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EXTENDING QUASI-FREE DERIVATIONS ON THE CAR ALGEBRA

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1. INTRODUCTION

Let $\mathscr{A}$ be the CAR algebra over a Hilbert space $\mathscr{H}$. Suppose A is a symmetric operator on $\mathscr{H}$ with deficiency indices (m, n), and δ_A is the corresponding quasi-free derivation on $\mathscr{A}$. A conjecture of R. T. Powers asserts that δ_A has generator extensions only if $m = n$ and in this case all such extensions are quasi-free. Only partial results have been obtained to date. In [7] it was shown that if A is maximal symmetric but not selfadjoint then δ_A has no generator extensions. In [12] the symmetric operator A with deficiency indices $(1, 1)$ was considered, where A is $-\mathrm{i}(\mathrm{d}/\mathrm{d}x)$ on $L^2[0, 1]$ whose domain consists of the set of absolutely continuous functions which vanish at 0 and 1. It was shown that the only generator extensions of δ_A are quasi-free derivations.

In this paper we look only at those generator extensions δ of a quasi-free derivation which commute with the action of the gauge group on the CAR algebra, and annihilate the Fock state. By the latter statement we mean $\omega(\delta(x)) = 0$ for all x in the domain of δ, where ω is the Fock state. We obtain the following results.

THEOREM 1.1. *Suppose A is a symmetric operator on a Hilbert space $\mathscr{H}$ with unequal deficiency indices. Then there are no generator extensions of δ_A commuting with the gauge group and annihilating the Fock state.*

THEOREM 1.2. *Let A be a symmetric operator on $\mathscr{H}$ with deficiency indices $(1, 1)$. Suppose δ is a generator extension of δ_A commuting with the gauge group action on the* CAR *algebra $\mathscr{A}$ and annihilating the Fock state. Then δ is a quasi-free derivation δ_H, where H is a selfadjoint extension of A.*

REMARK. Symmetric operators of deficiency indices $(1, 1)$ are associated with the Hamburger moment problem for a moment sequence for which the representing measure is non-unique (i.e., the indeterminate moment problem [1, 14]).

2. THE FOCK REPRESENTATION OF THE CAR ALGEBRA

We present below some of the properties of the Fock representation of the CAR algebra which are needed for our results. More detailed accounts appear in [4, 6, 11]. We begin by defining Fock space. Let $\mathscr{H}$ be a separable complex Hilbert space. By $\mathscr{H}^n$ we denote the n-fold tensor product space $\mathscr{H} \otimes \ldots \otimes \mathscr{H}$ of $\mathscr{H}$ with itself, and by $\mathscr{H}^0$ the vacuum space $\{\lambda\Omega : \lambda \in \mathbf{C}\}$. We form the Hilbert space $\mathscr{F}(\mathscr{H}) = \bigoplus_{n \geq 0} \mathscr{H}^n$. On $\mathscr{F}(\mathscr{H})$ there is an operator P_-, the antisymmetrizing projection, defined on each summand $\mathscr{H}^n$ by

$$P_-(f_1 \otimes \ldots \otimes f_n) =$$

$$= (n!)^{-1/2}(f_1 \wedge f_2 \wedge \ldots \wedge f_n) = (n!)^{-1} \sum_\sigma \operatorname{sgn}(\sigma) f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)},$$

where σ is a permutation of $\{1, 2, \ldots, n\}$. We say that $f_1 \wedge \ldots \wedge f_n$ is the wedge product of $f_1, \ldots, f_n$ and denote $P_-(\mathscr{H}^n)$ by $\mathscr{H}_n = \mathscr{H} \wedge \ldots \wedge \mathscr{H}$, the n-th wedge product space of $\mathscr{H}$ with itself. The (anti-symmetric) Fock space, $\mathscr{F}_-(\mathscr{H})$, is $\bigoplus_{n \geq 0} \mathscr{H}_n$. For brevity we write $\mathscr{F} = \mathscr{F}_-(\mathscr{H})$.

We may extend the domain of definition of a symmetric operator A on $\mathscr{H}$ to $\mathscr{F}$ by the method of second quantization. On $\mathscr{H}_n$ define an operator A_n by $A_0 = 0$ and set

$$A_n(f_1 \wedge \ldots \wedge f_n) = \sum_j f_1 \wedge \ldots \wedge Af_j \wedge \ldots \wedge f_n \tag{1}$$

where $f_j \in D(A)$ for all j. Extend to finite sums of wedge products by linearity. If A is a selfadjoint operator then its amplification A_n is essentially selfadjoint, [5, p. 8]. The closure of the sum $\oplus A_n$ on Fock space, $d\Gamma(A) = \oplus A_n$ is the second quantization of A.

Next we define the algebra of the canonical anti-commutation relations (CAR algebra), $\mathscr{A}$, by identifying it with its Fock representation. Consider the conjugate-linear (respectively, linear) mapping $a: f \to a(f)$ (respectively, $a^*: f \to a(f)^*$) from $\mathscr{H}$ to $B(\mathscr{F})$. The creation operators $a(f)^*$ map $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$ and are defined on wedge products by $a(f)^*\Omega = f$ and

$$a(f)^*(f_1 \wedge \ldots \wedge f_n) = f \wedge f_1 \wedge \ldots \wedge f_n.$$

The annihilator $a(f)$ (the adjoint of $a(f)^*$) satisfies $a(f)\Omega = 0$ and

$$a(f)(f_1 \wedge \ldots \wedge f_n) = \sum_j (-1)^{j+1}(f, f_j)(f_1 \wedge \ldots \wedge f_{j-1} \wedge f_{j+1} \wedge \ldots \wedge f_n).$$

As a consequence of the preceding equations one observes that the creation and annihilation operators satisfy the following anticommutation relations, for $f, g \in \mathscr{H}$:

(2.1) $$\{a(f)^*, a(g)\}_+ = (g,f)I$$

(2.2) $$\{a(f), a(g)\}_+ = 0.$$

Let $P(\mathscr{H})$ be the algebra of all polynomials in the creation and annihilation operators. We say that a polynomial p is Wick-ordered if each summand of p has all creators appearing to the left of all annihilators, and anti-Wick-ordered if all annihilators appear to the left of all creators. Using the relations (2) any polynomial may be brought into either Wick or anti-Wick order.

Let $\{f_j : j \in \mathbf{N}\}$ be an orthonormal basis for $\mathscr{H}$ and let P_n be the subalgebra of $P(\mathscr{H})$ generated by creators and annihilators in $f_1, \ldots, f_n$. P_n is isomorphic to a $2^n \times 2^n$ matrix algebra, [11]. The uniform completion of $\bigcup_{n \geqslant 1} P_n$ is therefore a UHF algebra of Glimm type 2^∞, the CAR algebra $\mathscr{A} = \mathscr{A}(\mathscr{H})$.

The Fock state ω_0 on $\mathscr{A}$ is the vector state associated with the vacuum Ω, i.e., $\omega_0(x) = (x\Omega, \Omega)$, for all x in $\mathscr{A}$. The Fock state annihilates any product of annihilators and creators in which an annihilator appears to the right.

3. DERIVATIONS COMMUTING WITH THE GAUGE ACTION

Let G be the circle group $G = \mathbf{R}/2\pi\mathbf{Z}$. The gauge group action on $\mathscr{A}$ is the representation $\gamma: G \to \mathrm{Aut}(\mathscr{A})$ defined by setting $\gamma_\theta(a(f)) = a(\mathrm{e}^{\mathrm{i}\theta}f)$. For $n \in \mathbf{Z} = \hat{G}$, we define the linear subspace $\mathscr{A}^\gamma(n)$ of $\mathscr{A}$, the n-th spectral subspace, to be

$$A^\gamma(n) = \{x \in \mathscr{A} : \gamma_\theta(x) = \mathrm{e}^{\mathrm{i}n\theta}x\}.$$

A Wick-ordered monomial lies in the n-th spectral subspace if and only if there are n more creators than annihilators in the product. Since $P(\mathscr{H})$ is dense in $\mathscr{A}$ the observation above shows that the linear span of $\bigcup_n \mathscr{A}^\gamma(n)$ is dense in $\mathscr{A}$. Moreover $ab \in \mathscr{A}^\gamma(m+n)$ if $a \in \mathscr{A}^\gamma(m)$, $b \in \mathscr{A}^\gamma(n)$. In particular the gauge-invariant subspace $\mathscr{A}^\gamma(0) = \mathscr{A}^\gamma$ is a C^*-subalgebra of $\mathscr{A}$, the GICAR algebra.

DEFINITION 3.1. A $*$-derivation δ on $\mathscr{A}$ is said *to commute with the gauge action* γ if $\gamma_\theta(D(\delta)) = D(\delta)$ for all $\theta \in G$, and for $x \in D(\delta)$, $\delta(\gamma_\theta(x)) = \gamma_\theta(\delta(x))$.

The quasi-free derivations on $\mathscr{A}$ commute with the gauge group action. They may be defined as follows. Let A be a symmetric operator on $\mathscr{H}$. Define a linear mapping δ_A on the annihilation operators $a(f)$, $f \in D(A)$, by $\delta_A(a(f)) = -a(\mathrm{i}Af)$. If we set $\delta_A(I) = 0$, and extend the domain of definition to polynomials in $a(f)^*$, $a(g)$, $f, g \in D(A)$, by the Leibniz rule, we obtain a closable $*$-derivation on $\mathscr{A}$, [10]. The closure δ_A is called the quasi-free derivation corresponding to A. It is known that δ_A is a generator of a one-parameter group of strongly continuous $*$-automorphisms if and only if A is selfadjoint.

Since $\gamma_\theta(\delta_A(x)) = \delta_A(\gamma_\theta(x))$ for x in the core $D(\delta_A) \cap P(\mathscr{H})$ of δ_A, δ_A commutes with the gauge action. Although we are only considering extensions of δ_A in this paper which commute with the gauge action, there are examples known of generator extensions of certain quasi-free derivations which do not have this property. (See [12, Theorem 1], where quasi-freeness is defined in a slightly more general sense. See also [2] for a description of quasi-freeness.)

LEMMA 3.2. *Let δ be a generator of a strongly continuous one-parameter group of $*$-automorphisms $\{\alpha_t : t \in \mathbf{R}\}$ on $\mathscr{A}$. Then if δ commutes with the gauge group action, the linear span of the sets $\mathscr{A}^\gamma(n) \cap D(\delta)$, $n \in \mathbf{Z}$, forms a core for δ.*

Proof. For x in $\mathscr{A}$ and $n \in \mathbf{Z}$ define $\Phi_n(x) = (1/2\pi)\int_0^{2\pi} e^{-in\theta}\gamma_\theta(x)\,d\theta$. Note that $\Phi_n(x) \in \mathscr{A}^\gamma(n)$. If $x \in D(\delta)$ then $\Phi_n(x) \in D(\delta)$ follows from the identity

$$(3) \qquad (1/2\pi)\int_0^{2\pi} e^{-in\theta}\gamma_\theta(\delta(x))\,d\theta = (1/2\pi)\int_0^{2\pi} e^{-in\theta}\delta(\gamma_\theta(x))\,d\theta,$$

and we have $\Phi_n(\delta(x)) = \delta(\Phi_n(x))$. Moreover if x is an analytic element [4, Definition 1] it follows easily from (3) that $\Phi_n(x)$ is also. Let D be the linear span of the analytic elements in $\mathscr{A}^\gamma(n)$, $n \in \mathbf{Z}$; then D is dense in $D(\delta)$, and from the identity $\delta \circ \Phi_n = \Phi_n \circ \delta$, $\delta(D) \subseteq D$. Hence D is a core for δ, [4, Corollary 1]. Since $\bigvee \{\mathscr{A}^\gamma(n) \cap D(\delta) : n \in \mathbf{Z}\}$ contains D, the proof is complete.

COROLLARY 3.3. *With the same assumptions as above,*

$$\alpha_t(\mathscr{A}^\gamma(n)) \subseteq A^\gamma(n) \quad \textit{for all } t \in \mathbf{R}.$$

Proof. Let $x \in \mathscr{A}^\gamma(n)$ be an analytic element for δ. Then there is an $R > 0$ such that for $|t| < R$,

$$\alpha_t(x) = \sum_{m=0}^{\infty} (t^m/m!)\delta^m(x).$$

Since $\gamma_\theta(\delta^m(x)) = \delta^m(\gamma_\theta(x)) = e^{in\theta}\delta^m(x)$, $\alpha_t(x) \in \mathscr{A}^\gamma(n)$, for $|t| < R$. For general t choose k in $\mathbf{N}$ such that $|t/k| < R$. Iterating k times, $\alpha_t(x) = \alpha_{t/k}\circ\ldots\circ\alpha_{t/k}(x)$ lies in $A^\gamma(n)$. Q.E.D.

REMARK. The two preceding results are implicit in [8], but the context and proofs are somewhat different. We choose to give self-contained proofs here for the sake of completeness.

The proof of the following result is motivated by the proof of [9, Lemma 4.2].

LEMMA 3.4. *Let δ be a generator on $\mathscr{A}$ commuting with the gauge group, and suppose $\omega_0 \circ \alpha_t = \omega_0$ for all $t \in \mathbf{R}$. Then for each n there is a one-parameter group $\{U_n(t)\}$ of unitary operators on $\mathscr{H}_n$ such that $U_n(t)(x\Omega) = \alpha_t(x)\Omega$, for all $x \in \mathscr{A}^\gamma(n)$.*

Proof. We first observe that $P(\mathscr{H}) \cap \mathscr{A}^\gamma(n)$ is dense in $\mathscr{A}^\gamma(n)$. For if $p \in P(\mathscr{H})$, an easy calculation shows that $\Phi_n(p)$ lies in $P(\mathscr{H}) \cap \mathscr{A}^\gamma(n)$. Then for $x \in \mathscr{A}^\gamma(n)$, $\|x - \Phi_n(p)\| = \|\Phi_n(x - p)\| \leqslant \|x - p\|$. The assertion now follows from the density of $P(\mathscr{H})$ in $\mathscr{A}$.

We observe also that for $p \in P(\mathscr{H}) \cap \mathscr{A}^\gamma(n)$, $p\Omega \in \mathscr{H}_n$, so by continuity we conclude $\mathscr{A}^\gamma(n)\Omega \subseteq \mathscr{H}_n$. In fact, $\overline{\mathscr{A}^\gamma(n)\Omega} = \mathscr{H}_n$. This follows immediately from the fact that for any wedge product vector $f_1 \wedge \ldots \wedge f_n$ in $\mathscr{H}_n, f_1 \wedge \ldots \wedge f_n = a(f_1)^* \ldots$ $\ldots a(f_n)^*\Omega$.

For each t define a linear operator $U_n(t) : \mathscr{A}^\gamma(n)\Omega \to \mathscr{H}_n$ by

$$U_n(t)(x\Omega) = \alpha_t(x)\Omega. \tag{4}$$

Since ω_0 is α_t-invariant,

$$\|U_n(t)(x\Omega)\|^2 = (\alpha_t(x)\Omega, \alpha_t(x)\Omega) = \omega_0(\alpha_t(x^*x)) = \omega_0(x^*x) = \|x\|^2,$$

so $U_n(t)$ extends by continuity to an isometry on $\mathscr{H}_n$. In fact, $U_n(t)$ is unitary, since $U_n(-t)U_n(t)(x\Omega) = \alpha_{-t}(\alpha_t(x))\Omega = x\Omega$ shows that $U_n(-t) = U_n(t)^{-1}$. Replacing $-t$ by $s \in \mathbf{R}$ in the equation above establishes the group property $U_n(t+s) = U_n(t)U_n(s)$. Q.E.D.

LEMMA 3.5. *Let δ_A be a quasi-free derivation on $\mathscr{A}$. Let δ be a generator extension of δ_A commuting with the gauge group action and annihilating the Fock state. Then the infinitesimal generator H of $\{U_1(t)\}$ is a selfadjoint extension of A.*

Proof. From (4) we have $U_1(t)(x\Omega) = \alpha_t(x)\Omega$ for x in $\mathscr{A}^\gamma(1)$. For f in $D(A)$, $a(f)^* \in D(\delta_A) \subseteq D(\delta)$, so

$$iAf = \delta_A(a(f)^*)\Omega = \delta(a(f)^*)\Omega = \lim_{t\to 0} [(\alpha_t(a(f)^*) - a(f)^*)/t]\Omega =$$

$$= \lim_{t\to 0} (U_1(t)f - f)/t = iHf.$$

Hence H is a selfadjoint extension of A. Q.E.D.

Proof of Theorem 1.1. If δ is a generator extension of δ_A then by the preceding lemma A has a selfadjoint extension. But this is impossible if A has unequal deficiency indices. Q.E.D.

4. GENERATOR EXTENSIONS OF QUASI-FREE DERIVATIONS

Suppose H is a selfadjoint operator on $\mathscr{H}$. As above we use H to define a symmetric operator H_n on $\mathscr{H}_n$ by setting $H_n(f_1 \wedge \ldots \wedge f_n) = \sum_j f_1 \wedge \ldots \wedge f_{j-1} \wedge Hf_j \wedge \wedge f_{j+1} \wedge \ldots \wedge f_n$, for f_i in $D(H)$. H_n is essentially selfadjoint on its domain $D(H) \wedge \ldots \wedge D(H)$ consisting of linear combinations of wedge products as above. For if D is the set of analytic vectors for H, and $D \wedge \ldots \wedge D \subseteq D(H) \wedge \ldots \wedge D(H)$ consists of linear combinations of wedge products $f_1 \wedge \ldots \wedge f_n$, with $f_i \in D$, then $D \wedge \ldots \wedge D$ is a dense set of analytic vectors for H_n and $H_n : D \wedge \ldots \wedge D \to D \wedge \ldots \wedge D$. So we may conclude, upon taking the graph closure, that H_n is a selfadjoint operator on $\mathscr{H}_n$.

Let A be a closed symmetric operator with deficiency indices (1,1) having H as an extension. Then [13, Section 123] there is a vector h in $D(H)$ such that $D(H) = \{f + \lambda h : f \in D(A),\ \lambda \in \mathbf{C}\}$. Let $f_1 \wedge \ldots \wedge f_n$ be in $D(H) \wedge \ldots \wedge D(H)$, and suppose that $f_i = g_i + \lambda_i h$ for some g_i in $D(A)$. Then

$$f_1 \wedge \ldots \wedge f_n = (g_1 + \lambda_1 h) \wedge \ldots \wedge (g_n + \lambda_n h) =$$

$$= g_1 \wedge \ldots \wedge g_n + \sum_j \lambda_j (g_j \wedge \ldots \wedge g_{j-1} \wedge h \wedge g_{j+1} \ldots \wedge g_n) =$$

$$= g_1 \wedge \ldots \wedge g_n + \sum_j (-1)^{n-j} g_1 \wedge \ldots \wedge g_{j-1} \wedge g_{j+1} \wedge \ldots \wedge g_n \wedge \lambda_j h.$$

Therefore $D(H) \wedge \ldots \wedge D(H)$ coincides with $D(A) \wedge \ldots \wedge D(A) \wedge D(H)$, and since H_n is essentially selfadjoint on the former domain, we have the following:

LEMMA 4.1. *Let H be a selfadjoint extension of a symmetric operator A with deficiency indices* (1,1). *Then H_n is essentially selfadjoint on the submanifold $D(A) \wedge \ldots \wedge D(A) \wedge D(H)$ of $D(H_n)$.*

CONJECTURE. *The result above is true for any symmetric operator A having equal deficiency indices.*

COROLLARY 4.2. *Let δ be a generator extension of δ_A which commutes with the gauge group and annihilates the Fock state. If A has deficiency indices* (1,1) *there is a selfadjoint extension H such that H_n is the generator of the group $\{U_n(t)\}$ on $\mathscr{H}_n$.*

Proof. Let $H(n)$ be the (selfadjoint) generators of the one-parameter groups $\{U_n(t)\}$. If $x \in D(\delta) \cap \mathscr{A}^\gamma(n)$ then since $\alpha_t(x)\Omega = U_n(t)(x\Omega)$, $x\Omega \in D(H(n))$ and $\delta(x)\Omega = iH(n)(x\Omega)$.

Consider the case $n = 1$, and let $H = H(1)$. The set of analytic elements for δ in $\mathscr{A}^\gamma(1) \cap D(\delta)$ is dense in $\mathscr{A}^\gamma(1)$ and invariant under δ, by Corollary 3.3 and its proof. If $x \in \mathscr{A}^\gamma(1) \cap D(\delta)$ is analytic for δ then $x\Omega$ is analytic for H. Hence $[\mathscr{A}^\gamma(1) \cap D(\delta)]\Omega$ contains a core of analytic vectors for H, invariant under H, so H is essentially selfadjoint on $[\mathscr{A}^{\gamma}(1) \cap D(\delta)]\Omega$. Hence if $f \in D(H)$ there exists a sequence $\{x_k : k \in \mathbf{N}\}$ of elements in $\mathscr{A}^{\gamma}(1) \cap D(\delta)$ such that

(i) $x_k\Omega \to f$, and

(ii) $-i\delta(x_k)\Omega \to Hf$.

Fix $n \in \mathbf{N}$ and fix vectors $f_1, \ldots, f_{n-1}$ in $D(A)$. For each k let $y_k = a(f_1)^* \ldots a(f_{n-1})^* x_k$. Then y_k lies in $\mathscr{A}^{\gamma}(n) \cap D(\delta)$, so $y_k\Omega \in D(H(n))$. Moreover, $\lim_{k\to\infty} y_k\Omega = f_1 \wedge \ldots \wedge f_{n-1} \wedge f$, and

$$\lim_{k\to\infty} iH(n)y_k\Omega = \lim_{k\to\infty} \delta(y_k)\Omega =$$

$$= \lim_{k\to\infty} \delta(a(f_1)^* \ldots a(f_{n-1})^*)(x_k\Omega) + a(f_1)^* \ldots a(f_{n-1})^*\delta(x_k)\Omega =$$

$$= \delta_A(a(f_1)^* \ldots a(f_{n-1})^*)f + a(f_1)^* \ldots a(f_{n-1})^* (iHf) =$$

$$= i\sum_{j=1}^{n-1} f_1 \wedge \ldots \wedge f_{j-1} \wedge Af_j \wedge f_{j+1} \wedge \ldots \wedge f_{n-1} \wedge f + i(f_1 \wedge \ldots \wedge f_{n-1} \wedge Hf),$$

so $f_1 \wedge \ldots \wedge f_{n-1} \wedge f \in D(H(n))$ and $H(n)(f_1 \wedge \ldots \wedge f_{n-1} \wedge f) = H_n(f_1 \wedge \ldots \wedge f_{n-1} \wedge f)$. Since linear combinations of wedge products as above form a core for H_n, by the lemma, $H(n)$ must be a selfadjoint extension of H_n. But H_n is itself selfadjoint, so $H_n = H(n)$. Q.E.D.

COROLLARY 4.3. *Let $\{\beta_t\}$ be the group of Bogoliubov automorphisms on $\mathscr{A}$ defined by $\beta_t(a(f)) = a(U_1(t)f)$, $f \in \mathscr{H}$. Then for any n and for $f_1, \ldots, f_n$ in $\mathscr{H}$,*

$$\alpha_t(a(f_1)^* \ldots a(f_n)^*)\Omega = \beta_t(a(f_1)^* \ldots a(f_n)^*)\Omega,$$

where $\{\alpha_t\}$ is the group of automorphisms generated by δ.

Proof. The proof of the preceding corollary shows $U_n(t) = e^{itH_n}$. It is straightforward to show that $e^{itH_n}(f_1 \wedge \ldots \wedge f_n) = (e^{itH}f_1) \wedge \ldots \wedge (e^{itH}f_n)$. Hence

$$\alpha_t(a(f_1)^* \ldots a(f_n)^*)\Omega = U_n(t)(a(f_1)^* \ldots a(f_n)^*\Omega) =$$

$$= U_n(t)(f_1 \wedge \ldots \wedge f_n) = (e^{itH}f_1) \wedge \ldots \wedge (e^{itH}f_n) = \beta_t(a(f_1)^* \ldots a(f_n)^*)\Omega.$$

Q.E.D.

COROLLARY 4.4. *$\alpha_t = \beta_t$, for all $t \in \mathbf{R}$.*

Proof. By the preceding corollary we have $\alpha_t(a(f)^*)\Omega = \beta_t(a(f)^*)\Omega$, for any f in $\mathscr{H}$. Also by the preceding corollary we have, for any vectors $f_1, \dots, f_n$ in $\mathscr{H}$,

$$\alpha_t(a(f)^*)f_1 \wedge \dots \wedge f_n = \alpha_t(a(f)^*)\,(\alpha_t \circ \alpha_{-t})\,(a(f_1)^* \dots a(f_n)^*)\Omega =$$

$$= \alpha_t(a(f)^*)\alpha_t(a(\mathrm{e}^{-\mathrm{i}tH}f_1)^* \dots a(\mathrm{e}^{-\mathrm{i}tH}f_n)^*)\Omega =$$

$$= \alpha_t(a(f)^*a(\mathrm{e}^{-\mathrm{i}tH})f_1)^* \dots a(\mathrm{e}^{\mathrm{i}tH}f_n)^*)\Omega =$$

$$= a(\mathrm{e}^{\mathrm{i}tH}f)^*a(f_1)^* \dots a(f_n)^*\Omega = \beta_t(a(f)^*)(f_1 \wedge \dots \wedge f_n).$$

Therefore, by linearity, $\alpha_t(a(f)^*)$ and $\beta_t(a(f)^*)$ agree on $\mathscr{H}_n$, for all n, so they agree on all of $\mathscr{F}$. But then $\alpha_t(a(f)^*) = \beta_t(a(f)^*)$, so $\alpha_t = \beta_t$. Q.E.D.

Proof of Theorem 1.2. Let $\{\alpha_t\}$ be the one-parameter group of automorphisms generated by δ. By the preceding corollary there is a one-parameter group of unitary operators $\{U_1(t) : t \in \mathbf{R}\}$ such that α_t is the Bogoluibov automorphism corresponding to $U_1(t)$. Moreover, the generator H of $\{U_1(t)\}$ is a selfadjoint extension of A by Lemma 3.5. Let $f \in D(H)$, then

$$\lim_{t \to 0} (\alpha_t(a(f)) - a(f))/t = \lim_{t \to 0} (a(U_1(t)f) - a(f))/t = a(\mathrm{i}Hf),$$

so $a(f) \in D(\delta)$ and $\delta(a(f)) = \delta_H(a(f))$. Hence $\delta \supseteq \delta_H$. But δ, δ_H are both generators, and so $\delta = \delta_H$. Q.E.D.

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