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SURGERY ON SPECTRAL SETS

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Let $B(\mathscr{H})$ denote the algebra of bounded linear operators on the Hilbert space $\mathscr{H}$. Let $R(M)$ denote the uniform closure of the rational functions with poles off M.

DEFINITION. A compact set $M \subset \mathbf{C}$ is a *K-spectral set* for $T \in B(\mathscr{H})$ if

$$\|f(T)\| \leqslant K\|f\|_{\infty}^{M}$$

for $f \in R(M)$ when K is a constant and

$$\|f\|_{\infty}^{M} = \sup\{|f(z)| : z \in M\}.$$

When $K = 1$, the set M is called a *spectral set* for T.

The notion of spectral set was introduced by John von Neumann [9] where he proved that the unit disc was a spectral set for any contraction. He also observed that if $M_j \supset M_{j+1} \ldots$ were spectral sets for T then $\cap M_j$ was also a spectral set for T but that in general the intersection of two spectral acts is *not* a spectral set. (See [14] for simple examples of this phenomenon.)

The areas of spectral sets, dilation theory and similarity of homomorphisms of C^*-algebras are intimately related. For a very clear, concise discussion of this relationship the reader is urged to consult [3]. For a wide range of recent results involving spectral sets and K-spectral sets we recommend [1], [2], [3], [6], [7], [11] and [12].

We will need two facts in the proofs below and record them here for convenience.

THEOERM vN. [9]. *Let M be a spectral set for $T \in \mathscr{B}(\mathscr{H})$. Let $f \in R(M)$. Then $f(M)$ is a spectral set for $f(T)$.*

THEOERM N-F. [8]. *Let $T \in \mathscr{B}(\mathscr{H})$ with $\sigma(T) \subset \overline{\mathbf{D}}$. Assume $\|(T - \lambda)^{-1}\| \leqslant \leqslant (|\lambda| - 1)^{-1}$ for $1 < |\lambda| < 1 + \delta$. Then T is similar to a contraction.*

Throughout this paper we will use $\mathbf{D}$ to designate the closed unit disc $\{z : |z| \leqslant 1\}$, and $\Delta(a, r)$ denotes the open disc with center a and radius r.

The next lemma is the basis for the entire paper. It, and variants of it, will be used again and again. Once the reader has seen the proof in the simplest case, he should have no trouble modifying it to cover other situations. Roughly speaking, the lemma says that if we have a spectral set we can excise a small disc which does not touch the spectrum of the operator and we still have a K-spectral set.

If the operator T is similar to an operator S for which M is a spectral set, it is obvious that M is a K-spectral set for T. The converse remains one of the most difficult and important open questions in the area. For this reason and because it is often crucial in later arguments we explicitly point out those situations in which T is similar to an operator for which M is a spectral set.

LEMMA 1. *Let* $\mathbf{D}$ *be a spectral set for* $T \in \mathscr{B}(\mathscr{H})$. *Choose* $a \in \mathbf{C} \setminus \mathbf{D}$ *and* $0 < r < 1/10$ *so that*

$$\frac{r}{2} < |a| - 1 < r.$$

Assume $\operatorname{dist}[\sigma(T), \Delta(a, r)] > 0$. *Then* $M = \mathbf{D} \setminus \Delta(a, r)$ *is a* K-*spectral set for* T *and indeed* T *is similar to an operator for which* M *is a spectral set.*

Proof. Let φ be the Riemann map of M to $\mathbf{D}$. By the way in which r and a were chosen we can extend φ to $\mathbf{D}$ by the Schwarz reflection principle. Thus $\varphi(T)$ is well defined.

Consider $\|(\varphi(T) - \lambda)^{-1}\|$ for

$$\lambda \in C = \{z : \arg\beta + \varepsilon < \arg z < \arg\alpha - \varepsilon,\ 1 < |z| < \gamma\}$$

where $\{z_1, z_2\} = \partial\mathbf{D} \cap \Delta(a, r)$, $\varphi(z_1) = \alpha$, $\varphi(z_2) = \beta$ and $\varepsilon > 0$ is chosen so that $\partial\varphi(\sigma(T)) \cap \partial\mathbf{D} \subset \partial C$. See Figure A. For $\lambda \in C$ we see that

$$\|(\varphi(T) - \lambda)^{-1}\| = \|(z - \lambda)^{-1}\|_\infty^{\varphi(\mathbf{D})} = \operatorname{dist}[\lambda, \varphi(\mathbf{D})]^{-1} = (|\lambda| - 1)^{-1}.$$

Now consider $\lambda \in C' = \{1 < |z| < \gamma\} \setminus C$. The function $(\varphi(T) - \lambda)^{-1}$ is analytic on C' and hence uniformly bounded there. On the other hand $(|\lambda| - 1)^{-1} \to \infty$ as $|\lambda| \to 1$. Hence by a new choice of γ say γ' we may assume that

$$\|(\varphi(T) - \lambda)^{-1}\| \leqslant (|\lambda| - 1)^{-1}$$

for all $1 < |\lambda| < \gamma'$. Thus the hypotheses of Theorem N-F are satisfied and hence

$$\varphi(T) = QSQ^{-1} \quad \text{where } S \text{ is a contraction.}$$

Thus $T = \varphi^{-1}[\varphi(T)] = Q\varphi^{-1}(S)Q^{-1}$. Since $\mathbf{D}$ is a spectral set for S, $\varphi^{-1}(\mathbf{D}) = M$ is a spectral set for $\varphi^{-1}(S)$. Thus T is similar to an operator $\varphi^{-1}(S)$ for which M is a spectral set.

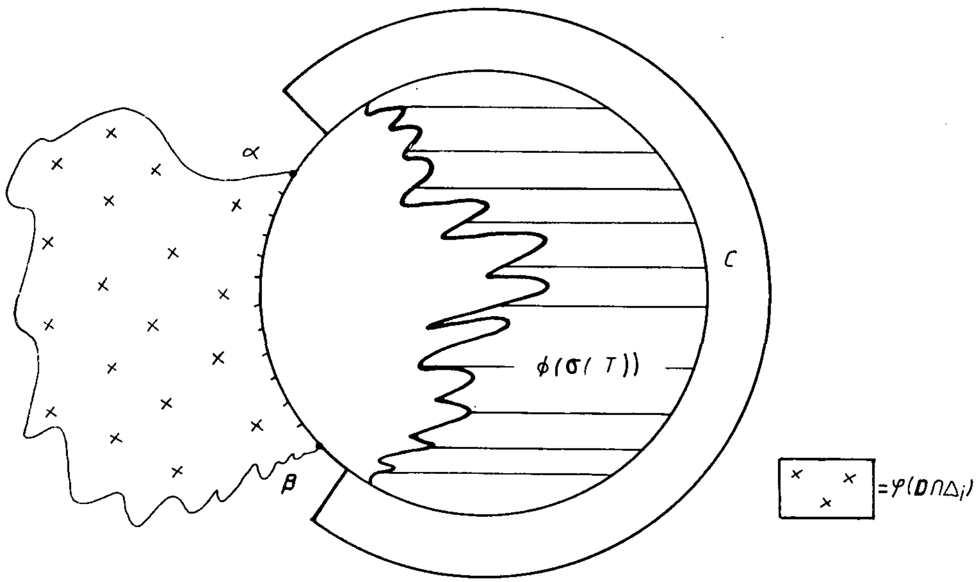

Figure A

REMARK. When we use Lemma 1 in the next Theorem and elsewhere in this paper we will do so in a different geometric environment. However we will always be reflecting our map φ across a line or a circular arc and we will carefully tailor the circumstance to ensure that such a reflection is easily achieved.

Roughly speaking, the next theorem says that given a spectral set M for T we may cut away any part of M that does not touch $\sigma(T)$ and we still have a K-spectral set.

THEOREM 1. *Let M be a spectral set for $T \in \mathcal{B}(\mathcal{H})$ where M is connected, simply connected and every boundary point of M is accessible. Let G be a simply connected open set which contains $\sigma(T)$. Then $\bar{G} \cap M$ is a K-spectral set for T and in fact T is similar to an operator for which $\bar{G} \cap M$ is a spectral set ($^-$ denotes closure).*

Proof. First map M to the unit disc $\mathbf{D}$ conformally. If ψ is this map then $\psi \in R(M)$ since ψ is analytic on int M and continuous on the boundary and $A(M) = R(M)$. We may thus assume $M = \mathbf{D}$ to begin with. Since G was arbitrary let us still call it G. We now apply Lemma 1 repeatedly following the procedure described in the Appendix. After one application of Lemma 1 we have T similar to T_1 where $\mathbf{D}\setminus\Delta_0$ is a spectral set for T_1 (Δ_i denotes the disc $\Delta(q_i, r_i)$). We may forget about T and concentrate on T_1. By applying Lemma 1 to $(\mathbf{D}\setminus\Delta_0)$ we see that T_1 is similar to T_2 where $(\mathbf{D}\setminus\Delta_0)\setminus\Delta_1$ is a spectral set for T_2. Following the directions in the appendix we can reduce $\mathbf{D}$ to $\bar{G} \cap M$ in a finite number of steps. There may be some question in the reader's mind as to whether we can always cut away more of our spectral set without becoming trapped in an impossible geometric situation or whether we can continue cutting away forever without converging to the desired goal. The procedure detailed in the appendix circumvents such difficulties and guarantees a solution in a finite number of steps.

As mentioned earlier, spectral sets can be used to obtain normal dilations.

COROLLARY 1. *Let T, M and G have the properties in Theorem 1. Then there exists:*

1) *an invertible operator $Q \in \mathscr{B}(\mathscr{H})$,*
2) *a Hilbert space $\mathscr{K} \supset \mathscr{H}$,*
3) *a normal operator $N \in \mathscr{B}(\mathscr{K})$ with $\sigma(N) \subset \partial(M \cap \bar{G})$ such that*

$$T^k f = QPN^kQ^{-1}f$$

for $k = 1, 2, \ldots$, and for all $f \in \mathscr{H}$, where P is the projection of $\mathscr{K}$ on $\mathscr{H}$.

Proof. The corollary follows immediately from the dilation theorem of Foiaş [4] or Lebow [6].

COROLLARY 2. *Let T be a quasinilpotent operator in $\mathscr{B}(\mathscr{H})$ where $\operatorname{Re} T \geqslant 0$. Then for any $R > 0$, the semi-disc $\{\operatorname{Re} z \geqslant 0\} \cap \{|z| \leqslant R\}$ is a K-spectral set for T.*

COROLLARY 3. *Let $V \in \mathscr{B}(L^2[0, 1])$ be the Volterra operator.* $\left((Vf)(x) = \int_0^x f(t)\, dt\right)$.

Then for any $R > 0$ the set $\{\operatorname{Re} z \geqslant 0\} \cap \{|z| \leqslant R\}$ is a K-spectral set for V.

In [5, p. 281], Kato proves that an accretive operator has a square root. We present a proof with a different approach.

COROLLARY 4. *Let $T \in \mathscr{B}(\mathscr{H})$ with $W(T) \subset \{\operatorname{Re} z \geqslant 0\}$. Then T has a square root. Indeed T^α exists for $\alpha > 0$.*

Proof. Since $W(T) \geqslant 0$, it follows that $\operatorname{Re} z \geqslant 0$ is a spectral set for T. Since $\sigma(T) \subset \{z : |z| \leqslant \|T\|\}$ it follows from Theorem 1, that the semi-disc $M = \{\operatorname{Re} z \geqslant 0\} \cap \{z : |z| \leqslant 2\|T\|\}$ is a K-spectral set for T. Since $f(z) = z^{1/2} \in R(M)$, the closure of the rational functions on M, it follows from the functional calculus that $A = f(T) = T^{1/2}$ is well defined and $A^2 = T$. The argument for the existence of T^α is similar.

THEOREM 2. *Let $T \in \mathscr{B}(\mathscr{H})$. Let $D_j = \{z : |z - a_j| \leqslant R_j\}$ for $j = 1, 2, \ldots$. Assume*

1. *D_j is a spectral set for T for $j = 1, 2, \ldots$;*
2. $\partial D_1 \cap \partial D_2 \cap \sigma(T) = \emptyset$;
3. $|a_1 - a_2|^2 < R_1^2 + R_2^2$.

Then $D_1 \cap D_2$ is a K-spectral set for T and T is similar to S where $D_1 \cap D_2$ is a spectral set for S.

Proof. Let $M = D_1 \cap D_2$. Let φ be the Riemann map from M to $\mathbf{D}$. Let $N = D_2 \setminus D_1$ and observe that Condition 3 ensures that the reflection Q of N across ∂D_1 does not cover all of M. Choose a point $\lambda_0 \in M \setminus Q$ and modify the map φ so that $\varphi(\lambda_0) = 0$. We may then extend φ to D_2 as a well defined analytic function by the Schwarz reflection principle. Since $\varphi \in R(\bar{D}_2)$, $\varphi(T)$ is well defined. Let $\varphi(z_i) = w_i$ for $i = 1, 2$ where $\{z_1, z_2\} = \partial D_1 \cap \partial D_2$. Consider $f \circ \varphi(T)$ where $f(z) = \dfrac{1}{z - \lambda}$ for $|\lambda| > 1$. Then $f \circ \varphi(T) = (\varphi(T) - \lambda)^{-1}$. The function $f \circ \varphi \in R(D_2)$ as observed above and $f \circ \varphi \in R(D_1)$. This last assertion needs a word of explanation. We can extend φ to D_1 as above but φ may have a simple pole in D_1 if we do so. However a moments thought reveals that the composition $f \circ \varphi$ on D_1 is well behaved. Let $\lambda \in C_i$, $i = 1, 2$, where the C's are the collars illustrated in Figure B. The collars may be chosen as close to w_1, w_2 as desired while not including them but at the same time containing $\sigma(\varphi(T)) \cap \partial\mathbf{D}$. By Theorem vN

$$\|(\varphi(T) - \lambda)^{-1}\| = \|f \circ \varphi\|_\infty^{D_2} =$$

$$= \{\text{dist}[\lambda, f \circ \varphi(D_2)]\}^{-1} = 1/(|\lambda| - 1) \quad \text{for } \lambda \in C_1. \qquad \text{(See Figure B.)}$$

Similarly for $\lambda \in C_2$

$$\|(\varphi(T) - \lambda)^{-1}\| = \|f \circ \varphi\|_\infty^{D_1} =$$

$$= \{\text{dist}[\lambda, f \circ \varphi(D_1)]\}^{-1} = 1/(|\lambda| - 1).$$

We would like to apply Theorem N-F but we have not established the resolvent estimate for λ near $\partial\mathbf{D} \setminus (C_1 \cup C_2)$. However for λ near and on $\partial\mathbf{D} \setminus (C_1 \cup C_2)$, $(\varphi(T) - \lambda)$ is invertible. Thus $\|(\varphi(T) - \lambda)^{-1}\|$ is uniformly bounded. However the expression $(|\lambda| - 1)^{-1}$ goes to ∞ as $|\lambda| \to 1$. Thus

$$\|(\varphi(T) - \lambda)^{-1}\| \leqslant (|\lambda| - 1)^{-1}$$

for λ near $\partial\mathbf{D} \setminus (C_1 \cup C_2)$. Thus it follows from Theorem N-F that $\varphi(T) = G^{-1}SG$ where S is a contraction. Hence

$$T = \varphi^{-1}[\varphi(T)] = G^{-1}\varphi^{-1}(S)G.$$

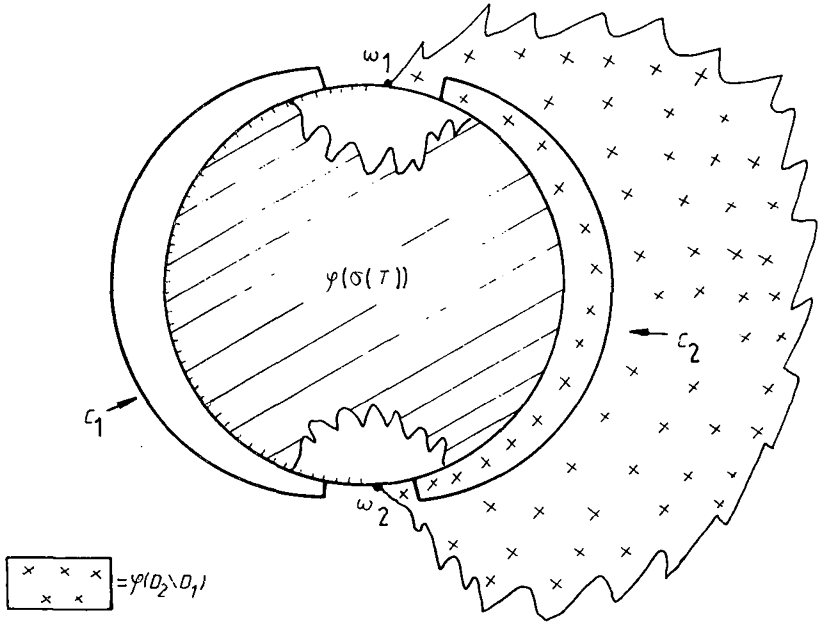

Figure B

Since $N = \varphi^{-1}(\mathbf{D})$ is a spectral set for $\varphi^{-1}(S)$ we have proved that T is similar to an operator for which N is a spectral set and hence N is K-spectral for T.

The technique employed in the last theorem is effective in other situations as well. We mention an example to illustrate this point.

Corollary 1. *Let $T \in \mathscr{B}(\mathscr{H})$. Let* $\mathbf{D}$ *be a spectral set for T. Let $\{\operatorname{Re} z \geqslant \alpha\}$ be a spectral set for T where $\alpha < 0$. Assume finally that the points $\alpha \pm i\sqrt{1-\alpha^2} \notin \sigma(T)$. Then $\{\operatorname{Re} z \geqslant \alpha\} \cap \mathbf{D}$ is a K-spectral set for T.*

Proof. Argue directly as in Theorem 1 or transform this case to Theorem 1 by a linear fractional transformation.

Question. Let M be a K-spectral set for $T \in \mathscr{B}(\mathscr{H})$. Is T similar to an operator for which M is a spectral set?

This question has not even been settled for the case when M is the unit disc. (But see [11] for some very interesting results on this question as well as the discussion in [3].)

APPENDIX — BUILDING TUNNELS

In order to repeatedly apply Lemma 1 we must devise a systematic method for cutting off chunks from $M \setminus \sigma(T)$. We describe the method for doing this here.

STEP 1. Assume $M = \mathbf{D}$ the closed unit disc. Let $\gamma = \mathrm{dist}[G', \sigma(T)]$. Set $\delta = 10^{-1}\gamma$. Place a grid on $\mathbf{D}$ and near-by points consisting of squares 2δ on a side.

STEP 2. Straight tunnels.

We now apply Lemma 1 using the disc $\Delta(z_0, \delta)$. This leaves us with $G \diagdown \Delta(z_0, \delta)$ as a spectral set for $T_1 \sim T$ ($\sim$ indicates similarity).

We now use the disc $\Delta(z_1, \delta)$ where $z_1 = z_0 + \delta/2$. Under reflection no point is further than 2δ from z_1 so we are well within $\sigma(T)'$. See Figure 1. Thus

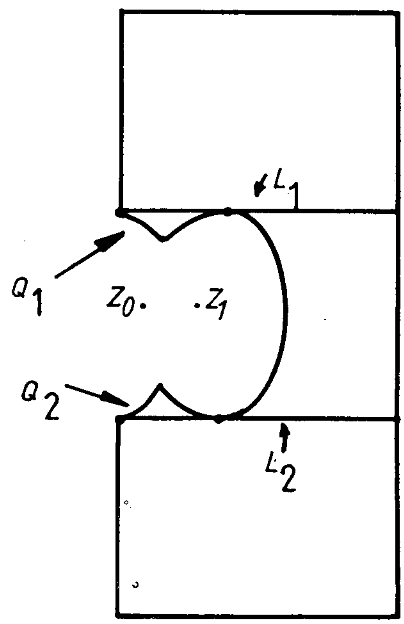

Figure 1

$M_1 = G \diagdown [\Delta(z_0, \delta) \cup \Delta(z_1, \delta)]$ is a spectral set for $T_2 \sim T_1 \sim T$. Notice that M_1 has two cusp shaped protrusions Q_1, Q_2. We may remove Q_1 by applying a slight modification of Lemma 1 where we reflect across the *line* L_1 instead of across a circular arc. The Schwarz reflection principle still applies in this case as does the rest of the proof of Lemma 1. Having done this we are left with a "clean" tunnel. We may continue this process to obtain a straight line tunnel of any length (staying within $\sigma(T)'$ and at least two grid squares away from $\sigma(T)$). When completing a straightline tunnel it is a good idea to over shoot by half a grid square. This insures that no points are left in the penultimate square in the tunnel.

STEP 3. Turns.

One may at any stage build a tunnel at angle 90° to the original tunnel.

STEP 4. Widening Tunnels.

Suppose we have already built one tunnel as indicated and wish to remove the adjacent row. We can not proceed as in Step 1 because reflection would land us in a region where the Riemann map was not defined. However it is easy to overcome this difficulty: First build a tunnel in Row 2 with half the width of our original grid as indicated in Figure 2. We may now remove the rest of the rectangles in

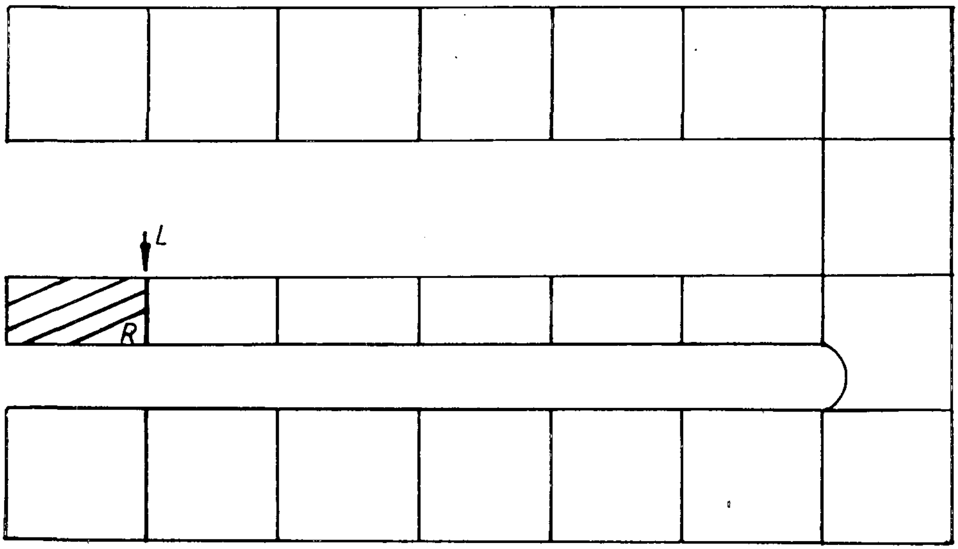

Figure 2

Row 2 by reflecting across straight lines. Thus by defining the Riemann map on the set $P \setminus R$ and then reflecting across the line L we can delete R. Proceeding in this way we can remove the rest of Row 2.

CONCLUSION. Since the process described above enables us to remove any grid square which is two squares away from $\sigma(T)$; it is clear we can complete the program proposed in Lemma 1.

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