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# NESTS AND INNER FLOWS

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## INTRODUCTION

The purpose of this article is to prove that if $M$ is a von Neumann algebra with separable predual, then every $\sigma$-weakly closed subalgebra of $M$ which contains a nest subalgebra of $M$ is itself a nest subalgebra of $M$. Via the work of Loebl and Muhly [5], this proves that if $\{\alpha_t\}_{t\in\mathbf{R}}$ is a $\sigma$-weakly continuous representation of $\mathbf{R}$ as inner $*$-automorphisms of $M$ (an inner flow on $M$) then every $\sigma$-weakly closed subalgebra $\mathscr{B}$ of $M$ that contains $H^\infty(\alpha)$ (the algebra of analytic elements, in $M$, with respect to $\alpha$) is $H^\infty(\gamma)$ for some inner flow $\gamma$. In [6] the second author proved the analogous result for arbitrary (not necessarily inner) flows $\alpha$ which are periodic. Thus we show that the condition of periodicity can be dropped provided the flow $\alpha$ is taken to be inner. The question of whether this condition can be dropped for arbitrary flows remains open.

In Section 0 we note that the general problem reduces to the case in which $M$ is a factor via the direct integral theory of [2]. In Section 1 we reduce the factor problem to the case in which $\mathscr{B}$ is relatively transitive in $M$. Some of the ideas we utilize concerning relative reflexivity arose in joint work of the first author with Jon Kraus. We thank him for his part. In Section 2 the proof is completed by solving the transitive case. In Section 3 an alternate lattice theoretic proof is given for the transitive case. We are taking the unusual step in this paper of giving two different proofs for this case because the techniques utilized in each proof are new, and we believe, have independent interest.

## 0. AN INITIAL REDUCTION

Let $M$ be a von Neumann algebra with separable predual and let $\mathscr{N} \subseteq M$ be a nest. We may assume that $M$ acts on a separable Hilbert space $H$. Let $\mathscr{A} := (\operatorname{alg}\mathscr{N}) \cap M$ be the associated nest subalgebra, and suppose $\mathscr{A}_1$ is a $\sigma$-weakly closed algebra with $M \supseteq \mathscr{A}_1 \supseteq \mathscr{A}$.

As in the direct integral theory developed in [2], we may decompose

$$M = \int_{\Lambda}^{\oplus} M(\lambda)\mu(\mathrm{d}\lambda), \quad \mathscr{A} = \int_{\Lambda}^{\oplus} \mathscr{A}(\lambda)\mu(\mathrm{d}\lambda),$$

$$\mathscr{A}_1 = \int_{\Lambda}^{\oplus} \mathscr{A}_1(\lambda)\mu(\mathrm{d}\lambda), \quad \mathscr{N} = \int_{\Lambda}^{\oplus} \mathscr{N}(\lambda)\mu(\mathrm{d}\lambda),$$

where for each $\lambda \in \Lambda$, $M(\lambda)$ is a factor, $\mathscr{N}(\lambda)$ is a nest contained in $M(\lambda)$, $\mathscr{A}(\lambda) := \mathrm{alg}(\mathscr{N}(\lambda)) \cap M(\lambda)$, and $\mathscr{A}_1(\lambda)$ is a $\sigma$-weakly closed subalgebra of $M(\lambda)$ which contains $\mathscr{A}(\lambda)$. Once we show that under these hypotheses $\mathscr{A}_1(\lambda)$ is necessarily a nest subalgebra of $M(\lambda)$, then since $\lambda \to \mathscr{A}_1(\lambda)$ is then a measurable field of nest subalgebras of von Neumann algebras, we may conclude by Theorem 4.4 in [2] that $\mathscr{A}_1$ is the nest subalgebra of $M = W^*(\mathscr{A}_1)$ with respect to some nest $\mathscr{N}_1$ contained in $M$, as desired. $\mathscr{N}_1$ will decompose as $\mathscr{N}_1 = \int_{\Lambda} \mathscr{N}_1(\lambda)\mu(\mathrm{d}\lambda)$, where $\mu$-a.e. we have $\mathscr{N}_1(\lambda) \subseteq M(\lambda)$ and

$$\mathscr{A}_1(\lambda) = M_1(\lambda) \cap \mathrm{alg}\,(\mathscr{N}_1(\lambda)).$$

The balance of this paper is thus devoted to the case in which $M$ is a factor.

## 1. REDUCTION TO THE TRANSITIVE CASE

Let $M$, $\mathscr{N}$, $\mathscr{A}$, $\mathscr{A}_1$ be as in Section 0, and assume $M$ is a factor. We first dispense with the type I case.

PROPOSITION 1. *If $M$ has type* I, *then $\mathscr{A}_1$ is a nest subalgebra of $M$.*

*Proof.* We may then assume without loss of generality that $M = L(H)$, so that $\mathscr{A}$ is a nest algebra. Then by [1, Appendix], $\mathscr{A}$ is the $\sigma$-weakly closed linear span of the rank-1 operators it contains. Since $I \in \mathscr{A}$ we have $\mathscr{A}_1\mathscr{A} = \mathscr{A}_1$, so $\mathscr{A}_1$ is also a unital algebra $\sigma$-weakly generated by rank-1 operators. Thus by [4, Theorem 2.2], $\mathscr{A}_1$ is reflexive. Since $\mathrm{lat}(\mathscr{A}_1) \subseteq \mathscr{N}$, $\mathrm{lat}(\mathscr{A}_1)$ is also a nest. Hence $\mathscr{A}_1$ is a nest subalgebra, as desired.

If $M$ is a von Neumann algebra and $\mathscr{B} \subseteq M$ is a unital algebra, we write $\mathrm{lat}_M(\mathscr{B}) = (\mathrm{lat}\,\mathscr{B}) \cap M$, and for $\mathscr{L} \subseteq M$ a projection lattice we write $\mathrm{alg}_M(\mathscr{L}) = (\mathrm{alg}(\mathscr{L})) \cap M$. We write $\mathrm{ref}_M(\mathscr{A}) = \mathrm{alg}_M(\mathrm{lat}_M(\mathscr{A}))$, and say that $\mathscr{A}$ is *relatively reflexive in $M$* (or *$M$-reflexive*) if $\mathscr{A} = \mathrm{ref}_M(\mathscr{A})$. We extend these definitions to include notions of $M$-reflexive, and $\mathrm{ref}_M(\mathscr{S})$, for arbitrary linear subspaces $\mathscr{S} \subseteq M$.

Define $\mathrm{ref}_M(\mathscr{S}) = \{T \in M : P^\perp T Q = 0$ whenever $P$, $Q$ are projections in $M$ with $P^\perp \mathscr{S} Q = 0\}$, and say that $\mathscr{S}$ is $M$-reflexive if $\mathscr{S} = \mathrm{ref}_M(\mathscr{S})$. These subspace definitions agree with the original if $\mathscr{S}$ is a unital algebra. (Use the fact that $I \in \mathscr{S}$.) It is easily verified that $\mathrm{ref}_M(S)$ is always $M$-reflexive. We say that $\mathscr{S}$ is *$M$-transitive* if $\mathrm{ref}_M(\mathscr{S}) = M$.

The next lemma is very useful, especially when $A$ and $B$ are projections.

LEMMA 2. *Let $\mathscr{S} \subseteq M$ be an arbitrary linear subspace and suppose $T \in M$ with $T \in \mathrm{ref}_M(\mathscr{S})$. Then if $A, B \in M$ are arbitrary operators, we have $ATB \in \mathrm{ref}_M(A\mathscr{S}B)$, where $A\mathscr{S}B = \{ASB : S \in \mathscr{S}\}$.*

*Proof.* Suppose $P$, $Q$ are projections in $M$ with $P^\perp A\mathscr{S}BQ = 0$. Let $E$ be the support projection of $P^\perp A$, and let $P_1 = I - E$. Let $Q_1$ be the range projection of $BQ$. Then $P_1^\perp \mathscr{S} Q_1 = 0$, and $P_1$, $Q_1 \in M$. So by hypothesis, $P_1^\perp T Q_1 = 0$. Since $P^\perp A = P^\perp A P_1^\perp$ and $BQ = Q_1 BQ$, this shows that $P^\perp ATBQ = 0$, as desired.

Next, let $M, \mathscr{N}, \mathscr{A}, \mathscr{A}_1$ be as in the introduction to Section 0, with $M$ a factor of type II or III. We wish to show that $\mathscr{A}_1$ is a nest subalgebra of $M$. If $\mathscr{N}_0$ is a nest in $M$ containing $\mathscr{N}$, then $(\mathrm{alg}\mathscr{N}_0) \cap M \subseteq (\mathrm{alg}\mathscr{N}) \cap M$, so without loss of generality we may assume that $\mathscr{N}$ is a maximal nest in $M$. Since $M$ has type II or III, this means that $\mathscr{N}$ is continuous.

We note that $\mathrm{lat}_M(\mathscr{A}_1) \subseteq \mathrm{lat}_M(\mathscr{A}) = \mathscr{N}$, using the (easily proven) fact that a nest subalgebra of a factor is relatively reflexive in the factor. So $\mathrm{lat}_M(\mathscr{A}_1)$ is a nest. Hence to show that $\mathscr{A}_1$ is a nest subalgebra of $M$ it will suffice to prove that $\mathscr{A}_1$ is $M$-reflexive.

Let $\mathscr{N}_1 = \mathrm{lat}_M(\mathscr{A}_1)$, a subnest of $\mathscr{N}$. Since we assume that $\mathscr{A}_1 \neq \mathscr{A}, \mathscr{N}_1$ is a proper subnest of $\mathscr{N}$, so cannot be continuous. If $\{E_n\}$ denotes the set of minimal $\mathscr{N}_1$-intervals (the "gap" projections), it follows that $\mathrm{alg}_M(\mathscr{N}_1)$ is the $\sigma$-weakly closed linear span of $\mathscr{A}$ together with the subspaces $E_n M E_n$, $n = 1,2, \ldots$ . (Just note that $E_n M E_n \subseteq \mathrm{alg}_M(\mathscr{N}_1)$, and that if $T \in \mathrm{alg}_M(\mathscr{N}_1)$, and if $E_n T E_n = 0$ for each $n$, then $T$ leaves the larger nest $\mathscr{N}$ invariant, so $T \in \mathscr{A}$.) Since each $E_n \in \mathscr{A}_1$ we have $E_n \mathscr{A}_1 E_n \subseteq$ $\subseteq \mathscr{A}_1$. So to prove that $\mathscr{A}_1 = \mathrm{alg}_M(\mathscr{N}_1)$ as desired, it will suffice to prove that $E_n \mathscr{A}_1 E_n = E_n M E_n$ for each $n$.

LEMMA 3. *For each minimal $\mathscr{N}_1$-interval $E_n$ we have* $\mathrm{ref}_M(E_n \mathscr{A}_1 E_n) = E_n M E_n$.

*Proof.* For any pair of projections $P$, $Q \in M$ it is trivially true that the subspace $PMQ$ is $M$-reflexive. So $E_n M E_n$ is $M$-reflexive. We have $E_n M E_n \subseteq \mathrm{ref}_M(\mathscr{A}_1)$, so by Lemma 2 we have

$$E_n M E_n = E_n(E_n M E_n)E_n \subseteq \mathrm{ref}_M(E_n \mathscr{A}_1 E_n) \subseteq E_n M E_n.$$

Now fix $n$ and let $\tilde{M}$ be the factor $\tilde{M} = E_n M \,|\, E_n H$, and let $\tilde{\mathscr{A}} = E_n \mathscr{A} \,|\, E_n H$, $\tilde{\mathscr{A}}_1 = E_n \mathscr{A}_1 \,|\, E_n H$, $\tilde{\mathscr{N}} = \mathscr{N}_{E_n} = \{E_n N \,|\, E_n H : N \in \mathscr{N}\}$. (The latter is a nest since

$E_n \in (\mathcal{N})'$, and is continuous since $\mathcal{N}$ is continuous.) It is easily verified that $\tilde{\mathcal{A}} = \tilde{M} \cap \text{alg}(\tilde{\mathcal{N}}) = \text{alg}_{\tilde{M}}(\tilde{\mathcal{N}})$. Also, $\tilde{\mathcal{A}}_1$ is a $\sigma$-weakly closed algebra since $E_n$ is a projection *contained* in $\mathcal{A}_1$, and will be properly contained in $\tilde{M}$ unless $E_n\mathcal{A}_1E_n = E_nME_n$.

LEMMA 4. *The algebra $\tilde{\mathcal{A}}_1$ is $\tilde{M}$-transitive.*

*Proof.* We need only show that if $\tilde{P}$, $\tilde{Q}$ are projections in $\tilde{M}$ with $\tilde{P}\tilde{\mathcal{A}}_1\tilde{Q} = 0$, then either $\tilde{P} = 0$ or $\tilde{Q} = 0$. Given $\tilde{P}$ and $\tilde{Q}$, extend to projections $P$, $Q$ in $M$ with $\tilde{P} = E_nP \mid E_nH$, $\tilde{Q} = E_nQ \mid E_nH$, and note that $PE_n\mathcal{A}_1E_nQ = 0$. By Lemma 3 we have $\text{ref}_M(E_n\mathcal{A}_1E_n) = E_nME_n$, so this implies that $PE_nME_nQ = 0$, so $E_nPE_nME_nQE_n = 0$, and so $\tilde{P}\tilde{M}\tilde{Q} = 0$. Since $\tilde{M}$ is a factor, and $\tilde{P}, \tilde{Q} \in \tilde{M}$, this implies that either $\tilde{P} = 0$ or $\tilde{Q} = 0$.

To complete the proof it will suffice to show that $\tilde{\mathcal{A}}_1 = \tilde{M}$. We observe that $\tilde{M}$, $\tilde{\mathcal{N}}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_1$ satisfy the hypotheses in the introduction to Section 1, and satisfy the additional restrictions that $\tilde{\mathcal{N}}$ is continuous and that $\tilde{\mathcal{A}}_1$ is $\tilde{M}$-transitive. Since $\tilde{\mathcal{A}}_1$ is $\tilde{M}$-transitive, $\tilde{\mathcal{A}}_1$ will be a nest subalgebra of $\tilde{M}$ if and only if $\tilde{\mathcal{A}}_1 = \tilde{M}$. The general problem has thus been reduced to the special case where $M, \mathcal{N}, \mathcal{A}, \mathcal{A}_1$ are such that $M$ is a factor of type II or III, $\mathcal{N}$ is continuous, and $\mathcal{A}_1$ is $M$-transitive.

## 2. THE TRANSITIVE CASE: A STRUCTURAL SOLUTION

Let $M, \mathcal{N}, \mathcal{A}, \mathcal{A}_1$ be as in the introduction to Section 0 with $M$ a factor of type II or III, $\mathcal{N}$ continuous, and $\mathcal{A}_1$ $M$-transitive. We will show that $\mathcal{D}_1 = \text{diag}(\mathcal{A}_1) = \mathcal{A}_1 \cap \mathcal{A}_1^*$ is also $M$-transitive, hence is a subfactor of $M$ with trivial relative commutant in $M$. The fact that $\mathcal{D}_1$ is a *factor* containing $\mathcal{N}$ and contained in $\mathcal{A}_1$, and that $\mathcal{A}_1$ contains a nest subalgebra of $M$, will then be used to show that $\mathcal{D}_1 = M$, hence $\mathcal{A}_1 = M$.

LEMMA 5. *Let $E$, $F$ be $\mathcal{N}$-intervals with $E \ll F$ (*i.e., *the upper endpoint of $E$ is contained in the lower endpoint of $F$). Then $F\mathcal{A}_1E \subseteq \mathcal{D}_1$.*

*Proof.* Since $E \ll F$ it follows that $EMF \subseteq \mathcal{A} \subseteq \mathcal{A}_1$. Thus $(F\mathcal{A}_1E)^* = E\mathcal{A}_1^*F \subseteq \mathcal{A}_1$, so $F\mathcal{A}_1E \subseteq \mathcal{A}_1^*$, so since $E, F \in \mathcal{A}_1$ we have $F\mathcal{A}_1E \subseteq \mathcal{A}_1 \cap \mathcal{A}_1^* = \mathcal{D}_1$.

PROPOSITION 6. *$\mathcal{D}_1$ is an $M$-transitive factor.*

*Proof.* Let $E$, $F$ be $\mathcal{N}$-intervals with $E \ll F$. Then $F\mathcal{A}_1E \subseteq \mathcal{D}_1$ by Lemma 5, so since $M = \text{ref}_M(\mathcal{A}_1)$, by Lemma 2 we have $FME \subseteq \text{ref}_M(F\mathcal{A}_1E) \subseteq \text{ref}_M(\mathcal{D}_1)$.

But $\mathscr{D}_1$ is selfadjoint, hence $\mathrm{ref}_M(\mathscr{D}_1)$ is selfadjoint, so also $(FME)^* = EMF \subseteq \subseteq \mathrm{ref}_M(\mathscr{D}_1)$.

If $E$, $F$ are orthogonal $\mathscr{N}$-intervals then either $E \ll F$ or $F \ll E$. Thus the above shows that $FME \subseteq \mathrm{ref}_M(\mathscr{D}_1)$ for every such pair.

Let $\mathscr{D} = \mathrm{diag}(\mathscr{A}) = (\mathscr{N})' \cap M$. Let $T \in M$ be arbitrary. By Proposition 2.1 in [3], and the construction in its proof, there is an operator $D \in \mathscr{D} \subseteq \mathscr{D}_1$ such that $T - D$ is in the $\sigma$-weakly closed linear span of $\{ETF : F,\ E$ are orthogonal $\mathscr{N}$-intervals$\}$. From above we have $ETF \in \mathrm{ref}_M(\mathscr{D}_1)$ for each pair $E$, $F$. Hence $T \in \mathrm{ref}_M(\mathscr{D}_1)$. This shows that $\mathrm{ref}_M(\mathscr{D}_1) = M$. That is, $\mathscr{D}_1$ is $M$-transitive. So the only projections in $M$ that commute with $\mathscr{D}_1$ are 0 and $I$. In particular, the center of $\mathscr{D}_1$ is trivial, so $\mathscr{D}_1$ is a factor.

LEMMA 7. *Suppose $E$, $F$ are $\mathscr{N}$-intervals with $E \ll F$, and suppose $P$, $Q$ are projections in $\mathscr{D}_1$ with $P \leqslant E$, $Q \leqslant F$, such that $P$ and $Q$ are equivalent relative to $\mathscr{D}_1$. (That is, suppose there exists a partial isometry $U \in \mathscr{D}_1$ such that $UU^* = P$, $U^*U = = Q$.) Then $(P + Q)M(P + Q) \subseteq \mathscr{D}_1$.*

*Proof.* Let $U$ be a partial isometry in $\mathscr{D}_1$ with $UU^* = P$, $U^*U = Q$. We have $PMQ \subseteq EMF \subseteq \mathscr{A} \subseteq \mathscr{A}_1$. Also $PMP = PMUU^* = (PMUQ)U^*$. Since $PMUQ \subseteq \subseteq \mathscr{A}_1$ and $U^* \in \mathscr{D}_1 \subseteq \mathscr{A}_1$, this shows that $PMP \subseteq \mathscr{A}_1$. Similarly, $QMQ = = U^*UMQ \subseteq U^*(PMQ) \subseteq \mathscr{A}_1$, and $QMP = U^*UMUU^* = U^*(PMQ)U^* \subseteq \mathscr{A}_1$. Hence $(P + Q)M(P + Q)$ is contained in $\mathscr{A}_1$, and is selfadjoint, so is contained in $\mathscr{D}_1$.

THEOREM 8. (Main result). *Let $M$ be a von Neumann algebra with separable predual, and let $\mathscr{N} \subseteq M$ be a nest. If $\mathscr{A}_1$ is a $\sigma$-weakly closed subalgebra of $M$ containing $\mathscr{A} = (\mathrm{alg}\,\mathscr{N}) \cap M$, then there is a nest $\mathscr{N}_1 \subseteq M$ such that $\mathscr{A}_1 = (\mathrm{alg}\,\mathscr{N}_1) \cap M$.*

*Proof.* By previous reductions we can assume that $M$ is a factor, $\mathscr{N}$ is continuous, $\mathscr{A}_1$ is $M$-transitive, and $\mathscr{D}_1 = \mathscr{A}_1 \cap \mathscr{A}_1^*$ is a factor. We must show that $\mathscr{D}_1 = M$. We consider cases.

$\mathscr{D}_1$ cannot be of finite type I since it contains a continuous nest. If $\mathscr{D}_1$ has type $\mathrm{II}_1$ then by continuity of the trace on $\mathscr{N}$ there is an $N \in \mathscr{N}$ with $N$ equivalent in $\mathscr{D}_1$ to $N^\perp$. Since $N + N^\perp = I$, Lemma 7 shows that $\mathscr{D}_1 = M$, as desired. If $\mathscr{D}_1$ is type III, let $N$ be any nontrivial projection in $\mathscr{N}$. Then $N$ and $N^\perp$ are $\mathscr{D}_1$-equivalent, so a similar argument shows that $\mathscr{D}_1 = M$. We have reduced to $\mathscr{D}_1$ of type $\mathrm{I}_\infty$ or $\mathrm{II}_\infty$.

Choose $N \in \mathscr{N}$, $N \neq 0$, $N \neq I$, arbitrarily. If $N$ and $N^\perp$ are $\mathscr{D}_1$-infinite, then they are $\mathscr{D}_1$-equivalent, so the argument above shows that $\mathscr{D}_1 = M$. Suppose $N$ is $\mathscr{D}_1$-finite. Then $N^\perp$ is $\mathscr{D}_1$-infinite. Let $P_1, P_2, \ldots$ be a sequence of mutually orthogonal projections in $\mathscr{D}_1$, all contained in $N^\perp$, and all $\mathscr{D}_1$-equivalent to $N$, such that $N^\perp = \sum_{n=1}^{\infty} P_n$. (We are using strongly the property that $\mathscr{D}_1$ is a factor.) By Lemma 7 we have $(N + P_n)M(N + P_n) \subseteq \mathscr{D}_1$. For each $n$, let $U_n$ be a partial isometry in

$\mathscr{D}_1$ with initial projection $P_n$ and final projection $N$. For each pair $n$, $m$ we have $P_nMP_m = U_n^*(NU_nMP_m) \subseteq \mathscr{D}_1$ since $U_n^* \in \mathscr{D}_1$. So since all "pieces" $NMN$, $P_nMN$, $NMP_n$, $P_nMP_m$ are contained in $\mathscr{D}_1$ for all $n$, $m$, we have $\mathscr{D}_1 = M$.

If $N^\perp$ is $\mathscr{D}_1$-finite, analogous argument shows that $\mathscr{D}_1 = M$ also.

## 3. THE TRANSITIVE CASE: A LATTICE-THEORETIC PROOF

Let $M, \mathscr{N}, \mathscr{A}, \mathscr{A}_1$ be as in the introduction to Section 0 with $M$ a factor and $\mathscr{N}$ a continuous nest (we do not assume yet that $\mathscr{A}_1$ is $M$-transitive).

Since $M$ is $*$-isomorphic to a factor von Neumann algebra with a separating vector (namely, $M \otimes I_k$ with $\dim k = \aleph_0$), we will assume here that $M$ has a separating vector and, consequently ([4]), every $\sigma$-weakly closed subalgebra of $M$ is reflexive. In particular,

$$\mathscr{A}_1 = \operatorname{alg} \operatorname{lat} \mathscr{A}_1 = \operatorname{alg}_M \operatorname{lat} \mathscr{A}_1 = \bigcap \{\operatorname{alg}_M P : P \in \operatorname{lat} \mathscr{A}_1\}.$$

If $\mathscr{A}_1$ is $M$-transitive then $\operatorname{alg}_M P$ is $M$-transitive for every $P \in \operatorname{lat} \mathscr{A}_1$. We will show, however, that, for $P \in \operatorname{lat} \mathscr{A}$ ($\supseteq \operatorname{lat} \mathscr{A}_1$), $\operatorname{alg}_M P$ is $M$-transitive if and only if $P \in M'$ (Corollary 11). This will show that if $\mathscr{A}_1$ is $M$-transitive, then $\mathscr{A}_1 = M$ completing the proof of the main result (Theorem 8) in the transitive case.

We shall need the following notation. Let $\mathscr{C}$ be the algebra $(M \cap \mathscr{N}')' = (\mathscr{A} \cap \mathscr{A}^*)'$ and for every projection $F \in \mathscr{C}$ write $R(F) = \bigvee \{uFu^* : u \text{ is in } U(M)$, the unitary operators in $M\}$. Clearly $R(F) \in M'$ and, if $F$ is a non zero projection in $M \cap \mathscr{C}$, then $R(F) = I$ (since $M$ is a factor).

LEMMA 9. *Let $P$ be in* lat $\mathscr{A}$ *and $N_1, N_2$ in $\mathscr{N}$ with $N_1 < N_2$. Then,*

(1) $$N_1R(P(1 - N_1)) = R(P(1 - N_1))N_1 \leqslant P$$

(2) $$q(N_1, N_2)N_1 \leqslant P, \quad \textit{where } q(N_1, N_2) = R(P(1 - N_1)) - R(P(1 - N_2))$$

*and*

(3) $$\textit{if } q(N_1, N_2) \neq 0, \quad \textit{then } (1 - N_2)(\operatorname{alg}_M P)N_1 = \{0\}.$$

*Proof.* (1) Let $u$ be a unitary operator in $M$. Then $N_1u(1 - N_1) \in \operatorname{alg}_M \mathscr{N} \subseteq \operatorname{alg}_M P$. Hence $(1 - P)N_1u(1 - N_1)Pu^* = 0$. Thus $(1 - P)N_1R(P(1 - N_1)) = 0$ and (1) follows. (Note that $R(P(1 - N_1))$ commutes with $N_1 \in \mathscr{N} \subseteq \mathscr{M}$).

(2) From (1) we see that $R(P(1 - N))N \leqslant P$ for $N = N_i$, $i = 1, 2$. Since $N_1 \leqslant N_2$, $R(P(1 - N))N_1 \leqslant P$ for $N = N_i$, $i = 1, 2$. Thus $q(N_1, N_2)N_1 \leqslant P$.

(3) Let $a$ be in $(1 - N_2)(\operatorname{alg}_M P)N_1$ ($\subseteq \operatorname{alg}_M P$) and write $q$ for $q(N_1, N_2)$. Then, for a vector $x$ in the range of the projection $qN_1P$ ($= qN_1$), $ax$ is in the range of $q(1 - N_2)P \leqslant qR(P(1 - N_2)) = 0$. Hence $aqN_1 = 0$. Since $M$ is a factor and $q$ is a non zero projection in $M'$, $a = aN_1 = 0$.

PROPOSITION 10. *For* $P \in \operatorname{lat}\mathscr{A}$ *and* $N \in \mathscr{N}\setminus\{0, I\}$, *the following are equivalent:*

(1) $N \in \operatorname{lat}\operatorname{alg}_M P$.

(2) *For every* $N_1, N_2$ *in* $\mathscr{N}$ *with* $N_1 \lneq N \lneq N_2$, $q(N_1, N_2) \neq 0$ *(i.e.* $R(P(1 - N_1)) \neq R(P(1 - N_2))$*).*

*Proof.* (1) implies (2): Suppose there are $N_1, N_2$ in $\mathscr{N}$ with $N_1 \lneq N \lneq N_2$ and $q(N_1, N_2) = 0$. Then,

$$(N_2 - N)R((N - N_1)P) \leqslant (N_2 - N)R((1 - N_1)P) = (N_2 - N)R((1 - N_2)P) \leqslant$$
$$\leqslant N_2 R((1 - N_2)P) \leqslant P.$$

(The last inequality follows from Lemma 9(1)). Since $(N_2 - N)R((N - N_1)P)$ is the projection onto the subspace $[(N_2 - N)M(N - N_1)PH]$, we have

$$(1 - P)(N_2 - N)M(N - N_1)P = 0.$$

Hence $(N_2 - N)M(N - N_1) \subseteq \operatorname{alg}_M P$. But $(1 - N)(N_2 - N)M(N - N_1)N = (N_2 - N)M(N - N_1)$ and $(N_2 - N)M(N - N_1) \neq \{0\}$ (since $M$ is a factor, $N_2 \neq N$ and $N \neq N_1$). Therefore $N \notin \operatorname{lat}\operatorname{alg}_M P$.

(2) implies (1): If $N$ satisfies (2) then it follows from Lemma 9(3) that for every $N_1, N_2$ in $\mathscr{N}$ with $N_1 \lneq N \lneq N_2$, $(1 - N_2)(\operatorname{alg}_M P)N_1 = 0$. By continuity of $\mathscr{N}$, $(1 - N)(\operatorname{alg}_M P)N = 0$ and (1) follows.

COROLLARY 11. *For* $P \in \operatorname{lat}\mathscr{A}$ *the following are equivalent:*

(1) $\operatorname{alg}_M P$ *is M-transitive.*

(2) $\mathscr{N} \cap \operatorname{lat}\operatorname{alg}_M P = \{0, I\}$.

(3) $P \in M'$.

*Proof.* (3) implies (1) is obvious and (1) implies (2) follows immediately from the fact that $M$ is a factor.

(2) implies (3). Suppose $\mathscr{N} \cap \operatorname{lat}\operatorname{alg}_M P = \{0, I\}$. Using Proposition 10, for every $N \in \mathscr{N}\setminus\{0, I\}$ there are $N_1, N_2 \in \mathscr{N}$ such that $N_1 \lneq N \lneq N_2$ and $R(P(1 - N_1)) = R(P(1 - N_2))$. It follows that the map $N \to R(P(1 - N))$ is constant on $\mathscr{N}\setminus\{0, I\}$.

Since $\bigvee\{(1 - N)P : N \in \mathscr{N}\setminus\{0, I\}\} = P$, we have $R(P) = R(P(1 - N))$ for every $N \in \mathscr{N}\setminus\{I\}$. But, then, from Lemma 9 (1), we have $NR(P) \leqslant P$ for all $N \in \mathscr{N}\setminus\{I\}$. Since $\bigvee\{N : N \in \mathscr{N}\setminus\{I\}\} = I$, $R(P) \leqslant P$. Hence $P = R(P) \in M'$.

As was mentioned in the introduction to this section, Corollary 11 completes the proof of the main result (Theorem 8) in the transitive case.

REMARKS. Since $\operatorname{alg}_M P \supseteq M \cap \operatorname{alg}\mathscr{N} = \mathscr{A}$ (for $P \in \operatorname{lat}\mathscr{A}$), we have $\operatorname{lat}_M \operatorname{alg}_M P \subseteq \operatorname{lat}_M \mathscr{A} = \mathscr{N}$. Thus Proposition 10 describes $\operatorname{lat}_M \operatorname{alg}_M P$ in terms of the map $N \to R(P(1 - N))$, $N \in \mathscr{N}$.

In particular, if $P_1$ and $P_2$ are projections in $\operatorname{lat}\mathscr{A}$ and for every $N \in \mathscr{N}$, $R(P_1(1-N)) = R(P_2(1-N))$, then $\operatorname{lat}_M \operatorname{alg}_M P_1 = \operatorname{lat}_M \operatorname{alg}_M P_2$.

Since we now know that $\operatorname{alg}_M P_i$, $i = 1, 2$, is $M$-reflexive (as a nest subalgebra of the factor $M$) we have that, if $R(P_1(1-N)) = R(P_2(1-N))$ for all $N \in \mathscr{N}$, then $\operatorname{alg}_M P_1 = \operatorname{alg}_M P_2$.

For $P \in \operatorname{lat}\mathscr{A}$ we let $\tilde{P}$ be the projection $\bigvee\{ER(P(1-E)) : E \in \mathscr{N}\}$. Note that, for every $N \in \mathscr{N}$,

$$R(\tilde{P}(1-N)) = \bigvee_E R(ER(P(1-E))(1-N)) = \bigvee_{E>N} R((E-N)R(P(1-N))) =$$

$$= \bigvee_{E>N} R(P(1-E)) = R(\bigvee_{E>N} P(1-E)) = R(P(1-N)).$$

Hence $\operatorname{alg}_M P = \operatorname{alg}_M \tilde{P}$.

Let $\mathscr{L}_0$ be the set of all elements of $\operatorname{lat}\mathscr{A}$ of the form $L = \bigvee_\lambda N_\lambda Q_\lambda$ where $N_\lambda \in \mathscr{N}$ and $Q_\lambda \in \operatorname{lat} M$. It was shown in [3, Theorem 4.6] that for every $P \in \operatorname{lat}\mathscr{A}$, the projection $\tilde{P} = \bigvee_N NR(P(1-N))$ is the unique element of $\mathscr{L}_0$ maximal with respect to containment in $P$. (It was also shown there that we might have $P \neq \tilde{P}$). We can, therefore, conclude that for every $P \in \operatorname{lat}\mathscr{A}$ there is a projection $\tilde{P} \in \mathscr{L}_0$ such that $\operatorname{alg}_M P = \operatorname{alg}_M \tilde{P}$.

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