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CYCLIC COHOMOLOGY OF THE GROUP ALGEBRA OF FREE GROUPS

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Dedicated to Professor O. Takenouchi on his sixtieth birthday

1. INTRODUCTION

Let $C_r^*(F_N)$ be the reduced group C^*-algebra of the free group F_N with N generators. We have an obvious group homomorphism $F_N \to \mathbf{Z}^N$ so that there is a co-action of $\mathbf{Z}^N$ on $C_r^*(F_N)$ which is $\mathbf{T}^N$ action. The C^*-algebra $C_r^*(F_N)$ has a canonical (normalized) trace τ which is easily seen to be invariant under the above $\mathbf{T}^N$ action. The method of A. Connes [2], [3] yields a family of n-traces out of commuting derivations $\delta_1, \ldots, \delta_N$ and the invariant trace τ. In particular, τ itself is an element of $H_\lambda^0(A)$ and $\varphi^j, j = 1, \ldots, N$ are elements of $H_\lambda^1(A)$, where A is a suitable dense $*$-subalgebra of $C_r^*(F_N)$ and

$$\varphi^j(a^0, a^1) = \tau(a^0\delta_j(a^1)), \quad a^0, a^1 \in A,\ j = 1, \ldots, N. \tag{1.1}$$

In view of the fact that the classifying space BF_N of the group F_N is homotopic to N-wedge sum of S^1, the (topological) projective dimension of the algebra is expected to be one. So we can expect that the spectral sequence associated with the exact couple of A. Connes will be stable after taking E_1-terms.

In this paper, we shall compute the cyclic cohomology of the group algebra $A = \mathbf{C}[F_N]$ in terms of explicitly constructed projective resolution. We show $H^{\text{even}}(A) \simeq \mathbf{C}^N$ with the generators above.

In Section 2, we give an explicit description of projective resolution. We compute the cyclic cohomology in Section 3. Section 4 and the Appendix will be devoted to the arguments with Fréchet topologies. However, our discussion does not apply for the ordinary Schwartz group algebra, which is known to have the same K-theory as $C_r^*(F_N)$ by P. Jolissaint. In Section 5, we shall give a brief discussion.

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2. PROJECTIVE RESOLUTION

Here we construct a projective resolution of $A = \mathbf{C}[F_N]$ with length one which will work also for the discussion if we endow some topologies on A. Here, we shall give a purely algebraic discussion.

$$0 \leftarrow A \overset{\varepsilon}{\leftarrow} B \overset{d}{\leftarrow} B \otimes \mathbf{C}^N \leftarrow 0, \tag{2.1}$$

where $B = A \otimes A^{\mathrm{opp}}$. The space $B \otimes \mathbf{C}^N$ is a left B-module in an obvious manner. By its construction. $B \otimes \mathbf{C}^N$ is a free module of rank N so finite and projective. The space A is also a left B-module and the map ε is B-linear, see [3]. The map d is the B-linear map determined by

$$d(1_B \otimes e_j) = U_j \otimes 1 - 1 \otimes \mathring{U}_j, \tag{2.2}$$

where $e_j, j = 1, \ldots, N$ are the fixed basis in $\mathbf{C}^N$ and $U_j, j = 1, \ldots, N$, are the generators of F_N so that they are viewed as the elements of the group algebra $A = \mathbf{C}[F_N]$.

We have to see that the sequence (2.1) is actually a projective resolution, i.e., to show that there exists a (continuous) $\mathbf{C}$-linear homotopy map which yields that the sequence (2.1) is acyclic. The existence of such a homotopy map is shown by the explicitly construction of quasi-isomorphisms between the resolution (2.1) and the canonical projective resolution of A as in [3].

DEFINITION 2.1. For invertible commuting elements U, V and $n \in \mathbf{Z}$,

$$\Phi(U, V, n) = (U^n - V^n)(U - V^{-1}). \tag{2.3}$$

It is easy to see that the right hand side makes sense algebraically, i.e.

$$\Phi(U, V, n) = \sum_{j=0}^{n-1} U^{n-1-j} V^j, \quad \text{for } n \geqslant 1, \tag{2.4}$$

$$\Phi(U, V, 0) = 0, \tag{2.5}$$

$$\Phi(U, V, n) = -U^{-1}V^{-1}\Phi(U^{-1}, V^{-1}, -n), \quad \text{for } n \leqslant -1. \tag{2.6}$$

Let $g \in F_N$. We can write g in terms of generators as follows:

$$g = U_1^{n_1^1} U_2^{n_1^2} \ldots U_N^{n_1^N} U_1^{n_2^1} \ldots U_N^{n_2^N} \ldots U_1^{n_M^1} \ldots U_N^{n_M^N} \tag{2.7}$$

with its reduced word length

$$|g| = \sum_{i=1}^{N} \sum_{j=1}^{M} |n_j^i|. \tag{2.8}$$

We define a map $k: B \otimes A \to B \otimes \mathbf{C}^N$ as the B-linear extension of the map defined on the generator by the formula:

$$k(1_B \otimes g) = \sum_{i=1}^{N} \sum_{j=1}^{M} (U_1^{n_1^1} \ldots U_N^{n_1^N} \ldots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} \ldots U_N^{n_M^N})^\circ \Phi(U_i, \mathring{U}_i, n_j^i) \otimes e_i. \tag{2.9}$$

Similarly, we define a map $h: B \otimes \mathbf{C}^N \to B \otimes A$:

$$h(1_B \otimes c_i) = 1_B \otimes U_i. \tag{2.10}$$

LEMMA 2.2 (1) *$k \circ h = 1$ on $B \otimes \mathbf{C}^N$.*

(2) *The following diagrams are commutative:*

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & A & \overset{\varepsilon}{\longleftarrow} & B & \overset{b_1}{\longleftarrow} & M_1 & \overset{b_2}{\longleftarrow} & M_2 & \longleftarrow & \ldots \\ & & \Big| \simeq & & \Big| \simeq & & \Big\uparrow h & & \Big\uparrow & & \\ 0 & \longleftarrow & A & \overset{\varepsilon}{\longleftarrow} & B & \overset{d}{\longleftarrow} & B \otimes \mathbf{C}^N & \longleftarrow & 0 & & \end{array} \tag{2.11}$$

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & A & \overset{\varepsilon}{\longleftarrow} & B & \overset{b_1}{\longleftarrow} & M_1 & \overset{b_2}{\longleftarrow} & M_2 & \longleftarrow & \ldots \\ & & \Big| \simeq & & \Big| \simeq & & \Big\downarrow k & & \Big\downarrow & & \\ 0 & \longleftarrow & A & \overset{\varepsilon}{\longleftarrow} & B & \overset{d}{\longleftarrow} & B \otimes \mathbf{C}^N & \longleftarrow & 0 & & \end{array} \tag{2.12}$$

where the upper rows of each diagram are the canonical projective resolution of A, see [3], § 5.

Proof. By (2.4) and (2.5), $\Phi(U_i, \mathring{U}_i, 1) = 1$ and $\Phi(U_i, \mathring{U}_i, 0) = 0$. This yields (1).

(2) By $b_1(1_B \otimes g) = g \otimes 1 - 1 \otimes \mathring{g}$, we have $b_1 \circ h(1 \otimes e_i) = U_1 \otimes 1 - 1 \otimes \mathring{U}_i$. This yields the commutativity of (2.11). Now we show the commutativity of (2.12). By (2.2), (2.3) and (2.9), we have

$$\begin{aligned} d \circ k(1_B \otimes g) &= \sum_{i=1}^{N} \sum_{j=1}^{M} (U_1^{n_1^1} \ldots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} U_N^{n_M^N})^\circ (U_i^{n_j^i} \otimes 1 - 1 \otimes \mathring{U}_i^{n_j^i}) = \\ &= (U_1^{n_1^1} \ldots U_N^{n_M^N}) \otimes 1 - 1 \otimes (U_1^{n_1^1} \ldots U_N^{n_M^N})^\circ = g \otimes 1 - 1 \otimes \mathring{g}. \end{aligned} \tag{2.13}$$

This shows the commutativity of the middle square of (2.12).

Let $g \in F_N$ be given by (2.7) and put

$$\tilde{g} = U_1^{n_{M+1}^1} \dots U_N^{n_{M+1}^N} \dots U_1^{n_J^1} \dots U_N^{n_J^N}. \tag{2.14}$$

By $b_2(1_B \otimes g \otimes \tilde{g}) = (g \otimes 1) \otimes \tilde{g} - 1 \otimes g\tilde{g} + (1 \otimes \mathring{\tilde{g}}) \otimes g$, we have

$$k \circ b_2(1_B \otimes g \otimes g) =$$

$$= \sum_{i=1}^{N} \sum_{j=M+1}^{J} g(U_1^{n_{M+1}^1} \dots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} \dots U_N^{n_J^N})^\circ \Phi(U_i, \mathring{U}_i, n_j^i) \otimes e_i -$$

$$- \sum_{i=1}^{N} \sum_{j=1}^{J} (U_1^{n_1^1} \dots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} \dots U_N^{n_J^N})^\circ \Phi(U_i, \mathring{U}_i, n_j^i) \otimes e_i + \tag{2.15}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{M} (U_1^{n_1^1} \dots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} \dots U_N^{n_M^N} \tilde{g})^\circ \Phi(U_i, \mathring{U}_i, n_j^i) \otimes e_i = 0.$$

This shows the commutativity of the right square of (2.12). Q.E.D.

COROLLARY 2.3. *The sequence* (2.1) *gives a projective resolution of A and hence the maps h and k induce quasi-isomorphisms.*

Proof. The upper rows of (2.11) and (2.12) are the canonical projective resolution with the explicity described homotopy maps, see [3], § 5. Let $S: A \to B$ $S_j: M_j \to M_{j+1}$ be the homotopy maps. We put $k \circ S_0: B \to B \otimes \mathbf{C}^N$. Then it is easy to see that this map gives a homotopy of (2.1). Q.E.D.

3. COMPUTATION OF CYCLIC COHOMOLOGY

In view of the fact that the projective dimension of $A = \mathbf{C}[F_N]$ is (less than or equal to) one, the long exact sequence associated with the exact couple of cyclic cohomology and Hochschild cohomology yields the following exact sequences:

$$0 \longrightarrow H_\lambda^0(A) \overset{I}{\underset{\sim}{\longrightarrow}} H^0(A, A^*) \longrightarrow 0, \tag{3.1}$$

$$0 \longrightarrow H_\lambda^1(A) \overset{I}{\longrightarrow} H^1(A, A^*) \overset{B}{\longrightarrow} H_\lambda^0(A) \overset{S}{\longrightarrow} H_\lambda^2(A) \longrightarrow 0, \tag{3.2}$$

$$0 \longrightarrow H_\lambda^j(A) \overset{S}{\underset{\sim}{\longrightarrow}} H_\lambda^{j+2}(A) \longrightarrow 0, \quad j \geqslant 1. \tag{3.3}$$

Thus the spectral sequence associated with the exact couple collapses by taking first derived couple. Let $D = I \circ B: H^{n+1}(A, A^*) \to H^n(A, A^*)$ be the de Rham differential of the exact couple then

$$0 \longrightarrow H^{\mathrm{odd}}(A) \longrightarrow H^1(A, A^*) \xrightarrow{D} H^0(A, A^*) \longrightarrow H^{\mathrm{even}}(A) \longrightarrow 0 \tag{3.4}$$

is exact and this gives a general strategy of computing cyclic cohomology.

We now rewrite all the corresponding maps in terms of the projective resolution (2.1). By Lemma 2.2, we obtain the following diagram

$$\begin{array}{ccc} A^* = \mathrm{Hom}_{\mathbf{C}}(A, \mathbf{C}) & \xrightarrow{b_1^*} & \mathrm{Hom}_{\mathbf{C}}(A \otimes A, \mathbf{C}) = (A \otimes A)^* \\ \| & & \| \\ \mathrm{Hom}_B(B, A^*) & \xrightarrow{b_1^*} & \mathrm{Hom}_B(B \otimes A, A^*) \\ \| & & h^* \downarrow \quad \uparrow k^* \\ \mathrm{Hom}_B(B, A^*) & \xrightarrow{d^*} & \mathrm{Hom}_B(B \otimes \mathbf{C}^N, A^*) \\ \| & & \| \\ A^* & \xrightarrow{d^*} & A^* \otimes \mathbf{C}^N \end{array} \tag{3.5}$$

with the differential on the cochain level, $D = B_1^* \circ k^* \colon A^* \otimes \mathbf{C}^N \to A^*$, where $B_1 \colon A \to A \otimes A$ given by $B_1(a) = 1 \otimes a + a \otimes 1$, $a \in A$, see [3], [4].

We identify an element of $\varphi \in A^*$ in terms of a $\mathbf{C}$-valued sequence indexed by $g \in F_N$, i.e., $\varphi = (\varphi_g)_{g \in F_N}$ together the pairing:

$$\varphi(a) = \sum_g \varphi_g a_g \tag{3.6}$$

where $a = \sum\limits_g a_g g \in A$.

LEMMA 3.1. *On the cochain level,*

(1) $$d^*(\varphi) = (\varphi^1, \ldots, \varphi^N), \quad \varphi_g^j = \varphi_{gU_j} - \varphi_{U_j g}, \quad 1 \leqslant j \leqslant N,$$

(2) $$\begin{cases} D(\varphi^1, \ldots, \varphi^N) = \varphi, \\ \varphi_{U_1^{n_1^1} \cdots U_N^{n_1^N} \cdots U_1^{n_M^1} \cdots U_N^{n_M^N}} = \\ = \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{M} \mathrm{Sum}(l : n_j^i) \varphi^i_{U_i^l U_{i+1}^{n_j^{i+1}} \cdots U_N^{n_M^N} U_1^{n_1^1} \cdots U_{i-1}^{n_j^{i-1}} U_i^{n_j^i - 1 - l}} \end{cases}$$

where $\mathrm{Sum}(l\colon M)a_l$ *is defined by* $\sum_{l=0}^{M-1} a_l$ *for* $M > 0$, *zero for* $M = 0$, *and* $-(a_{-1} + \dots + a_M) = -\sum_{l=-1}^{M} a_l$ *for* $M < 0$.

Proof. (1) Let $g \in F_N$ and $a = g \in A$. Then we get

$$(3.7)\quad \varphi_g^j = \varphi^j(a) = [(U_j \otimes 1 - 1 \otimes \mathring{U}_j)\varphi](a) = \varphi(aU_j) - \varphi(U_j a) = \varphi_{gU_j} - \varphi_{U_j g}.$$

(2) Let $(\varphi^1, \dots, \varphi^N) \in A^* \otimes \mathbf{C}^N$. Then $\theta \in \mathrm{Hom}_{\mathbf{C}}(A \otimes A, \mathbf{C})$ obtained by composing k is

$$(3.8)\quad \theta(a, b) = \sum_{i=1}^{N} \sum_{j=1}^{M} \varphi^i(\mu[(U_1^{n_1^1} \dots U_{i-1}^{n_j^{i-1}}) \otimes (U_{i+1}^{n_j^{i+1}} \dots U_N^{n_M^N})\Phi(U_i, \mathring{U}_i, n_j^i)](a))$$

where $b = g$ is of the form (2.7) and $\mu(a \otimes \mathring{b})(c) = bca$, $a, b, c \in A$. It follows that $\theta(a, 1) = 0$. Let $a = g \in F_N$ be of the form (2.7). Then

$$(3.9)\quad \begin{aligned} \varphi_g &= D(\varphi^1, \dots, \varphi^N)(a) = \theta(1, a) = \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} \mathrm{Sum}(l\colon n_j^i)\varphi^i_{U_i^l U_{i+1}^{n_j^{i+1}} \cdots U_N^{n_M^N} U_1^{n_1^1} \cdots U_{i-1}^{n_j^{i-1}} U_i^{n_j^i - 1 - l}}, \end{aligned}$$

where we used $\Phi(U, V, n) = \mathrm{Sum}(l\colon n)U^l V^{n-1-l}$. Q.E.D.

DEFINITION 3.2. For $g \in F_N$, we define

$$(3.10)\quad \|g\| = \inf_{h \in F_N} |hgh^{-1}|$$

and call *cyclic word length of* g.

By definition, $\|g\| \leqslant |g|$ and $\|g\| = \|hgh^{-1}\|$ for all $h, g \in F_N$. Further, $\|g\| = 0$ if and only if $g = e$.

COROLLARY 3.3. $H^0(A, A^*) \simeq H_\lambda^0(A)$ *is an infinite dimensional* $\mathbf{C}$-*vector space.*

Proof. By Lemma 3.1 (1), $\varphi \in \ker d^* = H^0(A, A^*)$ if and only if $\varphi_{gU_i} = \varphi_{U_i g}$ for any $g \in F_N$ and $1 \leqslant i \leqslant N$. Hence, any $\varphi = (\varphi_g)$, which is constant on each conjugate class, gives an element of $H^0(A, A^*)$. Since there are infinitely many conjugate classes (because the value of $\|g\|$ varies infinitely many) the assertion holds. Q.E.D.

We compute $H_\lambda^1(A)$ and $H_\lambda^2(A)$. Since $d^* D = 0$ and $Dd^* = 0$, we obtain $H_\lambda^1(A) = \ker D/\mathrm{Im}\, d$ and $H_\lambda^2(A) = \ker d/\mathrm{Im}\, D$.

LEMMA 3.4. *$H^2_\lambda(A) \simeq \mathbf{C}$. Furthermore, it is generated by the trace τ.*

Proof. It is easy to see that the trace $\tau = (\delta_{e,g})_{g \in F_N}$ is in $\ker d^*$. For any element in F_N, we can always choose g in its conjugate class such that $\|g\| = |g|$, i.e., there are no canceling factors on the front and the end of g. Let g be of the form (2.7). Then in view of the formula of Lemma 3.1 (2), the trace τ cannot be in the range of D because $\mathrm{Sum}(l: 0) = 0$. So, τ is a non-trivial element in $H^2_\lambda(A)$.

The basis of $\ker d^*$ corresponds to the conjugate class of $g \in F_N$. Suppose, $\varphi = (\varphi_h) \in \ker d^*$ such that $\varphi_h = 1$ if h is conjugate to g and $\varphi_h = 0$ if not. We assume $g \neq e$ and show that such φ is in the image of D. We can assume $\|g\| = |g| \geqslant 1$ and g be of the form (2.7). We define $\theta = (\theta^j_h)_{h \in F_N, 1 \leqslant j \leqslant N} \in A^* \otimes \mathbf{C}^N$ as follows:

$$\theta^i_{U_i^l U_{i+1}^{n_j^{i+1}} \cdots U_N^{n_M^N} U_1^{n_1^1} \cdots U_{i-1}^{n_j^{i-1}} U_i^{n_j^i - 1 - l}} = \tag{3.11}$$
$$= \begin{cases} \;\;|g|^{-1} & \text{for } l = 0, 1, \ldots, n_j^i \text{ if } n_j^i > 0 \\ -|g|^{-1} & \text{for } l = -1, \ldots, n_j^i \text{ if } n_j^i < 0. \end{cases}$$

The other components are defined to be zero. Then $D\theta = \varphi$ by Lemma 3.1 (2). Q.E.D.

LEMMA 3.5. *$H^1_\lambda(A) \simeq \mathbf{C}^N$. Furthermore, it is generated by $\tau^j \otimes e^j \in A^* \otimes \mathbf{C}^N$, $j = 1, \ldots, N$ where $\tau^j \in A^*$, $j = 1, \ldots, N$ are given by $\tau^j = (\delta_{gU_j^{-1}})_{g \in F_N}$.*

Proof. We have to show that τ^j, $j = 1, \ldots, N$ are not in the right hand side of the formula of Lemma 3.1 (2). For this, we have only to show that τ^j's do not appear on the right hand side of $\varphi_{U_j^n}$. This is seen by:

$$\varphi_{U_j^n} = \mathrm{Sum}(l: n)\varphi^j_{U_j^{n-1}} = n\varphi^j_{U_j^{n-1}}.$$

Thus, τ^j, $j = 1, \ldots, N$ are in $\ker D$.

Suppose that there exists $\xi_1, \ldots, \xi_N \in \mathbf{C}$ and $\varphi \in A^*$ such that $\sum_{j=1}^N \xi_j \tau^j \otimes e^j = d^*(\varphi) \in A^* \otimes \mathbf{C}^N$. We put $\Phi = \sum_{j=1}^N \xi_j U_j^{-1} \otimes e_j \in A \otimes \mathbf{C}^N$. By Lemma 3.1 (1), $\langle d^*(\varphi), \Phi \rangle = 0$. On the other hand, by the definition of τ^j,

$$\left\langle \sum_{j=1}^N \xi_j \tau^j \otimes e^j, \Phi \right\rangle = \sum_{j=1}^N \xi_j \zeta_j. \tag{3.12}$$

For any non-zero $(\xi_1, \ldots, \xi_N)$, we can always choose $\zeta_1, \ldots, \zeta_N$ such that $\sum_{j=1}^N \xi_j \zeta_j = 0$. This shows that $\tau^j \otimes e^j$, $j = 1, \ldots, N$ are independent in $H^1_\lambda(A)$.

We next show that any element $(\varphi^1, \ldots, \varphi^N) \in \ker D \subset A^* \otimes \mathbf{C}^N$ with $\varphi^j_{U_j^{-1}} = 0$, $j = 1, \ldots, N$, is in the image of d^*. We are going to use following basic formulas:

$$\varphi_{U_j^{-1} g U_j} = \varphi^j_{U_j^{-1} g} + \varphi_g, \tag{3.13}$$

$$\varphi_{U_j g U_j^{-1}} = -\varphi^j_{g U_j^{-1}} + \varphi_g, \quad j = 1, \ldots, N. \tag{3.14}$$

These formulas are simply the rewriting of $d^*\varphi = (\varphi^1, \ldots, \varphi^N)$. These formulas read that the value of φ_g at $g \in F_N$ determines automatically the value of φ_h for any h conjugate to g. The condition $(\varphi^1, \ldots, \varphi^N) \in \ker D$ guarantees that φ_g is well-defined in the following way. Suppose $g \in F_N$ is of the form (2.7). Then for example, by (3.13) we have

$$\varphi_{U_1^{n_1^1-1} U_2^{n_1^2} \cdots U_N^{n_M^N} U_1} = \varphi^1_{U_1^{n_1^1-1} U_2^{n_1^2} \cdots U_N^{n_M^N}} + \varphi_g \tag{3.15}$$

$$\varphi_{U_2^{n_1^2} \cdots U_N^{n_M^N} U_1^{n_1^1}} = \varphi^1_{U_2^{n_1^2} \cdots U_N^{n_M^N} U_1^{n_1^1-1}} + \varphi_{U_1 U_2^{n_1^2} \cdots U_N^{n_M^N} U_1^{n_1^1-1}} \tag{3.16}$$

if $n_1^1 > 0$. If $n_1^1 < 0$, then we use (3.14) instead of (3.13). This procedure enables us to come back to the redefinition of the value φ_g. The condition $(\varphi^1, \ldots, \varphi^N) \in \ker D$ guarantees that this value is the same as the given value φ_g. Then from its construction, $d^*(\varphi) = (\varphi^1, \ldots, \varphi^N)$ and this completes the proof. Q.E.D.

COROLLARY 3.6. *$H^1(A, A^*)$ is an infinite dimensional* $\mathbf{C}$*-vector space.*

Proof. By Corollary 3.3, Lemmas 3.4, 3.5 and the exactness of the sequence (3.2). Q.E.D.

THEOREM 3.7. *$H^{\text{even}}(A) \simeq \mathbf{C}$ and it is generated by the canonical trace τ. $H^{\text{odd}}(A) \simeq \mathbf{C}^N$ and it is generated by 1-traces φ^j, $j = 1, \ldots, N$, given by* (1.1).

Proof. The even part is Lemma 3.4. By using $h: B \otimes \mathbf{C}^N \to B \otimes A$ defined by (2.10), $h^*: \mathrm{Hom}_{\mathbf{C}}(A \otimes A, \mathbf{C}) \to A^* \otimes \mathbf{C}^N$ maps φ^i of (1.1) to τ^j of Lemma 3.5 by $\tau_j(a) = \varphi^j(a, U_j)$, $a \in A$, $j = 1, \ldots, N$, the statement follows. Q.E.D.

4. FRÉCHET TOPOLOGIES ON THE ALGEBRA

In this section, we consider two Fréchet topologies on the group algebra based on the word length (see (2.7) and (2.8)).

Let $\#(n)$, $n \in \mathbf{N}$, be the number of elements $g \in F_N$ such that $|g| = n$ i.e.,

$$(4.1) \qquad \#(n) = \#\{g \in F_N : |g| = n\} = \begin{cases} 2N(2N-1)^{n-1} & n \geqslant 1 \\ 1 & n = 0. \end{cases}$$

The second equality is easily seen by using an action of F_N on a tree EF_N.

Our first topology will be given by the following family of seminorms

$$(4.2) \qquad P_q^{\#}(a) = \sum_{g \in F_N} \#(|g|)^q |a_g|, \quad q \in \mathbf{N},$$

where $a = (a_g)_{g \in F_N} = \sum_{g \in F_N} a_g g$ is a $\mathbf{C}$-valued sequence on F_N. We define

$$(4.3) \qquad \mathscr{A}^{\#}(F_N) = \{a = \sum_g a_g g : P_q^{\#}(a) < \infty, \ \forall q \in \mathbf{N}\}.$$

Our next topology will be given by the following family of seminorms

$$(4.4) \qquad P_q(a) = \sum_{g \in F_N} |g|^q |a_g|, \quad q \in \mathbf{N}.$$

Similarly, we define

$$(4.5) \qquad \mathscr{A}(F_N) = \{a = \sum_g a_g g : P_q(a) < \infty, \ \forall q \in \mathbf{N}\}.$$

We will show in Appendix that both $\mathscr{A}^{\#}(F_N)$, $\mathscr{A}(F_N)$ are topological *-algebras.

We endow, as in [3], all the formulae with each of the above topologies. Especially, tensor products are viewed as projective tensor products.

LEMMA 4.1. *The projective resolution constructed in Section 2 is a topological projective resolution with respect to both topologies.*

Proof. In the discussion of Section 2, the maps d and h are easily seen to be continuous in both topologies. So, we show that the map $k : B \otimes A \to B \otimes \otimes \mathbf{C}^N$ is continuous. In view of (2.9), it is enough to show that the map $k_i : A \to B$ induced by

$$(4.6) \qquad k_i(g) = \sum_{j=1}^{M} \mathrm{Sum}(l : n_j^i)\,(U_1^{n_j^1} \ldots U_{i-1}^{n_j^{i-1}} U_i^l) \otimes (U_i^{n_j^i - 1 - l} U_{i+1}^{n_j^{i+1}} \ldots U_N^{n_M^N})^{\circ}$$

is continuous, where $g \in F_N$ is assumed to be of the form (2.7). Then for $a = \sum_g a_g g \in$

$\in \mathbf{C}[F_N]$, we have

$$(4.7)\quad \begin{aligned}(P_r\otimes_\pi P_s)(k_i(\sum_g a_g g)) &= \inf(\sum_l P_r(x_l)P_s(y_l): k_i(\sum_g a_g g) = \sum_l x_l \otimes y_l) \leqslant \\ &\leqslant \sum_g |a_g|(P_r \otimes P_s)(k_i(g)) \leqslant \sum_g |a_g| \sum_{l=1}^{|g|-1} l^r(|g|-l)^s \leqslant \sum_g |a_g|\, |g|^{r+s+1}.\end{aligned}$$

By the density $\mathbf{C}[F_N]$ in $\mathcal{s}(F_N)$, all k_i, $i = 1, \ldots, N$ are continuous with respect to P-topology.

The case of $P^\#$-topology is similar. For simplicity, we replace $\#(n)$ by $(2N - 1)^n$. It is clear that the replacement does not change the topology by $P^\#$.

$$(4.8)\quad \begin{aligned}(P_r^\# \otimes_\pi P_s^\#)(k_i(\sum_g a_g g)) &\leqslant \sum_g |a_g|(P_r \otimes P_s)(k_i(g)) \leqslant \\ &\leqslant \sum_g |a_g| \sum_{l=1}^{|g|-1} (2N-1)^{rl}(2N-1)^{s(|g|-l)}.\end{aligned}$$

If $r = s$ then (4.8) is dominated by $\sum_g |a_g|(2N-1)^{s|g|}|g| \leqslant \sum_g |a_g|(2N-1)^{(s+1)|g|}$. If $r \neq s$, then it is easy to see that (4.8) is dominated by Const. $\times \sum_g |a_g|(2N-1)^{t|g|}$, $t = \max(r, s)$. This proves the continuity. Q.E.D.

Next, we show that the proofs of Corollary 3.3, Lemmas 3.4 and 3.5 work also for the topological situation.

LEMMA 4.2. (1) $\varphi = (\varphi_g) \in (\mathcal{s}^\#(F_N))^*$ *if and only if there exists* $g \in \mathbf{N}$ *and a constant* $C^\#_{q,\varphi}$ *such that* $|\varphi_g| \leqslant C^\#_{q,\varphi}\ \#(|g|)^q$.

(2) $\varphi = (\varphi_g) = (\mathcal{s}(F_N))^*$ *if and only if there exists* $q \in \mathbf{N}$ *and a constant* $C_{q,\varphi}$ *such that* $|\varphi_g| \leqslant C_{q,\varphi}|g|^q$.

Proof. By the definition and the fact that $(l^1)^* = l^\infty$. Q.E.D.

By Lemma 4.2, $\varphi = (\varphi_g)$ which is constant on one conjugate class belongs to the dual in both senses, so that Corollary 3.3 holds for the topological situation.

Now, we show that Lemmas 3.4 and 3.5 hold for the topological situation. We have to show the following facts:

(a) in the proof of Lemma 3.4, the map $\{\varphi \in \ker d^*: \varphi_e = 0\} \to A^* \otimes \mathbf{C}^N$ defined by (3.11) is continuous,

(b) in the proof of Lemma 3.5, the map $\{(\varphi^1, \ldots, \varphi^N) \in \ker D: \varphi^j_{U_j^{-1}} = 0,$ $j = 1, \ldots, N\} \to A^*$ defined by (3.13) and (3.14) is continuous.

Proof. (a) By (3.11), $\theta_g^i = (|g| + 1)^{-1}\varphi_h$ for some $h \in F_N$ satisfying $|g| + 1 = |h|$ if $\theta_g^i \neq 0$. Hence, $|\theta_g^i| \leqslant |\varphi_h|$ so that by using $|\varphi_h| \leqslant \text{Const.}\ \#(|h|)^q$ (resp. $\leqslant \text{Const.}\ |h|^q$) $|\theta_g^i| \leqslant \text{Const.}\ \#(|g|)^{q+1}$ (resp. $\leqslant \text{Const.}\ |g|^{q+1}$). This proves (a).

(b) First, for the definition of the map, we choose $\varphi_g = 0$ for g in each conjugate class which satisfies $\|g\| = |g|$. The meaning of the formulas (3.13) and (3.14) is that the value φ_h is determined by φ_g if h and g are conjugate. Suppose g is conjugate to g_0 satisfying $\|g_0\| = |g_0|$ and $\varphi_{g_0} = 0$. Then,

$$\varphi_g = \pm \varphi_{h_1}^{j_1} \pm \varphi_{h_2}^{j_2} \pm \cdots \tag{4.9}$$

with number of terms less than $|g|$ and such that $|h_i| \leqslant |g|$. Hence, if we choose $q \in \mathbf{N}$ and $C > 0$ satisfying $|\varphi_h^j| \leqslant C|h|^q$ for $j = 1, \ldots, N$, then $|\varphi_g| \leqslant C|g|^{q+1}$ in the case of P-topology. In the case of $P^\#$-topology, we choose $q \in \mathbf{N}$ and $C > 0$ satisfying $|\varphi_h^j| \leqslant C\,\#(|h|)^q$, and we obtain $|\varphi_g| \leqslant C|g|\,\#(|g|)^q \leqslant C\,\#(|g|)^{q+1}$. Q.E.D.

THEOREM 4.3. *With any of these two topologies, the results in Section 3 remain true.*

5. DISCUSSIONS

The computation of cyclic homology of algebraic group algebra $A = R[G]$ with commutative ring R and discrete group G is already obtained by Burghelea [1] based on the method of classifying space and homotopy theory. Our approach is based on a straightforward method using projective resolution and spectral sequence.

The formal pairing with K-theory is as follows. The generator of $K_0(C_r^*(F_N))$ is $1 \in \mathscr{A}^\#(F_N) \subset C_r^*(F_N)$. This pairs with $[\tau] \in H^{\text{even}}(A)$. The generators of $K_1(C_r^*(F_n))$ are $U_1, \ldots, U_n \in \mathscr{A}^\#(F_N) \subset C_r^*(F_N)$. Each U_j pairs with $[\varphi^j] \in H^{\text{odd}}(A)$, $j = 1, \ldots, N$. But it is not yet known that $\mathscr{A}^\#(F_N)$ or $\mathscr{A}(F_N)$ give the same K-theory or not.

It is proved that $\mathscr{A}^\#(F_N)$ is nuclear and $\mathscr{A}(F_N)$ is non-nuclear as topological vector spaces (see Appendix). So, the nuclearity may not be important for cyclic cohomology.

We can also see that even if $N = \infty$, all the algebraic discussions together with P-topology work but $P^\#$-topology does not work.

APPENDIX

Here, we will show that $\mathscr{A}^\#(F_N)$ and $\mathscr{A}(F_N)$ defined by (4.3) and (4.5) are topological *-algebras.

LEMMA A.1. $\#(m + n) \leqslant \#(m)\,\#(n)$, $m, n \in \mathbf{N}$.

Proof. If $m = n = 0$ or one of these m or n is zero, then the inequality is trivial. So, we may assume $m \geqslant 1$ and $n \geqslant 1$, and then compute:

$$\begin{aligned} \#(m)\#(n) &= 2N(2N-1)^{m-1}2N(2N-1)^{n-1} = \\ &= 2N(2N-1)^{-1}\cdot 2N(2N-1)^{m+n-1} = 2N(2N-1)^{-1}\#(m+n) \geqslant \#(m+n). \end{aligned} \tag{4.4}$$

Q.E.D.

LEMMA A.2. *$\mathscr{S}^{\#}(F_N)$ is a topological *-algebra.*

Proof. Because $|g^{-1}| = |g|$, we have $P_q^{\#}(a^*) = P_q^{\#}(a)$.
We now compute:

$$\begin{aligned} P_q^{\#}(a*b) &= \sum_g \#(|g|)^q |(a*b)_g| \leqslant \\ &\leqslant \sum_{h,g} \#(|g|)^q |a_h|\,|b_{h^{-1}g}| \leqslant \sum_{h,g} \#(|h| + |h^{-1}g|)^q |a_h|\,|b_{h^{-1}g}| \leqslant \\ &\leqslant \sum_{h,g} \#(|h|)^1\, \#(|h^{-1}g|)^q |a_h|\,|b_{h^{-1}g}| = P_q^{\#}(a)P_q^{\#}(b), \end{aligned} \tag{A.1}$$

where we use Lemma A.1 for the last inequality. Q.E.D.

LEMMA A.3. *$\mathscr{S}(F_N)$ is a topological *-algebra.*

Proof. Because $|g^{-1}| = |g|$, we have $P_q(a^*) = P_q(a)$. We now compute

$$\begin{aligned} P_q(a*b) &= \sum_g |g|^q |(a*b)_g| \leqslant \sum_{h,g} |g|^q |a_h|\,|b_{h^{-1}g}| \leqslant \\ &\leqslant \sum_{h,g} (|h| + |h^{-1}g|)^q |a_h|\,|b_{h^{-1}g}| = \\ &= \sum_{r=0}^{q} \binom{q}{r} \sum_{h,g} |h|^r |h^{-1}g|^{q-r} |a_h|\,|b_{h^{-1}g}| = \sum_{r=0}^{q} \binom{q}{r} P_r(a)P_{q-r}(b). \end{aligned} \tag{A.2}$$

Q.E.D.

REMARK A.3. The seminorms to define the topological *-algebra $\mathscr{S}^{\#}(F_N)$ can also be chosen by the following family:

$$\tilde{P}_q^{\#}(a) = \sup_g (1 + \#(|g|))^q |a_g|, \quad q \in \mathbf{N}. \tag{A.3}$$

This is based on the fact that $\sum_g (1 + \#(|g|))^{-q}$ converges if $q \geqslant 2$. So, $\mathscr{S}^{\#}(F_N)$ will be an analog of the ordinary Schwartz space.

REMARK A.4. $\mathscr{S}^{\#}(F_N) \subset \mathscr{S}(F_N) \subset \mathscr{D} \subset C_r^*(F_N)$, where $\mathscr{D}$ is the set of all smooth elements with respect to the cannonical derivations $\delta_1, \ldots, \delta_N$. The inclusion $\mathscr{S}^{\#}(F_N) \subset \mathscr{S}(F_N) \subset \ell^1(F_N) \subset C_r^*(F_N)$ also holds.

REMARK A.5. The topological vector space $\mathscr{S}^{\#}(F_N)$ is nuclear but $\mathscr{S}(F_N)$ is not.

Proof. By the characterization of nuclearity of spaces of sequences (Theorem 6.1.2 of [4]), the space

(A.4) $$\mathcal{S}^b(F_N) = \{a = \sum_g a_g g : P_q^b(a) < \infty \quad \forall q \in \mathbf{N}\},$$

(A.5) $$P_q^b(a) = \sum_g w(g)^q |a_g|,$$

for given $w: F_N \to \mathbf{R}_+$ is nuclear if and only if for any $q \in \mathbf{N}$, there exists $r \in \mathbf{N}$ and $\mu = (\mu_g) \in \ell^1(F_N)$ such that $w(g)^q \leqslant \mu_g w(g)^r$. If $w(g) = \#(|g|)$, the condition is satisfied due to $\sum_g \#(|g|)^{-s} < \infty$ for $s \geqslant 2$. But if $w(g) = |g|$, it cannot be satisfied due to the fact that $\sum_g |g|^{-s} = \sum_n \#(n) \cdot n^{-s} = \infty$ for any $s \in \mathbf{N}$. Q.E.D.

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