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M-IDEALS AND IDEALS IN $L(X)$

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1. INTRODUCTION

Since Alfsen and Effros introduced the notion of an M-ideal [1], many authors have studied M-ideal structures in Banach algebras and especially in operator algebras with a view toward classifying the M-ideals and characterizing those Banach spaces X for which $K(X)$, the space of compact operators on X, is an M-ideal in $L(X)$, the space of continuous operators in X. In [20], Hennefeld checked that $K(X)$ is an M-ideal in $L(X)$ when $X = \ell_p$, $1 < p < \infty$. Smith and Ward [35] proved that M-ideals in a complex Banach algebra with identity are subalgebras and that they are two sided (algebraic) ideals if the algebra is commutative. They [35] also proved that M-ideals in a C^*-algebra are exactly the two sided ideals.

Later Flinn [16], and Smith and Ward [36] showed that for $1 < p < \infty$, $K(\ell_p)$ is the only nontrivial M-ideal in $L(\ell_p)$, and since 0 and $L(\ell_p)$ are both ideals and M-ideals, the M-ideals in $L(\ell_p)$ are exactly the two sided ideals in $L(\ell_p)$.

In Section 3, we will prove that for a uniformly convex space X, every M-ideal in $L(X)$ is a left ideal. Consequently if X^* is also uniformly convex, then every M-ideal in $L(X)$ is a two sided ideal. This verifies a special case of the conjecture of Smith and Ward [36] that if X is a uniformly convex space then every M-ideal in $L(X)$ is a two sided ideal. In fact, our argument is really a minor modification of the Smith-Ward proof [36] that every M-ideal in $L_p(\mu)$ is a left ideal. The main point is that the use of Clarkson's inequality in [36] can be replaced by one of the equivalent formulations of the definition of uniform convexity.

An application of this theorem is that for $X = (\sum \ell_p^{n_i})_r$ with $1 < p, r < \infty$ and $\{n_i\}$ a bounded sequence of positive integers, $K(X)$ is the only nontrivial M-ideal in $L(X)$.

Indeed, $K(X)$ is an M-ideal in $L(X)$ by a result of Lima [28] (see [8] for a generalization). Since both X and X^* are uniformly convex [11], by the theorem M-ideals in $L(X)$ are closed two sided ideals. Since X is isomorphic to ℓ_r and $K(\ell_r)$ is the only nontrivial two sided ideal in $L(\ell_r)$ [17], $K[X]$ is the only non-trivial two sided ideal in $L(X)$ and hence the M-ideals in $L(X)$ are exactly the closed two sided ideals in $L(X)$.

In Section 4 we show that if $X = \left(\sum_{n=1}^{\infty} \ell_p^n \right)_r$, $1 < p \neq r < \infty$, $p \neq 2$, then $L(X)$ contains a closed two sided ideal which is not an M-ideal. Since $L(X^*)$ is algebraically isometrically isomorphic to $L(X)$ when X is reflexive, it suffices to consider the cases when $p > 2$. When $p > 2$, we will prove that for this space X the closure $\overline{S_r(X)}$ of $S_r(X)$, the ideal of all operators in $L(X)$ which factor through a subspace of an L_r-space, is not proximinal in $L(X)$ and hence not an M-ideal in $L(X)$ [1]. The construction of the operator in $L(X)$ which has no best approximant in $S_r(X)$ uses a localization of the Benyamini-Lin [5] construction of an operator on $L_p[0.1]$ which has no best approximant in $K(L_p)$.

2. NOTATION AND PRELIMINARIES

A closed subspace J of a Banach space X is said to be an L-summand if there exists a closed subspace J' of X so that X is an algebraic direct sum of J and J', and if $j \in J$ and $j' \in J'$ then $\|j + j'\| = \|j\| + \|j'\|$. In this case we will write $X = J \oplus_1 J'$. Such a closed subspace J of X is called an M-summand if we have the norm condition $\|j + j'\| = \max\{\|j\|, \|j'\|\}$ in place of $\|j + j'\| = \|j\| + \|j'\|$. Here we will write $X = J \oplus_\infty J'$. A closed subspace J of a Banach space X is called an M-ideal in X if $J^\perp = \{x^* \in X^* : x^*(j) = 0 \text{ for all } j \in J\}$, the annihilator of J in X^*, is an L-summand in X^*.

For any Banach space X with $\dim X \geqslant 2$, the modulus of convexity $\delta_X(\varepsilon)$, $0 < \varepsilon \leqslant 2$, of X is defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} ; x, y \in X, \ \|x\| = \|y\| = 1, \ \|x - y\| = \varepsilon \right\}.$$

A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \leqslant 2$. In the definition of $\delta_X(\varepsilon)$ we can also take the infimum over all vectors $x, y \in X$ with $\|x\|, \|y\| \leqslant 1$ and $\|x - y\| \geqslant \varepsilon$ [32; p. 60].

If A is a Banach algebra, then the second dual A^{**} of A becomes a Banach algebra with respect to the Arens multiplication which is defined in the following fashion [6]. If $y \in A$, $f \in A^*$ and $F, G \in A^{**}$, then linear functionals $f_y, F_f \in A^*$ are defined by $f_y(x) = f(yx)$ and $F_f(x) = F(f_x)$ for $x \in A$. Then the Arens multiplication $GF \in A^{**}$ is defined by

$$(GF)(f) = G(F_f) \quad \text{for all } f \in A^*.$$

The canonical embedding of A into A^{**} is an isometric algebra isomorphism of A into A^{**}. Moreover, if A has identity e then the canonical image of e in A^{**} is the identity element of A^{**}.

In the rest of this section, A will denote a complex Banach algebra with unit e. In the dual space A^* of A, the state space S is defined to be $\{f \in A^* : f(e) = \|f\| = 1\}$. Obviously, this is weak*-closed and it is known [34] that A^* is algebraically spanned by S. If J is an L-summand in A^* with the complementary subspace J'; that is, $A^* = J \oplus_1 J'$, then J and J' are algebraically spanned by $F = J \cap S$ and $F' = J' \cap S$, respectively. More specifically:

PROPOSITION 1. [34]. *F and F' is a pair of complementary split faces of S and J and J' are algebraically spanned by F and F', respectively.*

An element $h \in A$ is said to be hermitian if $f(h)$ is real for each f in the state space S. It is known [6; p.46] that $h \in A$ is hermitian if and only if $\|e^{ith}\| = 1$ for all real numbers t. Of course, e^{ith} is defined by $\sum_{n=0}^{\infty} \frac{(ith)^n}{n!}$.

PROPOSITION 2. [35]. *Suppose $A = J_1 \oplus_\infty J_2$, $J_i \neq \{0\}$ $(i = 1,2)$, $P : A \to J_1$ is the natural projection onto J_1 and $z = P(e)$. Then z is hermitian and $z^2 = z$.*

If J is an M-ideal in a complex unital Banach algebra A, then $A^* = J^\perp \oplus_1 J^{\perp\prime}$ for some closed subspace $J^{\perp\prime}$ of A^* and it is easy to show that $A^{**} = (J^\perp \oplus_1 J^{\perp\prime})^* = J^{\perp\perp} \oplus_\infty J^{\perp\perp\prime}$, where $J^{\perp\perp} = (J^{\perp\prime})^*$ and $J^{\perp\perp\prime} = (J^{\perp\prime})^\perp = (J^\perp)^*$ up to an isometry. Let $P : A^{**} \to J^{\perp\perp}$ be the M-projection onto $J^{\perp\perp}$ and let $z = P(e)$; then by Proposition 2, z is a hermitian projection in A^{**}; that is, z is hermitian in A^{**} and satisfies $z^2 = z$. We shall need the following theorem of Smith and Ward in the next section.

THEOREM 3. [36]. *Let z be a hermitian projection in A^{**} associated with an M-ideal J in A. Then, given $\varepsilon > 0$, z is the weak*-limit of a net (e_α) in A such that*

$$\|e_\alpha\|,\ \|e - e_\alpha\|,\ \|e - 2e_\alpha\| \leqslant 1 + \varepsilon.$$

The following lemma is essentially due to Smith and Ward [36], although they restricted attention to right multiplication by a hermitian projection z associated with an M-ideal J in A.

LEMMA 4. *In the Banach algebra A^{**}, right multiplication by every element y in A^{**} is a weak*-continuous function on A^{**} and if $u \in A^{**}$ is the weak*-limit of a net $\{u_\alpha\}$ in A then, for every x in A, xu is the weak*-limit of $\{xu_\alpha\}$.*

Proof. To prove the first statement, let $\{v_\alpha\}$ be a net in A^{**} with the weak*-limit v in A^{**}. If $f \in A^*$ and $y \in A^{**}$, then, by the definition of Arens multiplication,

$$(vy)(f) = v(y_f) = \lim_\alpha v_\alpha(y_f) = \lim_\alpha (v_\alpha y)(f).$$

Thus $v_\alpha y \to vy$ in the weak*-topology and hence right multiplication by $y \in A^{**}$ is weak*-continuous.

To prove the second statement, let u and $\{u_\alpha\}$ be as above and $f \in A^*$. If $x \in A$, then

$$(xu)(f) = x(u_f) = u_f(x) = u(f_x) = \lim_\alpha u_\alpha(f_x) = \lim_\alpha f(xu_\alpha) = \lim_\alpha (xu_\alpha)(f)$$

and hence xu is the weak*-limit of $\{xu_\alpha\}$.

REMARK. G. Godefroy pointed out to us that since A is weak*-dense in A^{**}, the statement in Lemma 4 yields that left multiplication by an element of A is a weak*-continuous function on A^{**}.

3. M-IDEALS AND IDEALS IN $L(X)$

It is known [32; p. 66] that for every Banach space X, $\frac{\delta_X(\varepsilon)}{\varepsilon}$ is a nondecreasing function on $(0, 2]$. Thus if X is uniformly convex, then $\delta_X(\varepsilon)$ is a strictly increasing function on $(0, 2]$ and its inverse δ_X^{-1} is also a strictly increasing function on $(0, \delta_X(2)]$.

LEMMA 5. *Let X be a uniformly convex space. Then there is a nonnegative real valued function f on $(0, 2] \times (0, \infty)$ such that $\lim_{\lambda\to 0}\lim_{\varepsilon\to 0} f(\varepsilon, \lambda) = 0$, and for every A, T in $L(X)$ with $\|T\|, \|I - T\|, \|I - 2T\| < 1 + \varepsilon$, $\|A\| \leqslant 1$, we have $\|T + \lambda A(I - T)\| \leqslant 1 + \varepsilon + \lambda f(\varepsilon, \lambda)$, where I is the identity map on X.*

Proof. Fix $\varepsilon, \lambda > 0$ and $y \in X$ with $\|y\| = 1$. If $\|Ty\| \leqslant 1 - \lambda(1 + \varepsilon)$, then $\|(T + \lambda A(I - T))y\| \leqslant 1 + \varepsilon$. So we assume that $\|Ty\| \geqslant 1 - \lambda(1 + \varepsilon)$. Set $u = \frac{Ty}{1+\varepsilon}$ and $v = \frac{y - Ty}{1 + \varepsilon}$, then $\|u + v\| = \left\|\frac{y}{1+\varepsilon}\right\| \leqslant 1$ and $\|u - v\| = \left\|\frac{y - 2Ty}{1+\varepsilon}\right\| \leqslant 1$. Since $u = \frac{1}{2}\{(u + v) + (u - v)\}$ and $2v = (u + v) - (u - v)$, we have $\delta_X(\|2v\|) \leqslant 1 - \|u\|$. Hence $\|u\| \leqslant 1 - \delta_X(2\|v\|)$. By assumption, $\frac{1 - \lambda(1+\varepsilon)}{1+\varepsilon} \leqslant \|u\|$. Combining the last two inequalities, we have $\frac{1 - \lambda(1 + \varepsilon)}{1 + \varepsilon} \leqslant$ $\leqslant 1 - \delta_X(2\|v\|)$ and hence $\delta_X(2\|v\|) \leqslant \left(1 - \frac{1}{1+\varepsilon}\right) + \lambda$. Since δ_X^{-1} is an increasing function, $\|v\| \leqslant 2\|v\| \leqslant \delta_X^{-1}\left(1 - \frac{1}{1+\varepsilon} + \lambda\right)\cdot$ Then $\|(T + \lambda A(I - T))y\| \leqslant$ $\leqslant \|Ty\| + \lambda\|(I - T)y\| \leqslant 1 + \varepsilon + \lambda(1 + \varepsilon)\delta_X^{-1}\left(1 - \frac{1}{1+\varepsilon} + \lambda\right)$. Hence

$\|T + \lambda A(I - T)\| \leqslant 1 + \varepsilon + \lambda(1 + \varepsilon)\delta_X^{-1}\left(1 - \frac{1}{1+\varepsilon} + \lambda\right)$. Now let $f(\varepsilon, \lambda) = (1 + \varepsilon)\delta_X^{-1}\left(1 - \frac{1}{1+\varepsilon} + \lambda\right)$.

Now we are ready to prove the main theorem by using the Smith-Ward argument [36], but by replacing Clarkson's inequalities in ℓ_p, $1 < p < \infty$, with the inequality in Lemma 5.

THEOREM 6. *Let X be a uniformly convex space and J an M-ideal in $L(X)$. Then J is a left ideal in $L(X)$ and if X^*, the dual of X, is also uniformly convex then J is a two sided ideal in $L(X)$.*

Proof. Let $J^{\perp\prime}$ be the complementary subspace of the L-summand $J^\perp$ in $L(X)^*$; that is, $L(X)^* = J^\perp \oplus_1 J^{\perp\prime}$, and let $F = J^\perp \cap S$ and $F' = J^{\perp\prime} \cap S$ where S is the state space in $L(X)^*$. Let P be the M-projection of $L(X)^{**} = J^{\perp\perp} \oplus_\infty J^{\perp\perp\prime}$ onto $J^{\perp\perp}$ and $z = P(e)$ where e is the identity operator on X. Then z vanishes on $J^\perp$ and hence on F. Similarly $e - z$ vanishes on F'. For each $\varphi \in F'$, $1 = \varphi(e) = \varphi(e - z) + \varphi(z) = \varphi(z)$ and hence $z = 1$ on F'.

First we will show that $L(X)(e - z) \subseteq J^{\perp\perp\prime}$. In view of Proposition 1 and the equation $L(X)^{**} = J^{\perp\perp\prime} \oplus_\infty J^{\perp\perp}$, it suffices to show that if $A \in L(X)$ with $\|A\| \leqslant 1$ then $\varphi(A(e - z)) = 0$ for all $\varphi \in F'$. Suppose there is $\varphi \in F'$ and $A \in L(X)$ with $\|A\| \leqslant 1$ such that $\varphi(A(e - z)) \neq 0$. By multiplying A by a scalar we may assume that $\varphi(A(e - z)) = \lambda$, $0 < \lambda < 1$. Let $A_n = z + \lambda^n A(e - z) \in L(X)^{**}$. Then by Theorem 3 and Lemma 4, A_n is the weak*-limit of a net $\{e_\alpha + \lambda^n A(e - e_\alpha)\}_\alpha$ in $L(X)$ with $\|e_\alpha\|$, $\|e - e_\alpha\|$, $\|e - 2e_\alpha\| < 1 + \varepsilon$. By Lemma 5, $\|e_\alpha + \lambda^n A(e - e_\alpha)\| \leqslant 1 + \varepsilon + \lambda^n f(\varepsilon, \lambda^n)$ and hence we have

$$\|A_n\| \leqslant 1 + \varepsilon + \lambda^n f(\varepsilon, \lambda^n).$$

Since $\|\varphi\| = 1$ and $\varphi(z) = 1$, $1 + \lambda^{n+1} = \varphi(A_n) \leqslant \|A_n\| \leqslant 1 + \varepsilon + \lambda^n f(\varepsilon, \lambda^n)$. Letting $\varepsilon \to 0$, we have $\lambda \leqslant \lim_{\varepsilon \to 0} f(\varepsilon, \lambda^n)$, and letting $n \to \infty$, $0 < \lambda \leqslant \lim_{n\to\infty} \lim_{\varepsilon\to 0} f(\varepsilon, \lambda^n) = 0$. This is a contradiction. Hence $\varphi(A(e - z)) = 0$ for all $A \in L(X)$ and all $\varphi \in F'$, and we get that

$$L(X)(e - z) \subseteq J^{\perp\perp\prime}.$$

Since $J^{\perp\perp\prime}$ is weak*-closed and by Lemma 4 right multiplication by $e - z$ is a weak*-continuous function on $L(X)^{**}$, we have $L(X)^{**}(e - z) \subseteq J^{\perp\perp\prime}$ by the weak*-density of $L(X)$ in $L(X)^{**}$. Notice that if I is the identity map on $L(X)^{**}$ then $I - P$ is the M-projection of $L(X)^{**} = J^{\perp\perp} \oplus_\infty J^{\perp\perp\prime}$ onto $J^{\perp\perp\prime}$ and

$(I-P)e = e - Pe = e - z$. Thus by replacing z by $e - z$ in the above argument we get that

$$L(X)^{**}z \subseteq J^{\perp\perp}.$$

From the above two inclusions, we have $L(X)^{**}z = J^{\perp\perp}$ and $L(X)^{**}(e-z) = J^{\perp\perp'}$. Since $J^{\perp\perp} = L(X)^{**}z$ is a left ideal in $L(X)^{**}$, $J = J^{\perp\perp} \cap L(X)$ is a left ideal in $L(X)$.

Next suppose that X and X^* are uniformly convex. Let $\sigma : L(X) \to L(X^*)$ be defined by $\sigma(A) = A^*$, the adjoint of A. Then σ is an isometry and $\sigma(AB) = \sigma(B)\sigma(A)$ for every $A, B \in L(X)$. If J is a M-ideal in $L(X)$, then $\sigma(J)$ is an M-ideal in $L(X^*)$ and hence is a left ideal in $L(X^*)$ by the above result. Then $J = \sigma^{-1}\sigma(J)$ is a right ideal and hence a two sided ideal in $L(X)$.

4. AN EXAMPLE OF A SPACE $X = (\sum \ell_p^{n_i})_r$ FOR WHICH $L(X)$ CONTAINS A CLOSED TWO SIDED IDEAL WHICH IS NOT AN M-IDEAL IN $L(X)$

For a Banach space X and $1 < r < \infty$, $S_r(X)$ will denote the space of all operators in $L(X)$ which factor through a subspace of an L_r-space. Thus an operator T in $L(X)$ belongs to $S_r(X)$ if there exists a subspace E of an $L_r(\Omega)$ and bounded linear operators $A : X \to E$, $B : E \to X$ such that $T = BA$. It is easy to see that $S_r(X)$ is a two sided ideal in $L(X)$, hence the closure $\overline{S_r(X)}$ of $S_r(X)$ is also a two sided ideal in $L(X)$.

On $S_r(X)$, we put a norm which is defined for T in $S_r(X)$ by

$$S_r(T) = \inf\{\|A\|\,\|B\| : T = BA,\ A \in L(X, E),\ B \in L(E, X)\}$$

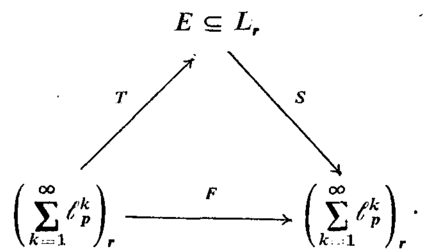
where the infimum is taken over all possible factorizations of T through subspaces E of L_r-spaces.

In this section we will heavily use the following lemma which is due to Figiel, Johnson and Schechtman.

LEMMA 7. [15]. *Suppose* $2 < p < \infty$, $T : \ell_p^k \to \ell_p^{2k}$ *with* $\|T\| \leqslant 1$ *and* $\operatorname{Average}_{\pm}\left\{\left\|\sum_{i=1}^{k} \pm Te_i\right\|\right\} \geqslant \delta k^{1/p}$, $\delta > 0$, *where the average is taken over all choices of* $+$ *and* $-$ *signs. Then there exist positive constants* $c = c(p, r, \delta)$ *and* $\alpha = \alpha(p, r)$ *such that* $S_r(T) \geqslant ck^{\alpha}$.

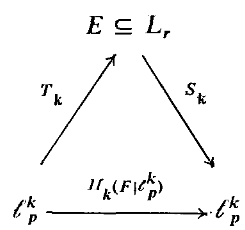
LEMMA 8. *Suppose* $1 < p \neq r < \infty$, $p \neq 2$, *and it is false that* $1 < r < p < 2$. *Then we have* $d(I, S_r(X)) = \inf\{\|I - T\| : T \in S_r(X)\} \geqslant 1$, *where* I *is the identity map on* $X = \left(\sum_{k=1}^{\infty} \ell_p^k\right)_r$.

Proof. If $d(I, S_r(X)) < 1$ then there is $F \in S_r(X)$ such that $\|F - I\| = 1 - \varepsilon$, $\varepsilon > 0$ and F factors through a subspace of L_r



Thus $\|ST - I\| = 1 - \varepsilon$.

Let Π_k be the projection from $\left(\sum_{k=1}^{\infty} \ell_p^k\right)_r$ onto ℓ_p^k, then $\Pi_k(F|\ell_p^k)$ has a factorization



where $T_k = T|\ell_p^k$ and $S_k = \Pi_k S$. Then $\|S_k T_k - I_k\| \leqslant 1 - \varepsilon$, where I_k is the identity map on ℓ_p^k. Thus $S_k T_k$ is invertible and by the Neumann series expansion of $(S_k T_k)^{-1}$, we have the estimates $\|(S_k T_k)^{-1}\| \leqslant \dfrac{1}{1 - (1 - \varepsilon)} = \varepsilon^{-1}$ and

$$S_r(I_k) \leqslant \|S_k\| \, \|T_k(S_k T_k)^{-1}\| \leqslant \|S\| \, \|T\| \varepsilon^{-1}.$$

To draw a contradiction, we need that $\sup_k S_k(I_k) = \infty$. It is known that ℓ_p cannot be (isomorphically) embedded in L_r under the hypothesis on p and r [3; p. 206], [25]. So standard considerations yield that ℓ_p^k's cannot be uniformly embedded in L_r.

Indeed, if ℓ_p^k's can be uniformly embedded in $L_r(\mu)$ for some measure μ then there exist positive numbers $a, b > 0$ and embeddings $T_k : \ell_p^k \to L_r(\mu)$ such that for every $k = 1, 2, 3 \ldots$

$$a\|x\| \leqslant \|T_k x\| \leqslant b\|x\| \quad \text{for any } x \in \ell_p^k.$$

By taking an ultraproduct of $\{T_k\}_{k=1}^{\infty}$, we get [32; p. 120] an operator

$$T = (T_k)_{\mathcal{U}} : (\ell_p^k)_{\mathcal{U}} \to (L_r(\mu))_{\mathcal{U}}$$

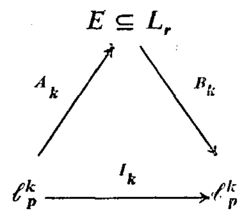
where $\mathcal{U}$ is an ultrafilter on $\mathbf{N}$, the set of all positive integers.

By definition, for any $(x_k)_\mathcal{U}$ in $(\ell_p^k)_\mathcal{U}$, $(T_k)_\mathcal{U}((x_k)_\mathcal{U}) = (T_k x_k)_\mathcal{U}$, $\|(x_k)_\mathcal{U}\| = \lim_\mathcal{U} \|x_k\|$, $\|(T_k x_k)_\mathcal{U}\| = \lim_\mathcal{U} \|T_k x_k\|$. Hence we have

$$a\|(x_k)_\mathcal{U}\| \leqslant \|T(x_k)_\mathcal{U}\| \leqslant b\|(x_k)_\mathcal{U}\|,$$

that is, T is an embedding of $(\ell_p^k)_\mathcal{U}$ in $(L_r(\mu))_\mathcal{U}$. Since $(L_r)_\mathcal{U}$ is also an L_r-space [32; p. 271] and $(\ell_p^k)_\mathcal{U}$ contains an isometric copy of ℓ_p, T yields an embedding of ℓ_p into L_r, which is a contradiction.

Going back to the main stream, for any $\delta > 0$ and each k, there is a factorization



so that $\|A_k\| \|B_k\| \leqslant S_r(I_k) + \delta$ and $\|B_k\| = 1$. Since A_k is an embedding $\sup_k \|A_k\| = \infty$, and hence $\sup_k S_r(I_k) = \infty$.

For $k = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots,$ and $1 \leqslant i \leqslant m$, let $\Omega_{k,m,i} = \{(s, t) : 1 \leqslant s \leqslant m,\ 1 \leqslant t \leqslant k, s$ and t are integers$\} \cup \{(1, 0)\}$ be a measure space with $\mu\{(1, 0)\} = m^{-1}$ and $\mu\{(s, t)\} = (km)^{-1}$ if $(s, t) \neq (1, 0)$.

For notational convenience we denote $L_p(\Omega_{k,m,i})$, $p > 2$ by $X(k, m, i)$, the indicator function of $\{(1, 0)\}$ by $e(k, m, i)$ and the indicator function of $\{(s, t)\}$ by $e_{s,t}(k, m, i)$ if $(s, t) \neq (1,0)$. So $\{e_{s,t}(k, m, i) : 1 \leqslant s \leqslant m,\ 1 \leqslant t \leqslant k\} \cup \{e(k, m, i)\}$ is the natural basis of $X(k, m, i)$. Usually the dependence on k, m and i will be suppressed.

Let $P_{k,m,i}$ be the projection on $X(k, m, i)$ defined by $P_{k,m,i}(e) = 0$ and $P_{k,m,i}(e_{s,t}) = m^{-1} \sum_{u=1}^{m} e_{u,t}$. We define a linear map $S_{k,m,i}$ on $X(k, m, i)$ by $S_{k,m,i}(e) = \sum_{t=1}^{k} e_{1,t}$ and $S_{k,m,i}(e_{s,t}) = 0$. We can easily see that both $P_{k,m,i}$ and $S_{k,m,i}$ have norm one.

Let $X(k, m) = \left(\sum_{i=1}^{m} X(k, m, i)\right)_p$ and $X = \left(\sum_{k,m=1}^{\infty} X(k, m)\right)_r$, $2 < p < \infty$, $1 < r \neq p < \infty$. Let $P : X \to X$ and $S : X \to X$ be the direct sum of the families $\{P_{k,m,i}\}$ and $\{S_{k,m,i}\}$ respectively. Since each $X(k, m)$ is isometric to $\ell_p^{km^2+m}$, X is isometric to $\left(\sum_{k,m=1}^{\infty} \ell_p^{km^2+m}\right)_r$.

Our goal is to prove that $\overline{S_r(X)}$ is not proximinal in $L(X)$ by showing that $P + S$ does not have a best approximant in $\overline{S_r(X)}$.

PROPOSITION 9. $d(P + S, \overline{S_r(X)}) = 1$.

Proof. It suffices to show that $d(P + S, S_r(X)) = 1$. For a fixed n, define an operator S_n on X so that S_n is the direct sum of operators $T_{k,m,i}$ on $X(k, m, i)$ where $T_{k,n,i} = S_{k,n,i}$ and $T_{k,m,i} = 0$ if $m \neq n$.

From the definition of S_n, it is easy to see that the range of $S_n|X(k, n)$ (S_n restricted to $X(k, n)$) is isometric to ℓ_p^n. Since ℓ_p^n is isomorphic to ℓ_r^n, $S_n|X(k, n)$ factors through ℓ_r^n. Thus it follows that for each fixed n, S_n, factors through $\ell_r = (\sum \ell_r^n)_r$, the ℓ_r-sum of infinitely many copies of ℓ_r^n, and hence $\tilde{S}_N = \sum_{n=1}^{N} S_n \in S_r(X)$ for all N.

Now we claim that $\|P + S - \tilde{S}_N\| \leqslant 1 + \left(\frac{1}{N}\right)^{1/p}$. To prove the claim, observe that

$$\|P + S - \tilde{S}_N\| = \sup\{\|(P + S - \tilde{S}_N) \,|\, X(k, m, i)\|\}$$

where the supremum is taken over all $k, m = 1, 2, 3, \ldots,$ and $1 \leqslant i \leqslant m$, and

$$\|(P + S - \tilde{S}_N) \,|\, X(k, m, i)\| = \begin{cases} \|P_{k,m,i}\| & \text{if } m \leqslant N \\ \|P_{k,m,i} + S_{k,m,i}\| & \text{if } m > N. \end{cases}$$

To prove that $\|P_{k,m,i} + S_{k,m,i}\| \leqslant 1 + \left(\frac{1}{N}\right)^{1/p}$ for all $m \geqslant N$, let $B = \{(1,0)\}$ and $A = \{(1, t) \in \Omega_{k,m,i} : 1 \leqslant t \leqslant k\}$. For $f \in X(k, m, i)$, let $f_1 = f \cdot 1_B$ and $f_2 = f - f_1$. Then $P_{k,m,i} f_1 = 0$ and $S_{k,m,i} f_2 = 0$.

Since $\|P_{k,m,i}\| = 1$ and $P_{k,m,i} f_2$ is constant on each row of $\Omega_{k,m,i} \setminus B$, $\|1_A P_{k,m,i} f_2\| = \left(\frac{1}{m}\right)^{1/p} \|P_{k,m,i} f_2\| \leqslant \left(\frac{1}{m}\right)^{1/p} \|f\|$.

Since $S_{k,m,i} f_1$ and $(1 - 1_A) P_{k,m,i} f_2$ have disjoint supports, $\|S_{k,m,i}\| = 1$ and $\|P_{k,m,i}\| = 1$, we have

$$\|S_{k,m,i} f_1 + (1 - 1_A) P_{k,m,i} f_2\| = (\|S_{k,m,i} f_1\|^p + \|(1 - 1_A) P_{k,m,i} f_2\|^p)^{1/p} \leqslant \|f\|.$$

Hence, for $f \in X(k, m, i)$,

$$\|(P_{k,m,i} + S_{k,m,i})f\| \leqslant \|P_{k,m,i} f_1\| + \|1_A P_{k,m,i} f_2\| +$$

$$+ \|S_{k,m,i} f_1 + (1 - 1_A) P_{k,m,i} f_2\| + \|S_{k,m,i} f_2\| \leqslant$$

$$\leqslant 0 + \left(\frac{1}{m}\right)^{1/p} \|f\| + \|f\| + 0.$$

Thus for $m \geqslant N$, $\|P_{k,m,i} + S_{k,m,i}\| \leqslant 1 + \left(\frac{1}{m}\right)^{1/p} \leqslant 1 + \left(\frac{1}{N}\right)^{1/p}$ and the proof of the claim is complete.

Since $\tilde{S}_N \in S_r(X)$, by letting $N \to \infty$, we infer that $d(P + S, S_r(X)) \leqslant 1$.

To prove the reverse inequality, notice that $P + S$ restricted to span $\left\{\sum_{s=1}^{m} e_{s,t} : t = 1, 2, \ldots, k\right\}$, which is isometric to ℓ_p^k, acts as the identity operator. Thus $P + S$ acts as the identity operator on an isometric copy of $\left(\sum_{k=1}^{\infty} \ell_p^k\right)_r$. So by Lemma 8 we have $d(P + S, S_r(X)) \geqslant 1$ and the proof of Proposition 9 is complete.

LEMMA 10. *Let $Q_{k,m,i}$ be the averaging projection of X onto* span $\left\{e_{1,t}(k, m, i), \sum_{s=2}^{m} e_{s,t}(k, m, i)\right\}_{t=1}^{k}$. *If T is in $S_r(X)$, then* $\limsup_{m\to\infty} \min_{k} \min_{1\leqslant i\leqslant m} \|m^{1/p}Q_{k,m,i}Te(k, m, i)\| = 0$.

Proof. Obviously it suffices to prove the lemma for T in $S_r(X)$ with $\|T\| \leqslant 1$. If the statement is false then there is $\delta > 0$ such that $\sup_k \min_{1\leqslant i\leqslant m} \|m^{1/p}Q_{k,m,i}Te(k, m, i)\| > 2\delta$ for infinitely many m. Fix such an m and choose $k = k(m)$ so that

$$\|m^{1/p}Q_{k,m,i}Te(k, m, i)\| > 2\delta \quad \text{for all } i = 1, 2, \ldots, m.$$

The map $\psi : \ell_p^m \to \operatorname{span}\{e(k,m,i)\}_{i=1}^m$ defined by $e_i \to m^{1/p}e(k, m, i)$ $(i = 1, 2, \ldots, m)$ is an isometry onto, where $\{e_i\}_{i=1}^m$ is the unit vector basis of ℓ_p^m.

Since the vectors $\{Q_{k,m,i}Te(k, m, i)\}_{i=1}^m$ have disjoint supports, $W = \operatorname{span}\{Q_{k,m,i}Te(k, m, i)\}_{i=1}^m$ is isometric to ℓ_p^n for some $n \leqslant m$ and hence the map U defined by $e_i \to Q_{k,m,i}T\psi(e_i)$ can be viewed to have values in ℓ_p^n. Since $\|Ue_i\| = \|m^{1/p}Q_{k,m,i}Te(k, m, i)\| > 2\delta$ for each $i = 1, 2, \ldots, m$, we get

$$\operatorname*{Average}_{\pm} \left\|\sum_{i=1}^{m} \pm Ue_i\right\| \geqslant \left\| \operatorname*{Average}_{\pm} \left|\sum_{i=1}^{m} \pm Ue_i\right| \right\| \geqslant$$

$$\geqslant 2^{-1}\left\|\left(\sum_{i=1}^{m} |Ue_i|^2\right)^{1/2}\right\| \geqslant \qquad \text{(by Khintchine's inequality [31; p.66])}$$

$$\geqslant 2^{-1}\left\|\left(\sum_{i=1}^{m} |Ue_i|^p\right)^{1/p}\right\| = 2^{-1}\left(\sum_{i=1}^{m} \|Ue_i\|^p\right)^{1/p} \geqslant \delta m^{1/p}.$$

Since $\|U\| \leqslant 1$, we conclude by Lemma 7 that there exist positive constants $c = c(p, r, \delta)$ and $\alpha = \alpha(p, \alpha)$ such that $S_r(U) \geqslant cm^{\alpha}$.

for infinitely many k. This is a contradiction and so (**) is true.

For a fixed vector $x \in X(k, m, 1)$ with the expansion $x = \sum x_{s,t} e_{s,}$ $+ x_0 e(k, m, 1)$ with respect to the natural basis for $X(k, m, 1)$, we have $\langle r_{\varepsilon,k,m}, x$ $= \int r_{\varepsilon,k,m} x \, d\mu = \frac{1}{km} \sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{m} x_{s,t}$ and so

$$(\operatorname*{Average}_{\varepsilon} |\langle r_{\varepsilon,k,m}, x\rangle|^2)^{1/2} = (2^{-k} \sum_{\varepsilon \in \{-1,1\}^k} |\langle r_{\varepsilon,k,m}, x\rangle|^2)^{1/2} =$$

$$= (km)^{-1}\left(2^{-k} \sum_{\varepsilon \in \{-1,1\}^k} \left| \sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{m} x_{s,t} \right|^2\right)^{1/2} = (km)^{-1}\left(\sum_{t=1}^{k} \left| \sum_{s=1}^{m} x_{s,t} \right|^2\right)^{1/2} \leqslant$$

$$\leqslant (km)^{-1}\left(\sum_{t=1}^{k} \left(\sum_{s=1}^{m} |x_{s,t}|^2 \right) m \right)^{1/2} = \text{(by Hölder's inequal}$$

$$= k^{-1/2}\left(\sum_{t=1}^{k} \sum_{s=1}^{m} (km)^{-1} |x_{s,t}|^2 \right)^{1/2} = k^{-1/2} \|x\|_2$$

where $\|x\|_2$ is the $L_2(\mu)$-norm of x.

It is easy to see that $\langle r_{\varepsilon,k,m}, x\rangle = \langle r_{\varepsilon,k,m}, Q_{k,m}x\rangle$ and hence from the ab inequality we have

$$(***) \qquad (\operatorname*{Average}_{\varepsilon} |\langle r_{\varepsilon,k,m}, Q_{k,m}x\rangle|^2)^{1/2} \leqslant \frac{1}{\sqrt{k}} \|x\|_2 .$$

Now we will show that $\|P + S - T\| > 1$ to finish the proof of Theorem

For each positive integer n, choose a positive integer $k(n)$ such that $k(n + 1)$ $> k(n) > 4n^2$ and for $k = k(n)$ the left hand side of (**) is smaller than $(4n)$ Then we have

$$\|Q_{k(n),m} T r_{\varepsilon,k(n),m}\| \leqslant n^{-1}$$

and $|\langle r_{\varepsilon,k(n),m}, Q_{k(n),m}x\rangle| \leqslant n^{-1}\|x\|_2$ for all $m > m_0$, for all $x \in X(k(n), m, 1)$, a some $\varepsilon = \varepsilon(n, m, x)$.

Now for fixed $n, m \geqslant m_0$, and $k = k(n)$, set $x = (S - T)e(k, m, 1)$ a $\varepsilon = \varepsilon(n, m, x)$.

If we define $g = r_{\varepsilon,k,m} + \lambda e(k, m, 1)$, $\lambda > 0$, then $\|g\| = (1 + \lambda^p m^{-1})^{1/p}$ a

$$\|Q_{k,m}(P + S - T)g\| = \|r_{\varepsilon,k,m} - Q_{k,m}Tr_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k, m, 1)\| \geqslant$$

$$\geqslant \|r_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k, m, 1)\| - \|Q_{k,m}Tr_{\varepsilon,k,m}\| \geqslant$$

$$\geqslant \|r_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k, m, 1)\|_2 - n^{-1} =$$

s gives a contradiction because the diagonal principle [31; p. 20] yields $S_r(T) \geqslant S_r(U)$. To see this, let V be the norm one projection from X onto W. ve consider U and $\tilde{U} = V_i \sum_{i=1}^{m} Q_{k,m,i} T\psi$ as operators from ℓ_p^m into ℓ_2^n, we e that U is just the diagonal of $\tilde{U}$ and hence $S_r(U) \leqslant S_r(\tilde{U})$ by the diagonal iciple. But

$$S_r(\tilde{U}) \leqslant \|V\| \left\| \sum_{i=1}^{m} Q_{k,m,i} \right\| S_r(T) \|\psi\| = S_r(T).$$

THEOREM 11. *$P + S$ has no best approximant in $\overline{S_r(X)}$.*

Proof. Suppose $P + S$ has a best approximant T in $\overline{S_r(X)}$, then by Proposition 9 $+ S - T\| = 1$. In view of Lemma 10, we do not lose anything by assuming, notational convenience, that for all k and all $m \geqslant m_0$,

$$\|m^{1/p} Q_{k,m,1} Te(k, m, 1)\| \leqslant 4^{-1}.$$

the sequel we will write $Q_{k,m,1}$ as $Q_{k,m}$. So the above inequality is

$$\|Q_{k,m} Te(k, m, 1)\| \leqslant 4^{-1} m^{-1/p}.$$

For each $k, m \geqslant m_0$ and $\varepsilon = (\varepsilon_i)_{i=1}^k$ with $\varepsilon_i = \pm 1$, we consider a Rademacher ıction $r_{\varepsilon,k,m}$ in the range of $Q_{k,m}$ defined by $r_{\varepsilon,k,m} = \sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{m} e_{s,t}(k, m, 1)$. Since rank of $Q_{k,m}$ is $2k$, by Lemma 7 and an approximation argument, we get that any $\delta > 0$, there is a $k(\delta)$ such that

$$\text{Average}_{\varepsilon} \|Q_{k,m} T r_{\varepsilon,k,m}\| < \delta \tag{**}$$

here the average is over all $\varepsilon \in \{-1,1\}^k$) for all $k \geqslant k(\delta)$ and all $m \geqslant m_0$.

Indeed, there exists $\tilde{T}$ in $S_r(X)$ such that $\|T - \tilde{T}\| < \delta/2$. So $\|Q_{k,m}T - Q_{k,m}\tilde{T}\| < \delta/2$ and $S_r(Q_{k,m}\tilde{T}) \leqslant S_r(\tilde{T})$ for any k and $m \geqslant m_0$. If (**) is violated then we get ıt

$$\text{Average}_{\varepsilon} \|Q_{k,m} \tilde{T} r_{\varepsilon,k,m}\| > \frac{\delta}{2}$$

infinitely many k and some $m = m(k)$. Notice that the map $e_i \to k^{1/p} \sum_{s=1}^{m} e_{s,i}(k, m, 1)$ defines an isometry from ℓ_p^k onto $\text{span}\left\{ \sum_{s=1}^{m} e_{s,i}(k, m, 1) \right\}_{i=1}^{k}$. nce the range of $Q_{k,m}$ is isometric to ℓ_p^{2k}, we can apply Lemma 7 to conclude that

$$S_r(\tilde{T}) \geqslant S_r(Q_{k,m}\tilde{T}) \geqslant ck^{\alpha}$$

$$= (1 + \lambda^2\|Q_{k,m}(S-T)e(k, m, 1)\|_2^2 + 2\lambda \langle r_{\varepsilon,k,m}, Q_{k,m}(S-T)e(k, m, 1)\rangle)^{1/2} - n^{-1} \geqslant$$

$$\geqslant (1 + \lambda^2\|Q_{k,m}(S-T)e(k, m, 1)\|_2^2 - 2\lambda n^{-1}\|(S-T)e(k, m, 1)\|_2)^{1/2} - n^{-1} \geqslant$$

$$\geqslant (1 + \lambda^2\|Q_{k,m}(S-T)e(k, m, 1)\|_2^2 - 2\lambda n^{-1}\|(S-T)e(k, m, 1)\|_p)^{1/2} - n^{-1} \geqslant$$

$$\geqslant (1 + \lambda^2\|Q_{k,m}(S-T)e(k, m, 1)\|_2^2 - 4\lambda n^{-1}m^{-1/p})^{1/2} - n^{-1}.$$

Since $\|Q_{k,m}Te(k, m, 1)\| \leqslant 4^{-1}m^{-1/p}$ by (*), by Chebyshev's inequality we have

$$\mu\left(\left\{(s, t) \in \Omega_{k,m,1} : |Q_{k,m}Te(k, m, 1)| > \frac{1}{2}\right\}\right) \leqslant$$

$$\leqslant 2^p\|Q_{k,m}Te(k, m, 1)\|^p \leqslant 2^p \cdot 4^{-p} \cdot m^{-1} < (4m)^{-1}.$$

Since $Q_{k,m}Se(k, m, 1) = 1_A$, where $A = \{(1, t) \in \Omega(k, m, 1) : 1 \leqslant t \leqslant k\}$, if we set $C = \{(1, t) \in A : |Q_{k,m}Te(k, m, 1)| \leqslant 1/2\}$ then $\mu(C) \geqslant \mu(A) - (4m)^{-1} = (3/4)m^{-1}$. Consequently we have

$$\|Q_{k,m}(S-T)e(k, m, 1)\|_2^2 \geqslant \int_C |1 - Q_{k,m}Te(k, m, 1)|^2 d\mu >$$

$$> \int_C 2^{-2} d\mu \geqslant (3/16)m^{-1}.$$

Thus for all $m \geqslant m_0$, we get

$$\|P + S - T\| \geqslant \overline{\lim_n} \, Q_{k(n),m}(P + S - T)\| \geqslant$$

$$\geqslant (1 + \lambda^2(3/16)m^{-1})^{1/2}(1 + \lambda^p m^{-1})^{-1/p} > 1 \quad \text{for small } \lambda.$$

This is a contradiction and the proof of Theorem 11 is complete.

REMARKS. 1. As mentioned in the introduction, the example presented here is a localization of the example in [5] and, indeed, we have here repeated some of the calculations in [5].

2. The reason we worked with $S_r(\cdot)$ instead of the more common $\gamma_r(\cdot)$ (factorization constant through an L_r space) is that if $\tilde{U}$ is an astriction of the oper-

ator U, then $S_r(\tilde{U}) = S_r(U)$. This simplified the exposition a bit. In fact, a simple variation of the argument presented here yields that $\gamma_r(X)$ is not proximinal in $L(X)$ where $X = \left(\sum_{n=1}^{\infty} \ell_p^n\right)_r$, $1 < r \neq p < \infty$, $p \neq 2$.

3. When $1 < r < p < 2$, $X = \left(\sum_{n=1}^{\infty} \ell_p^n\right)_r$ is isometric to a subspace of L_r and almost isometric to a subspace of ℓ_r; in fact, for each $N = 1, 2, 3, \ldots$, there are subspaces $E_{n,N}$ of ℓ_r with a 1-symmetric basis so that the basis-to-basis mapping from $E_{n,N}$ to ℓ_r yields an $1 + N^{-1}$ isomorphism from $E_{n,N}$ onto ℓ_p^n. It seems that the argument in Section 4 yields that for the subspace $Y = \left(\sum_{n,N=1}^{\infty} E_{n,N}\right)_r$ of ℓ_r, $\gamma_r(Y)$ is not proximinal in $L(Y)$; however, we did not attempt to check the details.

4. For $X = \left(\sum_{n=1}^{\infty} \ell_p^n\right)_r$, $1 < p, r < \infty$, we would guess that $K(X)$ is the only non-trivial ideal in $L(X)$. By the result of Section 3, all the M-ideals in $L(X)$ are ideals, but (except when $p = r$ or $p = 2$, which are the cases where X is isomorphic to ℓ_r) the ideal structure of $L(X)$ is so complex that the result of Section 3 appears to be of little use in verifying this conjecture.

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