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SIMILARITIES OF II_1 FACTORS WITH PROPERTY Γ

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Dedicated to Professor R. V. Kadison on his 60th birthday

1. INTRODUCTION

Shortly after Haagerup's proof of the Grothendieck-Ringrose-Pisier inequality [11] appeared I found the result stated in Proposition 2.1. The result tells that a representation Φ of a II_1 factor $\mathcal{M}$ can be extended to the bounded Hilbert-Schmidt operators in the semifinite properly infinite von Neumann algebra $\mathcal{M} \otimes B(L^2(\mathrm{N}))$. Unfortunately I have not been able to prove that Φ is in general completely bounded, but one can fairly easily see that Φ must be completely bounded when restricted to certain big subalgebras, e.g. the relative commutant of an abelian von Neumann subalgebra without atoms. A demonstration of this result is the main ingredient in the proof of Theorem 2.3. Recently Popa [15] has discussed derivations of a finite factor into the compacts, and he discovered also that the set of subalgebras, which behave nicely, do have a remarkable structure. I will not discuss the similar question for similarities but state that analogous results, for the set of nice subalgebras, are obtainable. When the above result concerning relative commutants has been shown one considers an ultraproduct $\mathcal{M}^\omega$ of the given factor and shows that the similarity lifts to a similarity of the ultraproduct. Since the von Neumann algebra is assumed to have property Γ, the algebra of central sequences in the ultraproduct commutes with the constant sequences, and we find that the constant sequences is contained in the relative commutant of something big and hence the similarity is completely bounded. It is of course a main point that the algebra of central sequences do not have atoms.

It is well known that the similarity question is connected to various other problems concerning derivations. In Section 3 we mention some consequences of our results.

2. SIMILARITIES

In this section $\mathcal{M}$ denotes a von Neumann algebra factor of type II_1 and Φ a representation of $\mathcal{M}$ on a Hilbert space H, such that $\Phi(I) = I$. It should be noted that we do consider $\mathcal{M}$ from an algebraic point of view and that H is the underlying space for the representation Φ, which is *not* assumed to be self-adjoint. We start with a general result on tensor products of Φ with the identity representation of the $n \times n$ complex matrices, and we remind the reader that Φ is said to be completely bounded, if the set of norms $\|\Phi \otimes \mathrm{id}_n\|$ is bounded. The sup of this set is called the complete bounded norm of Φ and denoted $\|\Phi\|_{cb}$.

2.1. PROPOSITION. *Let $n \in \mathbf{N}$ and let* tr *denote the normalized traces on $\mathcal{M}$ as well as on $\mathcal{M} \otimes M_n$, then for any x in $\mathcal{M} \otimes M_n$:*

$$\|(\Phi \otimes \mathrm{id}_n)(x)\| \leqslant \|\Phi\|^4(\|x\|^2 + n\,\mathrm{tr}(x^*x))^{\frac{1}{2}}.$$

Proof. Let $\gamma = \|\Phi\|$. By [11, Lemma 1.5] we get

$$\left\| \sum_{i=1}^{n} \Phi(m_i)^*\Phi(m_i) \right\| \leqslant \gamma^6 \left\| \sum_{i=1}^{n} m_i^* m_i \right\|$$

for all sets $m_1, \ldots, m_n$. In particular this means that if we restrict $\Phi \otimes \mathrm{id}_n$ to the first column in $\mathcal{M} \otimes M_n$, the norm here is dominated by γ^3. On the other hand $\bar{\Phi}(x) = \Phi(x^*)^*$ is also a homomorphism, so $\Phi \otimes \mathrm{id}_n$ must be bounded by γ^3 on row vectors as well. In order to estimate $\|(\Phi \otimes \mathrm{id}_n)(x)\|$ we consider a unit vector $\Xi = (\xi_1, \ldots, \xi_n)$ and evaluate

$$\|(\Phi \otimes \mathrm{id}_n)(x)\Xi\|^2 = \sum_i \| \sum_j \Phi(x_{ij})\xi_j \|^2.$$

The mappings $\Gamma_i : \mathcal{M} \otimes M_n \to H$ given by

$$\Gamma_i(x) = \sum_j \Phi(x_{ij})\xi_j$$

are all bounded by γ^3 so one finds by Haagerup's version of a Grothendieck-Ringrose-Pisier inequality [11, Theorem 3.2], that there exist states φ_i and ψ_i on $\mathcal{M} \otimes M_n$ such that

$$\|\Gamma_i(x)\|^2 \leqslant \gamma^6[\varphi_i(x^*x) + \psi_i(xx^*)].$$

It is easily seen that the Γ_i's are "permuted" copies of Γ_1 so we will stick to Γ_1 for a moment. Obviously Γ_1 depends only on the first row of x and therefore the above inequality is valid if we replace x on the right side with its first row x_{1-}. Since $x_{1-}x_{1-}^*$ is in $\mathcal{M}$ we find that we may consider ψ_1 to be a state ψ on $\mathcal{M}$, we will let φ denote φ_1 and we get

$$\|\Gamma_1(x)\|^2 \leqslant \gamma^6[\varphi((x_{1s}^* x_{1t})) + \psi(\sum x_{1t}x_{1t}^*)].$$

By permuting the rows of x we see that φ and ψ can be used for φ_i and ψ_i as well, so

$$\|(\Phi \otimes \mathrm{id}_n)(x)\|^2 \leqslant \gamma^6[\varphi((\sum_i x_{is}^* x_{it})) + \psi(\sum_{it} x_{it} x_{it}^*)] \leqslant$$

$$\leqslant \gamma^6\|x\|^2 + \gamma^6\psi(\sum_{it} x_{it} x_{it}^*).$$

For any unitary u in M and $\tilde{u} = u \otimes I$ we have $\|(\Phi \otimes \mathrm{id}_n)(\tilde{u}^*)\| = \|\Phi(u^*)\| \leqslant \gamma$ and we find when x is replaced by $\tilde{u}x$ in the inequality above

$$\|(\Phi \otimes \mathrm{id}_n)(x)\|^2 \leqslant \gamma^2\|(\Phi \otimes \mathrm{id}_n)(\tilde{u}x)\| \leqslant$$

$$\leqslant \gamma^8\|x\|^2 + \gamma^8\psi(u(\sum_{it} x_{it} x_{it}^*)u^*).$$

Dixmier's approximation theorem [7, III.5] or [13, Vol. II, 8.3] yields the desired inequality

$$\|(\Phi \otimes \mathrm{id}_n)(x)\|^2 \leqslant \gamma^8(\|x\|^2 + \sum_{it} \mathrm{tr}(x_{it} x_{it}^*)) =$$

$$= \gamma^8(\|x\|^2 + n\,\mathrm{tr}(x^*x)).$$

We are now going to prove that Φ is completely bounded if $\mathcal{M}$ has property Γ. This property was introduced by Murray and von Neumann in the early days in terms of an asymptotic commutation relation. Lately in [6, Section II] Connes has proven a number of equivalent definitions for property Γ. Some of the properties can be expressed in terms the ultrapower $\mathcal{M}^\omega$ of $\mathcal{M}$ and the algebra of central sequences. The reader is refered to [6, 8, 14] for exact details and definitions. Here we will just assume that ω is an ultrafilter on $\mathbf{N}$ finer than the Frechét filter, $\mathcal{M}^\omega$ the ultraproduct and $\bar{\mathcal{M}}$ the canonical embedding of $\mathcal{M}$ in $\mathcal{M}^\omega$ as constant sequences. The relative commutant of $\bar{\mathcal{M}}$ in the II_1 factor $\mathcal{M}^\omega$ is denoted $\mathcal{M}_\omega$ and could be refered to as the algebra of central sequences.

2.2. LEMMA. *If Φ is ultrastrongly continuous, then Φ can be extended to a representation Φ^ω of $\mathcal{M}^\omega$ on a bigger Hilbert space K, moreover $\|\Phi^\omega\| = \|\Phi\|$.*

Proof. In $\ell^\infty(\mathbf{N}, H)$ we consider the subspace spanned by sequences $(\xi_n) = (\Phi(m_n)\xi)$ where m_n is a bounded sequence in $\mathcal{M}$ and ξ belongs to H. We define a positive semidefinite inner product $((\xi_n), (\eta_n)) = \lim_\omega(\xi_n, \eta_n)$, and we get by dividing out the relevant null spaces and completing a Hilbert space K. By construction we get a representation Φ^ω of $\mathcal{M}^\omega$ on K by $\Phi^\omega([m_n])([\Phi(r_n)\xi]) = [\Phi(m_n r_n)\rho]$ and it is clear that $\|\Phi^\omega\| \leqslant \|\Phi\|$. On the other hand H imbeds isometrically in K since $\Phi(I) = I$ and we find that $\Phi^\omega | \bar{M}$ is an extension of Φ so $\|\Phi^\omega\| \geqslant \|\Phi\|$.

2.3. THEOREM. *Any representation Φ of a* II_1 *factor* $\mathcal{M}$ *with property* Γ *is completely bounded and*

$$\|\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|^{44}.$$

There exists a y in $\mathrm{GL}(H)$ *such that* $\|y\|\|y^{-1}\| \leqslant \|\Phi\|^{44}$ *and* $y\Phi(\cdot)y^{-1}$ *is a star--representation.*

Proof. When we have to estimate the complete bounded norm of Φ, we may and will assume that the representation has an at most countable cyclic set. In this case [5, Corollary 2.5 and Proposition 2.7] furnish a star-representation π of $\mathcal{M}$ on H such that for any vector ξ in H there exists a bounded injective operator x with dense range and a vector η such that

$$\forall m \in \mathcal{M}: \quad \Phi(m)x = x\pi(m); \quad \|x\| \leqslant 2\|\Phi\|^2; \quad x\eta = \xi; \quad \|\eta\| \leqslant \|\xi\|.$$

In particular the first property gives a homomorphism Ψ of $\pi(\mathcal{M})$ into $B(H)$ by $a \to \overline{xax^{-1}}$ and $\|\Psi\| = \|\Phi\|$, whereas the second shows that Ψ is ultrastrongly continuous since $\Psi(a)\xi = xa\eta$. We will also let Ψ denote the extension of Ψ to the von Neumann algebra $\pi(\mathcal{M})''$. In this algebra we will let f denote the maximal finite central projection and let $\mathcal{D}$ denote a copy of the compact operators placed inside $(I-f)\pi(\mathcal{M})''$, such that $I-f$ belongs to the weak closure of $\mathcal{D}$. Then $\mathcal{D} + \mathbf{C}f$ is a nuclear C^*-algebra, and by [5, Theorem 4.1] we can perturb Ψ with a z in $\mathrm{GL}(H)$ such that $\mathrm{Ad}(z)\circ\Psi$ is trivial on $\mathcal{D} \oplus \mathbf{C}f$ and $\|z\|\|z^{-1}\| \leqslant \|\Phi\|^2$. The new homomorphism $\mathrm{Ad}(z)\circ\Psi$ decomposes naturally into an orthogonal direct sum. The restriction to the properly infinite part is by construction completely bounded with complete bounded norm less than $\|\Phi\|^3$. The restriction to the finite part yields homomorphisms π_f and Δ of the finite von Neumann algebra $\mathcal{M}$ into $\mathcal{B}(fH)$ given by

$$\pi_f(m) = \pi(m)|fH \quad \text{and} \quad \Delta(m) = (zx)f\pi_f(m)(zx)^{-1}|fH.$$

Since a finite representation of a finite factor is ultrastrongly continuous because of the uniqueness of the trace we see that Δ is ultrastrongly continuous so we can extend Δ to a homomorphism Δ^ω of $\mathcal{M}^\omega$ with $\|\Delta^\omega\|$ say γ such that $\gamma \leqslant \|\Phi\|^3$. By Dixmier's result [8, Proposition 1.10] the algebra of central sequences $\mathcal{M}_\omega$ does not have minimal projections, and we may therefore find an abelian subalgebra $\mathcal{A}$ of $\mathcal{M}_\omega$ without minimal projections. According to [5, Theorem 4.1] there exists an invertible operator h in $B(K)$ such that $\|h\|\|h^{-1}\| \leqslant \|\Phi^\omega\|^2 = \gamma^2$, and $\Gamma = h^{-1}\Delta^\omega(\cdot)h$ is a star-representation when restricted to $\mathcal{A}$. Moreover the norm of Γ is dominated by γ^5. For any operator (m_{ij}) in $\mathcal{M} \otimes M_n$ and any set $(p_1, \ldots, p_k)$ of pairwise orthogonal projections in $\mathcal{A}$ with sum I, we find that $(\Gamma \otimes \mathrm{id})(p_s \otimes I)$

is a set of pairwise orthogonal projections which commute with $(\Gamma \otimes \mathrm{id})((\overline{m}_{ij}))$. Hence by Proposition 2.1

$$\|(\Gamma \otimes \mathrm{id})((\overline{m}_{ij}))\| = \sup_s \|(\Gamma \otimes \mathrm{id})((\overline{m}_{ij}))(\Gamma \otimes \mathrm{id})(p_s \otimes I)\| =$$

$$= \sup_s \|(\Gamma \otimes \mathrm{id})((\overline{m}_{ij} p_s))\| \leqslant$$

$$\leqslant \gamma^{12} \|(m_{ij})\| (\sup_s (1 + n \operatorname{tr}(p_s))^{\frac{1}{2}}).$$

Since $\mathscr{A}$ is diffuse, we see that Γ is completely bounded and $\|\Gamma\|_{\mathrm{cb}} \leqslant \gamma^{12}$, so Δ is completely bounded with $\|\Delta\|_{\mathrm{cb}} \leqslant \gamma^{14}$; and $\|\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|^2 \cdot \|\Phi\|^{42} = \|\Phi\|^{44}$.

The rest of the theorem follows from [11, Theorem 1.10].

2.4. Corollary. *Let $\mathscr{M}$ be a von Neumann factor of type* II_1 *with property* Γ *on a Hilbert space H and let x be a closed densely defined invertible operator (not necessarily bounded inverse) such that*

$$\forall m \in M: \quad mD(x^{-1}) \subseteq D(x^{-1}) \quad \textit{and} \quad x^{-1}mx \textit{ is bounded.}$$

There exists a y in $\mathrm{GL}(H)$ *such that* $\forall m \in M: ymy^{-1} = \overline{x^{-1}mx}$. *In particular* $xy \in \mathscr{M}'$.

Proof. The proof of [5, Proposition 2.7] shows after a minor change — E_n should be the spectral projection of $(xx^*)^{1/2}$ corresponding to the interval $[1/n, n]$ — that we get a bounded homomorphism $\Phi: \mathscr{M} \to B(H)$ by

$$\Phi(m) = \overline{x^{-1}mx}$$

and the corollary follows.

3. DERIVATIONS

In the papers [2, 3, 4] we discussed questions concerning derivations and the possibilities of measuring distances to the commutant in terms of norms of commutators. We recall the following definition.

3.1. Definition. A C^*-algebra $\mathscr{A}$ is said to have property D_k for some positive real k, if for each non degenerate star-representation π of $\mathscr{A}$ on a Hilbert space H we have

$$\forall x \in B(H): \quad \inf\{\|x - z\| \mid z \in \pi(\mathscr{A})'\} \leqslant k \sup\{\|x\pi(a) - \pi(a)x\| \mid a \in \mathscr{A}_1\}.$$

3.2. THEOREM. *Any von Neumann factor $\mathcal{M}$ of type II_1 with property Γ does have property* D_{60}.

Proof. Let $\pi:\mathcal{M} \to B(H)$ be a non degenerate star-representation and let x be in $B(H)$, then we get for any $t \in \mathbf{R}_+$ a homomorphism $\Phi_t:\mathcal{M} \to B(H \oplus H)$ by

$$\Phi_t(m) = \begin{pmatrix} \pi(m) & t[x, \pi(m)] \\ 0 & \pi(m) \end{pmatrix}.$$

Let $\alpha = \sup\{\|[x, \pi(m)]\| \mid m \in \mathcal{M}_1\}$ and let $\delta:\mathcal{M} \to B(H)$ be given by $\delta(m) = [x, \pi(m)]$. The results in the previous section show that δ is completely bounded, and it is clear that

$$t\|\delta\|_{\mathrm{cb}} \leqslant \|\Phi_t\|_{\mathrm{cb}} \leqslant \|\Phi_t\|^{44} \leqslant (1+t\|\delta\|)^{44}$$

so

$$\|\delta\|_{\mathrm{cb}} \leqslant \inf_{t \in \mathbf{R}_+} t^{-1}(1 + t\|\delta\|)^{44} \leqslant 119\|\delta\|.$$

An application of Corollary 2.2 of [2] yields $\inf\{\|x - z\| \mid z \in \pi(\mathcal{M})'\} = (1/2)\|\delta\|_{\mathrm{cb}} < 60\|\delta\|$ and the theorem follows.

3.3. DEFINITION. Let $\mathcal{A}$ be a C^*-algebra, π a star-representation of $\mathcal{A}$ and δ a linear mapping of $\mathcal{A}$ into $B(H)$. We say δ is a π-derivation if

$$\forall\, a, b \in \mathcal{A}: \quad \delta(ab) = \pi(a)\delta(b) + \delta(a)\pi(b).$$

3.4. COROLLARY. *Let π be a star-representation of $\mathcal{M}$, any π-derivation δ of $\mathcal{M}$ is implemented by an operator x in $B(H)$ such that $\|x\| \leqslant 60\|\delta\|$.*

Proof. One gets immediately a derivation $\hat{\delta}$ of $\pi(\mathcal{M})$ into $B(H)$ by $\hat{\delta}(\pi(m)) = \delta(m)$. The result follows now from [4, Corollary 3.2].

Our next result is an extension of previous works by Elliott [9, 10] and Akemann & Johnson [1].

3.5. THEOREM. *Let $\mathcal{M}$ be a II_1 factor with property Γ, π a star-representation of $\mathcal{M}$ on a Hilbert space H and $(\delta_n)_{n\in\mathbf{N}}$ a sequence of π-derivations of $\mathcal{M}$ into $B(H)$. If $\|\delta_n(x)\| \to 0$ for each x in $\mathcal{M}$, then $\|\delta_n\| \to 0$.*

Proof. Choose a hyperfinite II_1 subfactor $\mathcal{R}$ of $\mathcal{M}$, then by [1, Theorem 3.7] δ_n converges uniformly to zero on $\mathcal{R}$. By the previous corollary we can perturb δ_n by a sequence of π-derivations which tends uniformly to zero on $\mathcal{M}$ such that the perturbed sequence is trivial on $\mathcal{R}$. We will hence assume that each δ_n is trivial on $\mathcal{R}$.

Choose an increasing sequence $(\mathcal{S}_k)$ of subfactors of $\mathcal{R}$ such that $\mathcal{S}_k \simeq M_{2^k}$ and a sequence of projections e_k such that

$$e_k \in \mathcal{S}_k; \quad \mathrm{tr}(e_k) = 2^{-k}; \quad e_k e_j = 0 \quad \text{for } k \neq j.$$

Let us define $\mathcal{M}_k = e_k \mathcal{M} e_k$ then $\mathcal{M} \simeq \mathcal{S}_k \otimes \mathcal{M}_k$ and it is clear that the action of δ_n on $\mathcal{M}$ can be identified with the action of id $\otimes$ $(\delta_n | \mathcal{M}_k)$ on $\mathcal{S}_k \otimes \mathcal{M}_k$. If $\|\delta_n\| \nrightarrow 0$ we may as well assume that $\|\delta_n\| > 119$ for all $n \in \mathbf{N}$. Since π-derivations are completely bounded we get by the proof of Theorem 3.2

$$\forall n: \quad \|\delta_n | \mathcal{M}_n\| > 1.$$

We can now apply the rolling hump argument by choosing operators x_n in $\mathcal{M}_n$ such that $\|x_n\| \leqslant 1$, $\|\delta_n(x_n)\| > 1$. Let $x = \sum x_n$ then $\|x\| \leqslant 1$ and $\|\delta_n(x)\| \geqslant \|\delta_n(x)e_n\| = \|\delta_n(x_n)\| > 1$; a contradiction.

Recently S. Popa [15] has generalized results of Johnson & Parrott [12] concerning derivations of von Neumann algebras into the compact operators. S. Popa studies II_1 factors with a commutativity property which is weaker than Γ and generalizes Johnson and Parrotts result to this case. We have not been able to prove our results for algebras with this weaker commutativity property, but it is quite easy to adapt the technique from the proof above to show the following:

3.6. THEOREM. *Let $\mathcal{M}$ be a II_1 factor with property Γ on a Hilbert space H and let δ be a derivation of $\mathcal{M}$ into the compact operators $C(H)$ on H, then δ is implemented by a compact operator.*

Proof. Perturb δ by a derivation implemented by a compact operator such that the perturbed derivation is trivial on a hyperfinite II_1 subfactor $\mathcal{R}$. Then proceed as above — under the assumption $\|\delta\| > 119$ — and choose x_n in $\mathcal{M}_n$ such that $\|x_n\| \leqslant 1$ and $\|\delta(x_n)\| > 1$. Put $x = \sum x_n$ then $\delta(x) \in C(H)$ and $\|\delta(x)e_n\| = \|\delta(x_n)\| > 1$, a contradiction.

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