J. OPERATOR THEORY
16(1986), 203–212

SOME NEW ELEMENTS IN THE CLASS $\mathbf{A}_{\aleph_0}$

GEORGE EXNER

There has recently been considerable study of the classes $\mathbf{A}$ and $\mathbf{A}_n$ $(1 \leqslant n \leqslant \aleph_0)$ of bounded linear operators on Hilbert space. We exhibit in this paper some classes contained in $\mathbf{A}_{\aleph_0}$, notably from some direct sums and from within the class of compact perturbations of $\mathbf{A}_{\aleph_0}$ (see Section I for the relevant definitions). The organization of this paper is as follows: in Section I we give notation and preliminaries; in Section II we show that if $T \in \mathbf{A}_{\aleph_0}$ and F is a finite rank operator commuting with T for which $T + F$ does not leave $\mathbf{A}_{\aleph_0}$ for trivial reasons, then $T + F \in \mathbf{A}_{\aleph_0}$; in Section III we consider an "equation solving" property of $T \in \mathbf{A}_{\aleph_0}$; and in Section IV we make some remarks and raise some questions.

This paper is a portion of the thesis submitted in partial fulfillment of the degree requirements of The University of Michigan. It benefitted greatly from the comments of the reviewer of an earlier version. Utimately, however, I must express my debt to Professor Carl M. Pearcy, without whose encouragement and inspiration it would not have been written.

I.

We consider a separable complex Hilbert $\mathscr{H}$ of infinite dimension and $\mathscr{L}(\mathscr{H})$- the algebra of bounded linear operators on $\mathscr{H}$. Recall that $T \in \mathscr{L}(\mathscr{H})$ is a contraction if $\|T\| \leqslant 1$, and that T is completely non-unitary (c.n.u.) if its unitary part acts on the space (0). We say that T is an absolutely continuous contraction (AC contraction) if the unitary part of T is absolutely continuous or T is c.n.u.. Subspace will always mean closed linear manifold; recall that a subspace $\mathscr{K} \subset \mathscr{H}$ is invariant for T if $T\mathscr{K} \subset \mathscr{K}$. A subspace $\mathscr{K} \subset \mathscr{H}$ is semi-invariant for T if $\mathscr{K} = \mathscr{M} \ominus \mathscr{N}$, the orthogonal difference of some subspaces $\mathscr{M}$ and $\mathscr{N}$ invariant for T and with $\mathscr{N} \subseteq \mathscr{M}$. A subspace is reducing if it is invariant for both T and T^*, the adjoint of T. We use $P_{\mathscr{K}}$ for the orthogonal projection onto a subspace $\mathscr{K}$, and write $T|\mathscr{K}$ for the restriction of T to an invariant subspace $\mathscr{K}$; the operator $P_{\mathscr{K}}T|\mathscr{K}$ where $\mathscr{K}$ is a semi-invariant subspace for T, $\mathscr{K} = \mathscr{M} \ominus \mathscr{N}$, is the compression of T to $\mathscr{K}$. We say T dilates an operator B if B is unitarily equivalent to the compression of T to some semi--invariant subspace. Denote by $\sigma(T)$ the spectrum of T and by $\sigma_e(T)$ the essential (that is, Calkin) spectrum of T.

Our work takes place in the context of dual algebras, and we make some definitions and sketch some results; for amplification and an extensive bibliography see [3]. Recall that $\mathscr{L}(\mathscr{H})$ is the Banach dual of the ideal $\tau c(\mathscr{H}) = \tau c$ of trace class operators on $\mathscr{H}$ under the bilinear functional $\langle T, L\rangle = \operatorname{tr}(TL)$ $(T \in \mathscr{L}(\mathscr{H}), L \in \tau c)$. There is an induced weak* (or ultraweak) topology on $\mathscr{L}(\mathscr{H})$, and if $\mathscr{A}_T$ denotes the ultraweakly closed algebra generated by the polynomials in T, then we know from general facts that $\mathscr{A}_T$ is the Banach dual of $Q_T = \tau c/{}^{\perp}\mathscr{A}_T$, where ${}^{\perp}\mathscr{A}_T$ is the preannihilator of $\mathscr{A}_T$ in τc.

Denote by $H^\infty(\mathbf{D})$ the algebra of bounded functions analytic on the open unit disk $\mathbf{D}$. Let $\mathbf{A}$ be the class of AC contractions T for which the Sz.-Nagy — Foiaş functional calculus Φ_T for T is an isometry of $H^\infty(\mathbf{D})$ onto $\mathscr{A}_T$; Φ_T is in this case a weak*-weak* continuous isometric algebra isomorphism (see [8, Theorem III.2.1] and [5]). Assume in this, and the next two paragraphs, that $T \in \mathbf{A}$. Denote the elements of the quotient space Q_T by $[L]$, the quotient norm by $\|[L]\|$, and the dual action of $\mathscr{A}_T$ on Q_T by $\langle f(T), [L]\rangle$ where $f \in H^\infty(\mathbf{D})$ and thus $f(T) \in \mathscr{A}_T$ (since the functional calculus is onto, the $f(T)$ yield all of $\mathscr{A}_T$). Recall that if $x, y \in \mathscr{H}\setminus\{0\}$, the usual rank one operator $x \otimes y \in \mathscr{L}(\mathscr{H})$ is given by $(x \otimes y)(w) = \langle w, y\rangle x$ for $w \in \mathscr{H}$. We define the (very productive) classes $\mathbf{A}_n$ $(1 \leqslant n \leqslant \aleph_0)$ by $T \in \mathbf{A}_n$ if ($T \in \mathbf{A}$ and) we may "solve equations" of the form

$$[L_{ij}] = [x_i \otimes y_j] \quad (0 \leqslant i < n,\ 0 \leqslant j < n)$$

with

$$\{[L_{ij}]\}_{0 \leqslant i,j<n} \subseteq Q_T \text{ arbitrary}$$

and $\{x_i\}$, $\{y_j\} \subseteq \mathscr{H}$. We remark that $\langle T^n, [x \otimes y]\rangle = \langle T^n x, y\rangle$ $(n \geqslant 0)$ where $\langle\ ,\ \rangle$ is the inner product in $\mathscr{H}$. Since the Sz.-Nagy — Foiaş functional calculus Φ_T is an isometry, it is the adjoint of an isometry φ_T from Q_T onto $L^1(\mathbf{T})/H_0^1(\mathbf{T})$ where this quotient is the Banach pre-dual of $H^\infty(\mathbf{T})$ (identified as usual with $H^\infty(\mathbf{D})$).

Finally, a set $S \subseteq \mathbf{D}$ is said to be dominating for $\mathbf{T}$ (or just dominating) [4] if

$$\|f\| \equiv \sup_{\lambda \in \mathbf{D}} |f(\lambda)| = \sup_{\lambda \in S} |f(\lambda)| \quad \text{for all } f \in H^\infty(\mathbf{D}).$$

For $0 \leqslant \theta < 2\pi$ and $0 < \beta < \pi/2$, denote by $S_{\theta\beta}$ the set

$$S_{\theta\beta} = \{z \in \mathbf{D} : z = e^{i\theta} + re^{i(\theta+\gamma)} \text{ for some } r > 0,\ \gamma \in (\pi - \beta,\ \pi + \beta)\}.$$

A point $e^{i\theta} \in \mathbf{T}$ is said to be a non-tangential limit point of $S \subseteq \mathbf{D}$ if there exists $0 < \beta < \pi/2$ such that $e^{i\theta}$ is a limit point of $S \cap S_{\theta\beta}$. It is known [4] that $S \subseteq \mathbf{D}$ is dominating for $\mathbf{T}$ if and only if a.e. [Lebesgue] point of $\mathbf{T}$ is a non-tangential limit point of S.

The classes $\mathbf{A}_n$ have been studied in a number of papers; for a detailed summary of the progress and that in the more general area of dual algebras, see [3] which also has an extensive bibliography. The first known and best understood of the operators in these classes are the (BCP) operators, those AC contractions T with $\sigma_e(T) \cap \mathbf{D}$ dominating $\mathbf{T}$; we have (BCP) $\subseteq \mathbf{A}_{\aleph_0}$ (see [7], and the foundational paper [5] which used a slightly more restrictive definition). A more recent discovery is that the Bergman shift is in $\mathbf{A}_{\aleph_0}$ [3, Proposition 10.1]. The aim of much of this paper is to add some further classes of operators to these and others already known to be in $\mathbf{A}_n$ for some $1 \leqslant n \leqslant \aleph_0$.

II.

We turn to direct sums to produce some operators in the $\mathbf{A}_n$; the following is a generalization of [1, Proposition 3.3].

THEOREM 1. *Let $\{T_k\}_{k=1}^{\infty}$ acting on $\{\mathscr{H}_k\}_{k=1}^{\infty}$ be any collection of operators each in $\mathbf{A}_1$. Then $T = \bigoplus_{k=1}^{\infty} T_k$ acting on $\mathscr{H} = \bigoplus_{k=1}^{\infty} \mathscr{H}_k$ is in $\mathbf{A}_{\aleph_0}$.*

Proof. Let $\{\lambda_k\}_{k=1}^{\infty}$ be dense in $\mathbf{D}$. Since each T_k is in $\mathbf{A}_1$, produce for each T_k a semi-invariant subspace $\mathscr{M}_k \ominus \mathscr{N}_k \subset \mathscr{H}_k$ of dimension 1 such that T_k has compression

$$P_{\mathscr{M}_k \ominus \mathscr{N}_k} T_k \mid \mathscr{M}_k \ominus \mathscr{N}_k = \lambda_k I_{\mathscr{M}_k \ominus \mathscr{N}_k},$$

using [1]. Let $\mathscr{K} = \bigvee_{k=1}^{\infty} (\mathscr{M}_k \ominus \mathscr{N}_k)$ and $\mathscr{M} = \bigvee_{k=1}^{\infty} \mathscr{M}_k$; it is easy to see that since the various $\mathscr{M}_k \ominus \mathscr{N}_k$ are orthogonal and located in reducing subspaces for T that $\mathscr{K}$ is semi-invariant for T. We may compute that $P_{\mathscr{K}} T \mid \mathscr{K}$ is unitarily equivalent to $\operatorname{diag}(\lambda_k)$ and hence is (BCP). From [7] we then have $T \in \mathbf{A}_{\aleph_0}$.

The proof of the following is essentially as in [1, Proposition 3.3], is similar to steps in the proof of Theorem 1, and is omitted.

THEOREM 2. *Let $\{T_k\}_{k=1}^{n^2}$ acting on $\{\mathscr{H}_k\}_{k=1}^{n^2}$ be a collection of operators each in $\mathbf{A}_1$. Then $T = \bigoplus_{k=1}^{n^2} T_k$ acting on $\mathscr{H} = \bigoplus_{k=1}^{n^2} \mathscr{H}_k$ is in $\mathbf{A}_n$.*

We provide next a partial answer to a natural question: if $T \in \mathbf{A}_{\aleph_0}$ and C is compact, is $T + C \in \mathbf{A}_{\aleph_0}$? The answer is trivially no if $T + C \notin \mathbf{A}$ (say, $\|T + C\| > > 1$). Let us say that an operator S is *appropriate for* $T \in \mathbf{A}_{\aleph_0}$ if $S + T \in \mathbf{A}$. The following is immediate.

PROPOSITION 3. *If* $T \in (\mathrm{BCP})$ *and* C *is compact and appropriate for* T *then* $(T + C) \in (\mathrm{BCP})$ *and hence* $(T + C) \in \mathbf{A}_{\aleph_0}$.

Proof. We have $\sigma_e(T + C) = \sigma_e(T)$ dominating $\mathbf{T}$, so $T + C \in (\mathrm{BCP})$ as C is appropriate for T. But $(\mathrm{BCP}) \subset \mathbf{A}_{\aleph_0}$ by [7].

We will show that if $T \in \mathbf{A}_{\aleph_0}$ and R is a finite rank operator appropriate for T and commuting with T that $T + R \in \mathbf{A}_{\aleph_0}$. We begin with an observation independent of compactness.

PROPOSITION 4. *If* $T \in \mathbf{A}_{\aleph_0}$ *and* S *is appropriate for* T *and commuting with* T, *and we may solve arbitrary* $\aleph_0 \times \aleph_0$ *systems in* Q_T *using*

1) *only vectors in* $(\mathrm{ran}(S))^{\perp}$, *or*

2) *only vectors in* $\ker(S)$

then $T + S \in \mathbf{A}_{\aleph_0}$.

Proof. We consider only case 1 as case 2 is similar. We have Q_T isometrically isomorphic to L^1/H_0^1 via φ_T, and Q_{T+S} isometrically isomorphic to L^1/H_0^1 via φ_{T+S}, since $T \in \mathbf{A}$ and $T + S \in \mathbf{A}$ respectively. Therefore, Q_{T+S} is isometrically isomorphic to Q_T via $\varphi_T^{-1} \circ \varphi_{T+S}$. Recall also that for any $A \in \mathbf{A}$ we have $[L_1] = [L_2]$ in Q_A if and only if, for all $n \geq 0$, $\langle A^n, [L_1] \rangle = \langle A^n, [L_2] \rangle$. It is easy to check that for any $[\tilde{L}] \in Q_T$ and any $n \geq 0$, $\langle T^n, [\tilde{L}]_{Q_T} \rangle = \langle (T + S)^n, \varphi_{T+S}^{-1} \circ \varphi_T([\tilde{L}]_{Q_T}) \rangle$.

Given $\{[L_{ij}]\}_{i,j=1}^{\infty} \subset Q_{T+S}$, let

$$\{[\tilde{L}_{ij}]\}_{i,j=1}^{\infty} \subset Q_T$$

be chosen by

$$[\tilde{L}_{ij}] = \varphi_T^{-1} \circ \varphi_{T+S}([L_{ij}]) \quad (i, j \geq 1).$$

Choose sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ in $(\mathrm{ran}(S))^{\perp}$ satisfying

$$[x_i \otimes y_j]_{Q_T} = [\tilde{L}_{ij}]_{Q_T},$$

that is,

$$\langle T^n, [x_i \otimes y_j] \rangle = \langle T^n, [\tilde{L}_{ij}] \rangle \quad (\text{all } n \geq 0) \text{ in } Q_T.$$

Then for any $i, j \geqslant 1$ and $n \geqslant 0$,

$$\langle (T+S)^n, [x_i \otimes y_j]_{Q_{T+S}} \rangle = \langle (T+S)^n x_i, y_j \rangle =$$

$$= \langle T^n x_j, y_j \rangle + \langle S \cdot (\text{terms in } T \text{ and } S)\, x_i, y_j \rangle =$$

$$= \langle T^n x_j, y_j \rangle = \langle T^n, [x_i \otimes y_j]_{Q_T} \rangle = \langle T^n, [\tilde{L}_{ij}]_{Q_T} \rangle =$$

$$= \langle (T+S)^n, \varphi_{T+S}^{-1} \circ \varphi_T([\tilde{L}_{ij}])_{Q_T} \rangle = \langle (T+S)^n, [L_{ij}]_{Q_{T+S}} \rangle.$$

Thus $[x_i \otimes y_j] = [L_{ij}]$ in Q_{T+S}, and we can solve the $\aleph_0 \times \aleph_0$ system in Q_{T+S}

It is known that if $T \in A_{\aleph_0}$, then T has a (BCP) compression B to a semi-invariant subspace $\mathcal{M} \ominus \mathcal{N}$. We may then "solve equations" for T using only vectors in $\mathcal{M} \ominus \mathcal{N}$, since for $x, y \in \mathcal{M} \ominus \mathcal{N}$ we have $\langle T^n x, y \rangle = \langle B^n x, y \rangle$ for all $n \geqslant 0$. We omit the details, as they are similar to steps in the proof of Proposition 4. We have the following.

LEMMA 5. Let $T \in A_{\aleph_0}$, *and let* R *be a finite rank operator. Then we may solve* $\aleph_0 \times \aleph_0$ *systems of equations in* Q_T *using only vectors in* $\operatorname{ran}(R)^{\perp}$.

Proof. By the remark before this Lemma and $\operatorname{ran}(R)^{\perp}$ replaced by $P_{\mathcal{M} \ominus \mathcal{N}}(\operatorname{ran}(R)^{\perp})$ we may reduce to the case $T \in$ (BCP). It is known [7], but easy to see, that given **a** finite dimensional space F one may solve $\aleph_0 \times \aleph_0$ systems of equations for $T \in$ (BCP) using only vectors orthogonal to F. Since $\dim(\operatorname{ran}(R))$ is finite, we are done.

We now have the promised first step in the matter of compact perturbations immediately from Proposition 4 and Lemma 5.

COROLLARY 6. *Let* $T \in A_{\aleph_0}$ *and let* R *be a finite rank operator appropriate for* T *and commuting with* T. *Then* $T + R \in A_{\aleph_0}$.

III.

We consider a more general "equation solving" property of elements in $A_{\aleph_0}$ (roughly, in L^1 as opposed to L^1/H_0^1).

Fix $T \in A$ for the moment, and recall that

$$[L_1] = [L_2] \quad \text{in } Q_T$$

if and only if

$$\langle T^n, [L_1] \rangle = \langle T^n, [L_2] \rangle \quad (n \geqslant 0).$$

For any $l \in L^1$ let $\{C_n(l)\}_{n=-\infty}^{\infty}$ be the sequence of Fourier coefficients of l. If $\varphi_T([L]) = [l]$, l is not determined uniquely; one has, however, that if $[l] = [l^1]$ then $l - l^1 \in H_0^1$ and $C_{-n}(l) = C_{-n}(l^1)$ for all $n \geq 0$. One may deduce that

$$[x \otimes y] = \varphi_T^{-1}([l])$$

if and only if

$$\langle T^n x, y \rangle = C_{-n}(l) \quad (\text{all } n \geq 0).$$

Thus $T \in \mathbf{A}_{\aleph_0}$ is equivalent to $T \in \mathbf{A}$ and for any set $\{[l_{ij}]\}_{i,j=1}^{\infty} \subset L^1/H_0^1$ there exist sets $\{x_i\}_{i=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ contained in $\mathscr{H}$ such that

$$C_{-n}(l_{ij}) = \langle T^n x_i, y_j \rangle \quad (1 \leqslant i,\ 1 \leqslant j,\ 0 \leqslant n).$$

It is known that if $T \in$ (BCP) one may do considerably better. In [2] it is shown that for $T \in$ (BCP) and any $\{l_{ij}\}_1^{\infty} \subset L^1$ one may find sets $\{x_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ in $\mathscr{H}$ such that for all i and j,

$$C_{-n}(l_{ij}) = \langle T^n x_i, y_j \rangle \quad (n \geqslant 0)$$

and

$$C_n(l_{ij}) = \langle T^{*n} x_i, y_j \rangle \quad (n \geqslant 0)$$

that is, we "match" *all* the Fourier coefficients of the $\{l_{ij}\}$. This result is a corollary of deeper results, but it is fair to say the relevant proofs are hard; further, they rely on the Sz.-Nagy -- Foiaş functional model of a contraction, and this theory is itself hard.

We give here a direct proof of a more restricted result (which contains some of the more interesting cases) which is good for $T \in \mathbf{A}_{\aleph_0}$, which avoids the functional model, and which relies only on the ability of any element of $\mathbf{A}_{\aleph_0}$ to dilate a direct sum of strict contractions [1]. We begin with the case of a single equation.

THEOREM 1. *Let $T \in \mathbf{A}_{\aleph_0}$, and suppose $l \in L^1(\mathbf{T})$ satisfies*

$$\sum_{-\infty}^{\infty} |C_n(l)| < \infty.$$

Then there exist vectors x and y in $\mathscr{H}$ satisfying

$$C_{-n}(l) = \langle T^n x, y \rangle \quad (n \geqslant 0)$$

and

$$C_n(l) = \langle T^{*n} x, y \rangle \quad (n \geqslant 0).$$

Proof. Write C_n for $C_n(l)$ and let $a_n = \sqrt{C_{-n}}$ for $n \in \mathbf{Z}$; then $\{a_n\}_{n=-\infty}^{\infty}$ is in (2-sided) ℓ^2.

We define operators B_n acting on $\mathscr{H}_n$ where

$$\dim(\mathscr{H}_n) = |n| + 1.$$

Let $\mathscr{H}_n$ have the ordered orthonormal basis $J_n = \{e_k^n\}_{k=0}^{|n|}$.

If $n > 0$, let B_n have the matrix with $(\sqrt{2})^{-1/n}$ on the superdiagonal and zero elsewhere (with respect to the natural ordering of J_n). If $n < 0$, let B_n have the matrix $(\sqrt{2})^{-1/|n|}$ on the subdiagonal and zero elsewhere (again with respect to the natural ordering of J_n). Note each B_n is a strict contraction. For each $n \neq 0$, let

$$x_n = a_n\sqrt{2}e_{|n|}^n$$

and

$$y_n = \bar{a}_n e_0^n.$$

One may compute that for $n \geqslant 0$

$$\langle B_n^m x_n, y_n\rangle = \begin{cases} 0 & (m \geqslant 0,\ m \neq n) \\ a_n^2 = C_{-n} & (m = n) \end{cases}$$

and

$$\langle B_n^{*-m} x_n, y_n\rangle = 0 \quad (m > 0).$$

Similarly, for $n < 0$,

$$\langle B_n^m x_n, y_n\rangle = 0 \quad (m > 0)$$

and

$$\langle B_n^{*-m} x_n, y_n\rangle = \begin{cases} 0 & (m \leqslant 0,\ m \neq n) \\ a_n^2 = C_{-n} & (m < 0,\ m = n). \end{cases}$$

Finally, let B_0 be the zero operator on a one-dimensional space $\mathscr{H}_0$ and let $x_0 = a_0 e_0^0$, $y_0 = \bar{a}_0 e_0^0$ where e_0^0 is a unit vector in $\mathscr{H}_0$; then for $n > 0$, $\langle B_0^n x_0, y_0\rangle = \langle B_0^{*n} x_0, y_0\rangle = 0$, while $\langle B_0^0 x_0, y_0\rangle = a_n^2 = C_0$.

Let $B = \bigoplus_{-\infty}^{\infty} B_n$ acting on $\bigoplus_{-\infty}^{\infty} \mathscr{H}_n$, and let $X = \bigoplus_{-\infty}^{\infty} x_n$ and $Y = \bigoplus_{-\infty}^{\infty} y_n$. One may easily finish the details to show

$$\langle B^n X, Y\rangle = C_{-n} \quad (n \geqslant 0)$$

and

$$\langle B^{*n} X, Y\rangle = C_n \quad (n > 0).$$

Since B is a direct sum of strict contractions, T has a compression $\tilde{T}$ to a semi--invariant subspace $\mathscr{M} \ominus \mathscr{N}$ which is unitarily equivalent to B [1]; we may finish the proof using the approach indicated in the remarks before Lemma II.5 and omit the details.

The extension to the case of an $\aleph_0 \times \aleph_0$ system of equations uses an idea from [7].

THEOREM 2. *Let* $T \in \mathbf{A}_{\aleph_0}$ *and let* $\{l_{ij}\}_{i,j=1}^{\infty}$ *be any collection of functions in* L^1 *each with absolutely summable Fourier coefficients. Then there exist sets of vectors* $\{x_i\}_{i=1}^{\infty}$ *and* $\{y_j\}_{j=1}^{\infty}$ *in* $\mathscr{H}$ *satisfying*

$$C_{-n}(l_{ij}) = \langle T^n x_i, y_j \rangle \quad (1 \leqslant i,\ 1 \leqslant j \text{ and } n \geqslant 0)$$

and

$$C_n(l_{ij}) = \langle T^{*n} x_i, y_j \rangle \quad (1 \leqslant i,\ 1 \leqslant j, \text{ and } n > 0).$$

Proof. Use temporarily the notation $\|l\|_1$ for the ℓ^1 norm of the Fourier coefficients of such an $l \in L^1$. Consider first the case in which

(1) $$\sum_{j=1}^{\infty} \|l_{i_0 j}\|_1^{1/2} < \infty \quad (\text{each } i_0 \geqslant 1),$$

and

(2) $$\sum_{i=1}^{\infty} \|l_{i j_0}\|_1^{1/2} < \infty \quad (\text{each } j_0 \geqslant 1).$$

For each pair (i, j) with $i, j \geqslant 1$ construct as in Theorem 1 an operator B_{ij} acting on a space $\mathscr{H}_{ij}$ and vectors X_i, Y_j in $\mathscr{H}_{ij}$ such that

$$\langle B_{ij}^n X_i, Y_j \rangle = C_{-n}(l_{ij}) \quad (n \geqslant 0)$$

and

$$\langle B_{ij}^{*n} X_i, Y_j \rangle = C_n(l_{ij}) \quad (n > 0).$$

Observe that $B = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} B_{ij}$ acting on $\bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} \mathscr{H}_{ij}$ is a direct sum of strict contractions; we may as well assume B is the compression of T to a semi-invariant subspace $\mathscr{M} \ominus \mathscr{N}$ (by the remarks before Lemma II.5).

We now define some vectors in $\mathscr{M} \ominus \mathscr{N} = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} \mathscr{H}_{ij}$. Denote the components of such a v by $\{v(i,j)\}_{i,j=1}^{\infty}$, where $v(i,j) \in \mathscr{H}_{ij}$. For $i_0 \geqslant 1$, let x_{i_0} be the vector given by

$$x_{i_0}(i,j) = \begin{cases} 0 & (i \neq i_0) \\ X_{i_0,j} & (i = i_0). \end{cases}$$

For $j_0 \geqslant 1$, let y_{j_0} be given by

$$y_{j_0}(i,j) = \begin{cases} 0 & (j \neq j_0) \\ Y_{i,j_0} & (j = j_0). \end{cases}$$

One may check easily, if tediously, that these vectors are well defined and satisfy the conclusions of the Theorem.

In order to deal with the general case, in which (1) and (2) are not satisfied, we use an idea from [7] which amounts to scaling the $\{l_{ij}\}_{i,j=1}^{\infty}$ to produce new $\{\tilde{l}_{ij}\}_{i,j=1}^{\infty}$ which do satisfy (1) and (2). The vectors satisfying the conclusions of the theorem for the $\{\tilde{l}_{ij}\}_{i,j=1}^{\infty}$ may be scaled in turn to obtain vectors appropriate for the $\{l_{ij}\}_{i,j=1}^{\infty}$; the interested reader is referred to [7] for the details.

IV. REMARKS

We may improve (in the spirit of [6]) the results of Proposition II.4 concerning appropriate and commuting perturbations of a $T \in A_{\aleph_0}$. If we assume not that T and S commute, but that for S_1 commuting with T we have

$$(*) \qquad TS - ST = S_1$$

where we may solve systems of equations in Q_T using only vectors in $(\mathrm{Ran}(S) \cup \cup \mathrm{Ran}(S_1))^{\perp}$, we may modify the proof to obtain the same conclusion. The essential step is to use (∗) to ensure that every term in the expansion of $(T+S)^n$ except T^n has either S or S_1 as its first element. We have no further difficulty with the following:

PROPOSITION 1. *Suppose* $T \in A_{\aleph_0}$ *and* $\{S_i\}_{i=1}^{n}$ *is a collection of operators satisfying*

(1) S_1 *is appropriate for* T,
(2) $TS_i - S_iT = S_{i+1}$ $(1 \leqslant i \leqslant n-1)$,
(3) $TS_n = S_nT$,
(4) *we may solve* $\aleph_0 \times \aleph_0$ *systems in* Q_T *using only vectors in* $(\bigvee_{i=1}^{n} \mathrm{ran}(S_i))^{\perp}$.

Then $T + S_1 \in A_{\aleph_0}$. *In particular, if* (1)—(3) *hold and each* S_i *is of finite rank,* $T + S_1 \in A_{\aleph_0}$ *since* $\bigvee_{i=1}^{n} \mathrm{ran}(S_i)$ *is of finite dimension.*

We observe that the more general equation solving in Section III has some interesting consequences. For example, it is possible to find non-zero x and y in $\mathscr{H}$ with $\langle T^n x, y\rangle = 0$ $(n \geqslant 0)$ and $\langle T^{*n} x, y\rangle = 0$ $(n \geqslant 0)$. But then the subspaces $\bigvee_{n=0}^{\infty} \{T^n x\}$ and $\bigvee_{n=0}^{\infty} \{T^{*n} x\}$ are non-trivial invariant subspaces for T and T^* respectively which intersect non-trivially. For further results see [2].

Finally, in studying systems of equations in the predual of $\mathscr{A}_T$ (which may be formed for any T) we assume T is a contraction and T is absolutely continuous. The first assumption is without unpleasant loss of generality, but the second is not. Take $T \in \mathbf{A}_{\aleph_0}$, T c.n.u., and let $S = T \oplus I$ where I acts on a one-dimensional space. One element of Q_S is $[D]$ where

$$\langle S^n, [D] \rangle = 1 \quad \text{for all } n \geqslant 0.$$

We may clearly solve for $[D]$ with a unit vector from the space of I, but we cannot solve the 2×2 system $[L_{11}] = [L_{22}] = [D]$ and $[L_{12}] = [L_{21}] = 0$; if we could, the two-dimensional, semi-invariant subspace for S given by $\bigvee_{i=1}^{2} \{x_i\}$ would yield a compression $\tilde{S}$ of S which is the identity and hence a restriction and a unitary operator on a two-dimensional space. One might study the problem without the AC restriction, although we increase the difficulty enormously as we sacrifice the Sz.-Nagy — Foiaş functional calculus.

REFERENCES

1. Bercovici, H.; Foiaş C.; Pearcy, C., Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I, *Michigan Math. J.*, **30**(1983), 335–354.
2. Bercovici, H.; Foiaş, C.; Pearcy, C., Factoring trace-class operator valued functions with applications to the class $\mathbf{A}_{\aleph_0}$, *J. Operator Theory*, **14**(1985), 351–389.
3. Bercovici, H.; Foiaş, C.; Pearcy, C., *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS—AMS Lecture Notes, **56**, A.M.S., 1985.
4. Brown, L.; Shields, A.; Zeller, K., On absolutely convergent exponential sums, *Trans. Amer. Math. Soc.*, **96**(1960), 162–183.
5. Brown, S.; Chevreau, B.; Pearcy, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
6. Kim, H.; Pearcy, C.; Shields, A., Rank-one commutators and hyperinvariant subspaces, *Michigan Math. J.*, **22**(1975), 193–194.
7. Robel, G., On the structure of (BCP) - operators and related algebras. I; II, *J. Operator Theory*, **12**(1984), 23–45; 235–245.
8. Sz.-Nagy, B.; Foiaş, C., *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

GEORGE EXNER
Department of Mathematics,
Oberlin College,
Oberlin, Ohio 44074,
U.S.A.

Received April 1, 1984; revised March 31, 1986.