and } \mathrm{Id} - p \text{ are not identically equal } 0. \tag{1.3}$$

*Then the space* $\overline{\mathrm{Pdiff}}_p$ *is a classifying space for the* K*-functor.*

Much less we can say about the space $\mathrm{Pdiff}_p$. The only fact which follows trivially from Theorem 1.1 and from the results of Section 4, is that it has countably many path-connected components.

COROLLARY 1.3. *The space* $\mathrm{Pdiff}_p$ *has countably many path-connected components.*

The proof of the Theorem 1.1 depends on the fact that the group

$$\mathrm{GL}_{P_0} = \{g \in \mathrm{GL}(H) : gP_0 - P_0 g \in \mathscr{K}(H)\}, \tag{1.4}$$

where $\mathrm{GL}(H)$ is the group of invertible elements of $\mathscr{B}(H)$, is a classifying space for K-functor. It has the homotopy type of $\mathrm{Fred}(H)$ — the space of all Fredholm operators in $H$ (see [4], [8]). We show that there is a principal fibre bundle $\mathrm{GL}_{P_0} \xrightarrow{\pi} \mathscr{P}_{P_0}$ with contractible fibre $\mathrm{GL}(H) \oplus \mathrm{GL}(H)$ ([7]), Theorem 2).

Section 2 contains some technical results which are also of independent interest. Theorem 1.1 is proved in Section 3.

In Section 4 we analyse more closely the group $\pi_0(\overline{\mathrm{Pdiff}_p})$. We show that we are able to distinguish connected components of $\mathrm{Pdiff}_p$ and $\overline{\mathrm{Pdiff}}_p$ using the index of a suitable Fredholm operator, which is also a spectral invariant.

To make our exposition self-contained we show that $\mathrm{GL}_{P_0}$ is homotopically equivalent to the Fred$(H)$ in the Appendix.

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## 2. SOME TECHNICALITIES

LEMMA 2.1. *Let* $P_1, P_2 \in \mathscr{P}_{P_0}$ *and* $\|P_1-P_2\| < 1$*; then* $T=(1/2)[(2P_2-\mathrm{Id})\cdot(2P_1-\mathrm{Id})+\mathrm{Id}]$ *belongs to the group* $\mathrm{GL_c} = \{g \in \mathrm{GL}(H): (g-\mathrm{Id}) \in \mathscr{K}(H)\}$ *and the following equality holds*

$$TP_1T^{-1} = P_2. \tag{2.1}$$

*If* $P_1, P_2$ *are pseudodifferential then* $T$ *is elliptic and* $\sigma_L(T) = \mathrm{Id}_{\pi^*(E)}$, *where* $\sigma_L(T)$ *denotes the principal symbol of* $T$.

*Proof.* We have

$$(2P_i - \mathrm{Id})^2 = \mathrm{Id} \tag{2.2}$$

which gives us

$$\|T - \mathrm{Id}\| = \frac{1}{2}\|(2P_2 - \mathrm{Id})(2P_1 - \mathrm{Id}) - \mathrm{Id}\| =$$

$$= \frac{1}{2}\|(2P_2 - \mathrm{Id})(2P_1 - \mathrm{Id}) - (2P_2 - \mathrm{Id})^2\| \leqslant \tag{2.3}$$

$$\leqslant \frac{1}{2}\|2P_2 - \mathrm{Id}\|\,\|2(P_2 - P_1)\| = \|P_2 - P_1\| < 1$$

and we see that $T$ is invertible. Moreover

$$\sigma_L(T) = \frac{1}{2}[(2p - \mathrm{Id})(2p - \mathrm{Id}) + \mathrm{Id}] = \mathrm{Id}. \tag{2.4}$$

COROLLARY 2.2. *If* $P_1, P_2$ *lie in the same connected component of* $\mathscr{P}_{P_0}$ *then there exists* $v \in \mathrm{GL_c}$ *such that* $P_2 = vP_1v^{-1}$.

REMARK. It is trivial ($\mathrm{GL_c}$ is path-connected) that the converse theorem is also true, i.e. if $P_1, P_2$ lie in different components of $\mathscr{P}_{P_0}$ then such $v$ belonging to $\mathrm{GL_c}$ will not exist.

Now we consider a more general situation.

LEMMA 2.3. *Let* $P_1 \in \mathscr{P}_{P_0}$ *and* $\|P_1 - P_0\| = 1$*; then* $T = (1/2)[(2P_1 - \mathrm{Id})\cdot(2P_0 - \mathrm{Id}) + \mathrm{Id}]$ *is a Fredholm operator with the following properties*

$$\text{index}\, T = 0 \tag{2.5}$$

$$TP_0 = P_1 T \tag{2.6}$$

(2.7) $\ker T = \operatorname{coker} T = \{v : P_0 v = v,\ P_1 v = 0\} \oplus \{w : P_0 w = 0,\ P_1 w = w\}$.

*In the pseudodifferential case we still have* $\sigma_L(T) = \mathrm{Id}$.

*Proof.*

$$T = \frac{1}{2}\cdot[(2P_1 - \mathrm{Id})(2P_0 - \mathrm{Id}) + \mathrm{Id}] = \mathrm{Id} + 2P_1P_0 - P_1 - P_0 =$$

$$= \mathrm{Id} + 2(P_0 + P_1 - P_0)P_0 - P_1 - P_0 = \tag{2.8}$$

$$= \mathrm{Id} + 2(P_1 - P_0)P_0 - (P_1 - P_0) = \mathrm{Id} + (P_1 - P_0)(2P_0 - \mathrm{Id}).$$

So $T = \mathrm{Id} +$ compact operator. (2.6) is also trivial and in fact $T$ splits onto $T_+ = P_1P_0 : P_0H \to P_1H$ and $T_- = (\mathrm{Id} - P_1)(\mathrm{Id} - P_0) : (\mathrm{Id} - P_0)H \to (\mathrm{Id} - P_1)H$, so we have

$$\ker T = \ker T_+ + \ker T_-, \quad \operatorname{coker} T = \operatorname{coker} T_+ + \operatorname{coker} T_-. \tag{2.9}$$

In such situation everything is very simple

$$\begin{aligned} \ker T_+ &= \{w : P_0 w = w,\ P_1 w = 0\} = \\ &= \{w : (\mathrm{Id} - P_1)w = w,\ (\mathrm{Id} - P_0)w = 0\} = \operatorname{coker} T_-. \end{aligned} \tag{2.10}$$

The last equality is obvious. Assume that $(\mathrm{Id} - P_1)w = w$; if we have such $z$ that $w = (\mathrm{Id} - P_0)z$, then $w$ should belong to the image of $T_-$. Similarly, we have

$$\begin{aligned} \ker T_- &= \{w : (\mathrm{Id} - P_0)w = w,\quad (\mathrm{Id} - P_1)w = 0\} = \\ &= \{w : P_1 w = w,\ P_0 w = 0\} = \operatorname{coker} T_+. \end{aligned} \tag{2.11}$$

In Section 1 we introduced the group $\mathrm{GL}_{P_0}$. Here we summarize some of its elementary properties.

LEMMA 2.4. (1) *If* $Q \in \mathscr{P}_{P_0}$ *then* $\mathrm{GL}_Q = \mathrm{GL}_{P_0}$.

(2) *Let us consider the action of* $\mathrm{GL}_{P_0}$ *on* $\mathscr{P}_{P_0}$ *given by the formula*

$$g \cdot Q = gQg^{-1}; \tag{2.12}$$

*then we have* $\mathrm{GL}_{P_0} \cdot P_0 = \mathscr{P}_{P_0}$.

(3) *Denote by* $H_Q$ *the stationary group of* $Q$, $H_Q = \{g \in \mathrm{GL}_{P_0} : gQg^{-1} = Q\}$, *then* $H_Q$ *is isomorphic to the group* $\mathrm{GL}(QH) \oplus \mathrm{GL}((QH)^\perp)$, *where the last group is the set of operators from* $\mathrm{GL}(H)$ *which commute with the orthogonal projection onto the subspace* $QH$.

*Proof.* (1) Let $g \in \mathrm{GL}_Q$; then

$$gP_0 - P_0 g = g(P_0 - Q) + (P_0 - Q)g + (gQ - Qg) \in \mathscr{K}(H).$$

(2) We know that $P_1$ is a projection with infinite dimensional range and kernel; then there exists $g \in \mathrm{GL}(H)$ such that $gP_0g^{-1} = P_1$ and if $(P_0 - P_1) \in \mathscr{K}(H)$ then

$$gP_0 - P_0 g = (gP_0g^{-1} - P_0)g = (P_1 - P_0)g \in \mathscr{K}(H).$$

(3) Let $R$ be the orthogonal projection onto $QH$ and $h \in \mathrm{GL}(H)$ such that $hRh^{-1} = Q$; then

$$H_Q = h(\mathrm{GL}(RH) \oplus \mathrm{GL}((\mathrm{Id} - R)H))h^{-1}.$$

## 3. PROOF OF THEOREM 1.1

As we know from Lemma 2.4, $\mathrm{GL}_{P_0} \to \mathscr{P}_{P_0}$ is a surjection with fibre isomorphic to $\mathrm{GL}(H) \oplus \mathrm{GL}(H)$. We have to show that it is a fibre bundle.

The situation is even simpler than in the proof of the Theorem 2.3 from [8] (see also Appendix). The construction of local section is easy because we have Lemma 2.1. We take $\mathscr{P}_{P_0} \supset U = \{Q : \|Q - P_0\| < 1\}$ and then define $s : U \to \mathrm{GL}_c \subset \mathrm{GL}_{P_0}$ by the formula

$$s(Q) = \frac{1}{2}[(2Q - \mathrm{Id})(2P_0 - \mathrm{Id}) + \mathrm{Id}]. \tag{3.1}$$

If $gP_0 = P_0 g$ then $s(Q)gs(Q)^{-1}Q = Qs(Q)gs(Q)^{-1}$, in other words, if $g \in H_{P_0}$ then $s(Q)gs(Q)^{-1} \in H_Q$ so $s$ gives us local trivialisation. Thus we proved that $\mathrm{GL}_{P_0} \xrightarrow{\pi} \mathscr{P}_{P_0}$ is a principal fibre bundle with a structural group and fibre $F = \mathrm{GL}(H) \oplus \mathrm{GL}(H)$ and this is a contractible space ([7], Theorem 2).

$\mathrm{GL}_{P_0}$ is homotopically equivalent to the space Fred($H$) which is a classifying space for the K-functor (see Section 5 of [7] for further references). Now, $\pi$ induces isomorphisms on homotopy groups and since both $\mathrm{GL}_{P_0}$ and $\mathscr{P}_{P_0}$ are of the homotopy type of CW-complexes it follows that $\pi$ is a homotopy equivalence.

## 4. $\pi_0(\mathrm{Pdiff}_p)$

We study now closer an integer invariant which can be used to distinguish connected components of the space $\mathscr{P}_{P_0}$.

It should be mentioned that this invariant first appeared in the famous paper of Brown, Douglas and Fillmore ([5], Remark 4.9), where it was called the esential codimension.

We use notation from Section 2 and, although it is not necessary, we restrict our attention to the space $\overline{\mathrm{Pdiff}_p}$.

PROPOSITION 4.1. *Let $P$, $P_1 \in \overline{\mathrm{Pdiff}_p}$ and let $g$ be an element of $\mathrm{GL}_P$ such that $P_1 = g^{-1}Pg$. Then $(\mathrm{Id} - P) + gP$ is a Fredholm operator. Its index depends only on $P_1$, not on the choice of $g$, and in fact it is equal to the index of the operator*

$$P_1P : PH \to P_1H. \tag{4.1}$$

index$((\mathrm{Id} - P) + gP)$ *is* $0$ *if and only if $P_1$ and $P$ lie in the same connected component of* $\overline{\mathrm{Pdiff}_p}$.

*Proof.* $(\mathrm{Id} - P) + Pg^{-1}$ is a two-sided parametrix (quasi-inverse) for the operator $(\mathrm{Id} - P) + gP$.

$$\begin{aligned} \mathrm{index}((\mathrm{Id} - P) + gP) &= \mathrm{index}((\mathrm{Id} - P) + PgP + (\mathrm{Id} - P)gP) = \\ &= \mathrm{index}((\mathrm{Id} - P) + PgP) = \mathrm{index}\, PgP = \\ &= \mathrm{index}\, g(P_1P) = \mathrm{index}\, P_1P. \end{aligned} \tag{4.2}$$

Now if $P_1$ and $P$ belong to the same connected component of $\mathrm{Pdiff}_p$, then index$((\mathrm{Id} - P) + gP)$ is 0, because from Corollary 2.2 we know that in this case there exists $h \in \mathrm{GL}_c$ such that $h^{-1}Ph = P_1$ and

$$\begin{aligned} \mathrm{index}((\mathrm{Id} - P) + gP) &= \mathrm{index}((\mathrm{Id} - P) + hP) = \\ &= \mathrm{index}(\mathrm{Id} + \mathrm{compact}) = 0. \end{aligned} \tag{4.3}$$

Let us assume now that $\text{index}((\text{Id} - P) + gP) = \text{index}\, P_1P = 0$. This means (see (2.6), (2.7), (2.9) — (2.11)) that

$$\dim\{w : Pw = w,\ P_1w = 0\} = \dim\{w : Pw = 0,\ P_1w = w\}, \tag{4.4}$$

and we see that there exists $h$ — an automorphism of the space $\ker T = \ker(P_1P) \oplus \oplus \operatorname{coker}(P_1P)$ — such that $h^2 = \text{Id}$ and

$$\begin{aligned} &h|\ker(P_1P) : \ker(P_1P) \to \operatorname{coker}(P_1P) \\ &h|\operatorname{coker}(P_1P) : \operatorname{coker}(P_1P) \to \ker(P_1P). \end{aligned} \tag{4.5}$$

Now we define an operator $T_1$ equal to the operator $T$ on the orthogonal complement of $\ker T$ and $h$ on $\ker T$

$$T_1 = (T|(\ker T)^{\perp}) \oplus (h|\ker T). \tag{4.6}$$

This is an invertible operator and its symbol is equal to the identity, so it belongs to the group $\text{GL}_c$. Moreover, $P_1T_1 = T_1P$ and we can find a path $\{h_t\}_{t\in I} \subset \text{GL}_c$ such that $h_0 = \text{Id}$ and $h_1 = T_1$. The family $\{P_t = h_tPh_t^{-1}\}_{t\in I}$ joins $P$ with $P_1$.

REMARK. $T_1$ is still an elliptic pseudodifferential operator if $T$ is so. We can choose $h_t$ such that they belong to the group $\text{GL}(\infty) = \{g \in \text{GL}_c : g - \text{Id}$ is an operator with finite dimensional range$\}$. In this case $h_t - \text{Id}$ are operators with smooth kernels.

In the last part let us describe the connection with a spectral invariant of families over the circle. Any pair $P_0 = P$, $P_1$ defines a family of elliptic operators with essential spectrum $\{\pm 1\}$

$$\{B_t = t(\text{Id} - 2P_1) + (1 - t)(\text{Id} - 2P_0) = (\text{Id} - 2P_0) + 2t(P_0 - P_1)\}_{t\in I}. \tag{4.7}$$

The family joins the operators $B_i = (\text{Id} - P_i) - P_i$ $(i = 0,1)$ with spectrum equal to $\{-1,+1\}$. The spectral flow of such family is the difference between the number of eigenvalues which change sign from $-$ to $+$ when $t$ comes from 0 to 1 and the number of eigenvalues which change sign from $+$ to $-$ (see [1], [4], [8]). We denote it $\text{sf}\{B_t\}$. In our case an easy computation (see [4], Section 3, or [8], Section 2) shows that

$$\text{sf}\{B_t\}_{t\in I} = \text{index}\, P_1P. \tag{4.8}$$

REMARK. The family (4.7) defines a family of Fredholm operators with the essential spectrum $\{\pm 1\}$ over $S^1$. We know that $P_1 = g^{-1}Pg$ and we can choose the path $\{h_t\}_{t\in I}$ in the contractible space $\text{GL}(H)$, such that $h_0 = \text{Id}$ and $h_1 = g$.

Then we define

$$D_t = \begin{cases} B_{2t} & \text{for } \leqslant t \leqslant \frac{1}{2} \\ h_{2t-1} B_1 h_{2t-1}^{-1} & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{4.9}$$

The family $\{D_t\}$ is defined uniquely up to a homotopy. The spectral flow is the only homotopy invariant of families of such Fredholm operators over $S^1$.

Now let us assume that there exists a continuous path $\{P_t\}_{t \in I}$ joining $P_0$ with $P_1$. Then the family

$$\tilde{B}_t = \begin{cases} B_{2t} & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \mathrm{Id} - 2P_{2(1-t)} & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{4.10}$$

is a loop in a contractible space of elliptic operators acting on sections of fixed vector bundle with the same principal symbol. It is homotopy equivalent to the constant family, so

$$0 = \mathrm{sf}\{\tilde{B}_t\} = \mathrm{sf}\{B_t\} \tag{4.11}$$

($\mathrm{sf}\{\mathrm{Id} - 2P_t\}$ is equal to 0).

Now it is very easy to construct families with different spectral flows. We start from $P_0 = P$ and take any $\varphi \in L^2(E)$ with the following properties

$$P\varphi = 0, \quad \|\varphi\|_{L^2} = 1 \tag{4.12}$$

and define

$$P_1(f) = P_0(f) + ((\mathrm{Id} - P_0)(f); \varphi)\varphi. \tag{4.13}$$

It is seen at once that

$$\mathrm{sf}\{B_t\} = \mathrm{index}\, P_1 P = -1. \tag{4.14}$$

To get a family with positive spectral flow we take $\psi \in L^2(E)$ with $P_0\psi = \psi$ and $P_1(f) = P_0(f) - (P_0(f); \psi)\psi$.

Using these constructions we can get a family of pseudodifferential operators with an arbitrary spectral flow, which in fact ends the proof of Corollary 1.3.

## APPENDIX

To make our exposition self-contained we present here the proof of Theorem 2.3 from [8].

THEOREM. *Let $P$ be a projection in a separable Hilbert space $H$ with infinite dimensional range and kernel.* $GL_P$ *is homotopically equivalent to the space* Fred$(PH)$ *of all Fredholm operators in $PH$.*

*Proof.* Let $\mathscr{B}_P = \{A \in \mathscr{B}(H) : AP-PA \in \mathscr{K}(H)\}$. We have a surjection of Banach spaces $R : \mathscr{B}_P \to \mathscr{B}(PH)$ given by

$$R(A) = PAP$$

and $R$ restricted to $GL_P$ gives a surjection $R : GL_P \to \text{Fred}(PH)$. Let $B \in \text{Fred}(PH)$ and consider

$$R^{-1}(B) \cap GL_P \cong \left\{ \begin{matrix} g : (\mathrm{Id} - P)H \oplus \ker B \to (\mathrm{Id}-P)H \oplus \operatorname{coker} B \\ g \text{ is a continuous linear isomorphism.} \end{matrix} \right\}.$$

This space is isomorphic to $GL(H)$. It folllows now from a theorem of Bartle and Graves (see for instance: Bessaga, Cz.; Pełczyński, A., *Selected topics from infinite dimensional topology*, Warsaw, 1975) that we have a continuous map $S: \mathscr{B}(PH) \to \mathscr{B}_P$ such that $RS = \mathrm{Id}$. Hence for any $g \in GL_P$, $R(g)$ has a neighbourhood $U$ in Fred$(PH)$, such that there is a map $J_U : U \to GL_P$ with the property

$$RJ_U = \mathrm{Id}|U.$$

This facts implies that $R : GL_P \to \text{Fred}(PH)$ is a principal fibre bundle with fibre $GL(H)$. Fred$(PH)$ is an open subset of the Banach space $\mathscr{B}(PH)$, so it has a homotopy type of CW-complexes and now the theorem follows from Corollary 1 of Section 5 from [7].

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