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A SHORT PROOF OF "INJECTIVITY IMPLIES HYPERFINITENESS" FOR FINITE VON NEUMANN ALGEBRAS

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Dedicated to the anniversary of Connes' fundamental theorem

In [1] A. Connes proved a fundamental result in operator algebras which gives a powerful intrinsic characterization of the hyperfinite algebras as those von Neumann algebras that are norm one projections of the algebra of operators on the Hilbert space on which they act, i.e. as the injective von Neumann algebras. To prove this result A. Connes developed a large amount of new techniques, including, as a key part, a deep analysis of the automorphism group of II_1 factors.

In [6] U. Haagerup gave a new proof of Connes' theorem, more direct and without the automorphism group machinery, but using the property of semi-discreteness that the injective factors have. His proof solves separately the finite and the properly infinite cases, the proof of the infinite case being very short and elementary.

We present here another proof of the finite case, quite short and derived directly from the definition of injectivity.

Our proof is not related to [6], it is closer to certain ideas in [1], [3] and depends on the techniques of maximal abelian algebras that we developed in [9], more precisely on a general noncommutative local Rohlin lemma. To use this lemma we will consider the injective algebras as modules over their maximal abelian $*$-subalgebras. Before describing our approach, we first recall some facts from [1] and [3].

In [1] Connes pointed out an analogy between amenable groups and injective II_1 factors (for a detailed discussion see [2]). He notes that a finite injective factor M has a hypertrace which can be viewed as the analogue of the invariant mean of an amenable group. He uses Day's trick to get from it an operatorial Følner type condition for M, similarly to the way one obtains the classical Følner condition for amenable groups. In his proof the hypertrace of M is regarded as a state on $M \vee JMJ = \mathscr{B}(\mathscr{H})$, where J is the canonical conjugation in the standard form of M. The Følner condition obtained yields the existence of a finite dimensional projection in $\mathscr{H}$ almost invariant, in the Hilbert-Schmidt norm, under a given finite set of unitary elements of M. The analogy between the amenable groups and algebras was also exploited by Connes, Feldman and Weiss in [3] (see also [10]) where again Day's trick is used in an essential way to get a Følner type condition for the unitaries in the normalizer of a Cartan subalgebra A of an injective factor M. Using this condition they prove, by a simple maximality argument, the striking result that the normalizer of A is single generated as a full group. In the proof of the finite case of this Følner condition, Day's trick is applied to the hypertrace restricted to $A \vee JAJ$ and it gives a projection $e \in A$, almost invariant under a given finite set of unitaries in M, together with a matrix algebra which approximate them on e.

The proof of "injectivity implies hyperfiniteness" that we give here belongs to this circle of ideas. We also use Day's trick, but in a different setting. We consider an arbitrary maximal abelian $*$-subalgebra $A \subset M$ (instead of a regular one which aprioricaly may not exist) and apply Day's trick to the hypertrace restricted to $M \vee JAJ$. The advantage of using $M \vee JAJ$ instead of $M \vee JMJ = \mathscr{B}(\mathscr{H})$ is that the normal states of $M \vee JAJ$ are related in a nice way to the algebras A and M. From this first step we get finite elements in $M \vee JAJ$ almost invariant under a fixed finite set of unitaries in M. By Connes' noncommutative Namioka type trick these elements can be assumed finite projections. The almost invariance can then be translated into a set of 2-norm inequalities which only involve elements in A and M. The rest of the proof is the interpretation of these inequalities and it depends on a certain approximation property of maximal abelian algebras (1.2 in [9]). This property may be viewed as a noncommutative local Rohlin lemma. Its (trivial) commutative version says that, given n measure preserving transformations acting mutually free on a measure space X and a set of positive measure $A \subset X$ one can find a small subset of positive measure $A_0 \subset A$ which is transported into disjoint sets by the n transformations. From all these we deduce an appropriate Følner condition similar to the one in [3] and mentioned above. The proof ends with a maximality argument as in [3].

1. PRELIMINARIES ON THE ALGEBRA $M \vee JAJ$

1.1. Let M be a finite von Neumann algebra with a fixed faithful normal trace τ, $\tau(1) = 1$. Denote $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, and $L^2(M, \tau)$ the completion of M in this norm. We regard M as acting by left multiplication on $L^2(M, \tau)$ and denote by J the canonical conjugation on $L^2(M, \tau)$, uniquely determined by $Jx = x^*$ for $x \in M$. Then, if $x \in M$, JxJ is the multiplication to the right by x^*.

For $B \subset M$ a von Neumann subalgebra E_B denotes the unique normal τ-preserving conditional expectation onto B.

1.2. We fix a maximal abelian selfadjoint $*$-subalgebra $A \subset M$ and consider the von Neumann algebra $M \vee JAJ$ generated by M and JAJ. Since A is maximal abelian in M, JAJ is maximal abelian in JMJ, so that $(M \vee JAJ)' = (JAJ)' \cap \cap JMJ = JAJ$. Thus $M \vee JAJ = (JAJ)'$ and JAJ is the center of $M \vee JAJ$. In particular it follows that $M \vee JAJ$ is of type I.

1.3. If for $\xi \in L^2(M, \tau)$ we denote by p_ξ the cyclic projection on the closure $\overline{\xi A}$ of $JAJ\xi = \xi A$ in $L^2(M, \tau)$, then by [7] any such projection is majorised in $M \vee JAJ$ by p_1 (here $1 \in M \subset L^2(M, \tau)$ is regarded as a vector). Since $p_1(M \vee JAJ)p_1 = JAJp_1$ it follows that $p_\xi(M \vee JAJ)p_\xi = JAJp_\xi$, for all $\xi \in L^2(M, \tau)$.

1.4. If $z(\xi)$ denotes the conjugate by J of the central support of p_ξ in $M \vee JAJ$ then $z(\xi) \in A$ and it is the smallest projection e in A with $\xi e = \xi$. Remark that if $\xi \in L^2(M, \tau)$ then p_ξ is equivalent in $M \vee JAJ$ with $Jz(\xi)Jp_1 = p_{z(\xi)}$. From this and 1.2 it follows that if $X \in M \vee JAJ$ then there exists a unique $a \in Az(\xi)$ such that $p_\xi X p_\xi = JaJp_\xi$.

1.5. If $x \in M$ then $z(x)$ is just the support of $E_A(x^*x)$ in A. In particular if $E_A(x^*x)$ is a projection in A then $z(x) = E_A(x^*x)$ and for any $y \in M$, $p_x(y) = xE_A(x^*y) \in xA$. Indeed, for any $a \in A$,

$$\tau(ax^*(y - xE_A(x^*y))) = \tau(ax^*y) - \tau(ax^*xE_A(x^*y)) = \tau(ax^*y) - \tau(aE_A(x^*x)E_A(x^*y)) =$$

$$= \tau(ax^*y) - \tau(az(x)E_A(x^*y)) = \tau(ax^*y) - \tau(aE_A(z(x)x^*y)) =$$

$$= \tau(ax^*y) - \tau(aE_A(x^*y)) = 0.$$

Moreover, if $x_1, x_2 \in M$ are such that $E_A(x_1^*x_1)$, $E_A(x_2^*x_2)$ are projections then p_{x_1}, p_{x_2} are mutually orthogonal iff $E_A(x_2^*x_1) = 0$.

1.6. Let $x, y \in M$ be such that $E_A(x^*x)$ and $E_A(y^*y)$ are projections in A. Then $p_xp_yp_x = JaJp_x$ where $a = E_A(x^*y)E_A(y^*x)$. Indeed, if $z \in M$ then by 1.3

$$p_xp_yp_x(z) = p_xp_y(xE_A(x^*z)) =$$

$$= p_x(yE_A(y^*xE_A(x^*z))) = p_x(yE_A(y^*x)E_A(x^*z)) = xE_A(x^*yE_A(y^*x)E_A(x^*z)) =$$

$$= xE_A(x^*y)E_A(y^*x)E_A(x^*z) = xE_A(x^*z)E_A(x^*y)E_A(y^*x) =$$

$$= xE_A(x^*zE_A(x^*y)E_A(y^*x)) = (p_xJaJ)(z) = (JaJp_x)(z).$$

1.7. To define a normal semifinite faithful trace on $M \vee JAJ$ it is sufficient to define a normal semifinite faithful trace on the reduced algebra of $M \vee JAJ$ by a projection of central support 1 and to extend it spatially to all $M \vee JAJ$.

In this way we define φ to be the unique normal semifinite faithful trace on $M \vee JAJ$ satisfying $\varphi(JaJp_{\hat{1}}) = \tau(a)$, for $a \in A_+$. By 1.4 φ satisfies $\varphi(JaJp_{\xi}) = \tau(az(\xi))$, $a \in A_+$, $\xi \in L^2(M, \tau)$.

We denote $\|X\|_{2,\varphi} = \varphi(X^*X)^{1/2}$, $X \in M \vee JAJ$, the corresponding 2-norm.

We need to approximate the finite projections of $M \vee JAJ$ by certain nice projections.

1.8. LEMMA. *Let $P \in M \vee JAJ$ be a projection with $\varphi(P) < \infty$ and $\varepsilon > 0$. There exist $x_1, \ldots, x_n \in M$ such that $E_A(x_i^* x_j) = \delta_{ij} f_j$, for some projections $f_j \in A$, and $\left\| P - \sum_{i=1}^{n} p_{x_i} \right\|_{2,\varphi} < \varepsilon$.*

Proof. Let $P = \sum_{i \in I} p_{\xi_i}$ for some $\xi_i \in L^2(M, \tau)$. Then there exists a finite set of vectors $\xi_1, \ldots, \xi_n \in \{\xi_i\}_i$ such that $\left\| P - \sum_{j=1}^{n} p_{\xi_j} \right\|_{2,\varphi} < \varepsilon/2$.

Let $1/4 \geqslant \delta > 0$, put $\xi_0 = 0$, $x_0 = f_0 = 0$, and suppose that for some $k \geqslant 1$ we found $x_0, x_1, \ldots, x_{k-1} \in M$ and projections $f_0, f_1, \ldots, f_{k-1} \in A$ such that

$E_A(x_i^* x_j) = \delta_{ij} f_j$, $\quad 0 \leqslant i, j \leqslant k-1$, $\quad$ and $\left\| \sum_{j=0}^{k-1} p_{\xi_j} - \sum_{j=0}^{k-1} p_{x_j} \right\|_{2,\varphi} < \delta$. We show

there exists $x_k \in M$ such that $E_A(x_k^* x_i) = 0$, $1 \leqslant i \leqslant k-1$, $E_A(x_k^* x_k)$ is a projection and $\left\| \sum_{j=0}^{k} p_{\xi_j} - \sum_{j=0}^{k} p_{x_j} \right\|_{2,\varphi} < 2\delta^{1/2}$. Then it easily follows by induction that we can find $x_1, \ldots, x_n \in M$ so that $E_A(x_i^* x_j) = \delta_{ij} f_j$ with f_j projections, and $\left\| \sum_{j=1}^{n} p_{\xi_j} - \sum_{j=1}^{n} p_{x_j} \right\|_{2,\varphi} < \varepsilon/2$, which proves the lemma.

By regarding ξ_k as an operator affiliated with M (cf. [4]) it follows that there exists an increasing sequence of projections in M, $f_m \uparrow 1$, $f_m \xi_k \in M$. Thus $\|f_m p_{\xi_k} f_m - p_{\xi_k}\|_{2,\varphi} \underset{m}{\to} 0$, $\varphi(p_{\xi_k}) - \varphi(f_m p_{\xi_k} f_m) \underset{m}{\to} 0$ and since $p_{f_m \xi_k}$ is the support of $f_m p_{\xi_k} f_m$ it follows that

$$\|p_{f_m \xi_k} - p_{\xi_k}\|_{2,\varphi}^2 \leqslant 2\|p_{f_m \xi_k} - f_m p_{\xi_k} f_m\|_{2,\varphi}^2 + 2\|f_m p_{\xi_k} f_m - p_{\xi_k}\|_{2,\varphi}^2 \leqslant$$

$$\leqslant 2(\varphi(p_{f_m \xi_k}) - \varphi(f_m p_{\xi_k} f_m)) + 2\|f_m p_{\xi_k} f_m - p_{\xi_k}\|_{2,\varphi}^2 \to 0.$$

Let m be so that $\|p_{f_m \xi_k} - p_{\xi_k}\|_{2,\varphi} < \delta'$ and denote $y = f_m \xi_k$. Let F be the support projection of $\sum_{j=0}^{k-1} p_{x_j} + p_y$. Then, since $\left\| \sum_{j=0}^{k-1} p_{x_j} \right\|_{2,\varphi} \leqslant k^{1/2}$, $\|p_{\xi_k}\|_{2,\varphi} \leqslant 1$, by the

Cauchy-Schwartz inequality

$$\left\| F - \left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right)\right\|_{2,\varphi}^2 = \varphi(F) - 2\varphi\left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right) + \varphi\left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right) +$$

$$+ 2\varphi\left(p_y \sum_{j=0}^{k-1} p_{x_j}\right) = \varphi(F) - \varphi\left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right) + 2\varphi\left((p_y - p_{\xi_k}) \sum_{j=0}^{k-1} p_{x_j}\right) +$$

$$+ 2\varphi\left(p_{\xi_k}\left(\sum_{j=0}^{k-1} p_{x_j} - \sum_{j=0}^{k-1} p_{\xi_j}\right)\right) \leqslant \varphi(F) - \varphi\left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right) + 2k^{1/2}\delta' + 2\delta.$$

Since $\varphi(F) \leqslant \varphi\left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right)$ it follows that

$$\left\| F - \left(\sum_{j=0}^{k-1} p_{x_j} + p_y\right)\right\|_{2,\varphi}^2 < 2\delta + 2k^{1/2}\delta'$$

and thus

$$\left\| F - \sum_{j=0}^{k} p_{\xi_j}\right\|_{2,\varphi} < (2\delta + 2k^{1/2}\delta')^{1/2} + \left\|\sum_{j=0}^{k-1} p_{\xi_j} - \sum_{j=0}^{k-1} p_{x_j}\right\|_{2,\varphi} +$$

$$+ \|p_{\xi_k} - p_y\|_{2,\varphi} < (2\delta + 2k^{1/2}\delta')^{1/2} + \delta + \delta'.$$

But for δ' sufficiently small and $\delta \leqslant 1/4$, $(2\delta + 2k^{1/2}\delta')^{1/2} + \delta + \delta' \leqslant 2\delta^{1/2}$, so that we may assume y and F satisfy

$$\left\| F - \sum_{j=1}^{k} p_{\xi_j}\right\|_{2,\varphi} < 2\delta^{1/2}.$$

But $y_0 = \left(\sum_{j=1}^{k-1} p_{x_j}\right)(y) = \sum_{j=1}^{k-1} x_j E_A(x_j^* y)$ is in M and thus $y_1 = y - y_0$ is in M and we have $F = \sum_{j=1}^{k-1} p_{x_j} + p_{y_1}$. Let $a = E_A'(y_1^* y_1)$ and $e_m \in A$ the spectral projections of a corresponding to $(1/m, \infty)$. Then $ae_m \geqslant 1/m e_m$ and $\| e_m - s(a)\|_2 \to 0$, $s(a)$ being the support of a. By the definition of φ we get

$$\|p_{y_1 e_m} - p_{y_1}\|_{2,\varphi} = \|Je_m J p_{y_1} - p_{y_1}\|_{2,\varphi} = \|e_m - s(a)\|_2 \to 0.$$

Moreover $p_{y_1 e_m} \leqslant p_{y_1}$ so that $p_{y_1 e_m} p_{x_j} = 0$ for $0 \leqslant j \leqslant k-1$. So for large m, $\left\| F - \left(\sum_{j=1}^{k-1} p_{x_j} + p_{y_1 e_m}\right)\right\|_{2,\varphi}$ can be made small enough to insure that

$$\left\| \sum_{j=1}^{k} p_{\xi_j} - \left(\sum_{j=1}^{k-1} p_{x_j} + p_{y_1 e_m}\right)\right\|_{2,\varphi} < 2\delta^{1/2}.$$

But if $x_k = y_1 e_m a^{-1/2}$ then $p_{x_k} = p_{y_1 e_m}$ and $f_k = E_A(x_k^* x_k) = e_m$ is a projection.

Q.E.D.

The preceding lemma says that any finitely generated submodule of $L^2(M, \tau)$ over A can be "approximated" well enough by a finitely generated orthocomplemented projective submodule of M.

2. A FOLNER TYPE CONDITION

We now assume the finite algebra M of §1 is injective so that it has a hypertrace, i.e. there is a state on $\mathscr{B}(L^2(M, \tau))$ with M in its centralizer. We denote by ψ_0 the restriction of this state to $M \vee JAJ$. It satisfies $\psi_0(v^* x v) = \psi_0(x)$ for $x \in M \vee JAJ$ and unitary elements $v \in M$. The normal semifinite faithful trace φ on $M \vee JAJ$ that we use below is the one defined at 1.7. Let $\varepsilon > 0$ and $v_1, \ldots, v_m \in M$ some unitary elements. Let $\mathscr{L} = \{(\psi(v_k^* \cdot v_k) - \psi(\cdot))_{m \geqslant k \geqslant 1} \mid \psi$ normal state on $M \vee JAJ$ with Radon-Nykodim derivative with respect to φ a positive element in $M \vee JAJ$ of finite trace $\varphi\}$. Then $\mathscr{L}$ is a bounded convex set in $((M \vee JAJ)_*)^m$. Since the states ψ in the definition of $\mathscr{L}$ are norm dense in the space of normal states on $M \vee JAJ$ and this one is $\sigma((M \vee JAJ)^*, (M \vee JAJ))$ dense in the space of all states on $M \vee JAJ$, it follows that the $\sigma(((M \vee JAJ)^*)^m, (M \vee JAJ)^m)$ closure $\overline{\mathscr{L}}^{\mathrm{w}}$ of $\mathscr{L}$ contains any m-tuple of continuous forms $(\psi(v_k^* \cdot v_k) - \psi)_{m \geqslant k \geqslant 1}$ with ψ a state on $M \vee JAJ$. In particular $\overline{\mathscr{L}}^{\mathrm{w}}$ contains $(\psi_0(v_k^* \cdot v_k) - \psi_0)_{m \geqslant k \geqslant 1} = 0$. But since $0 = (0, \ldots, 0)$ is in $((M \vee JAJ)_*)^m$ and the dual of $((M \vee JAJ)_*)^m$ is $(M \vee JAJ)^m$, it follows that the $\sigma(((M \vee JAJ)_*)^m, (M \vee JAJ)^m)$ closure of $\mathscr{L}$ is equal to the norm closure of $\mathscr{L}$ and that 0 is norm adherent to $\mathscr{L}$. It follows that for any $\varepsilon > 0$ there exists $b \in (M \vee JAJ)_+$ with $\varphi(b) = 1$ such that

$$\| v_k b v_k^* - b \|_{1,\varphi} = \| \varphi(v_k^* \cdot v_k b) - \varphi(\cdot\, b) \| < \varepsilon/m, \tag{1}$$

where, for $y \in M \vee JAJ$, $\|y\|_{1,\varphi} = \varphi(|y|)$, and $m \geqslant k \geqslant 1$.

Let $a = b^{1/2}$. By the Powers-Stormer inequality and (1) we have for $1 \leqslant k \leqslant m$,

$$\|a - v_k a v_k^*\|_{2,\varphi}^2 \leqslant \|b - v_k b v_k^*\|_{1,\varphi} < \varepsilon/2m^2 = (\varepsilon/2m^2)\|a\|_{2,\varphi}^2$$

and thus

$$\|a - v_k a v_k^*\|_{2,\varphi}^2 < (\varepsilon/2m^2)\|a\|_{2,\varphi}^2. \tag{2}$$

If we put $\|(x_1, x_2, \ldots, x_m)\|_{2,\sim} = (\sum_k \|x_k\|_{2,\varphi}^2)^{1/2}$, for $x_1, \ldots, x_m \in M \vee JAJ$, and we denote $\tilde{a} = (a, \ldots, a)$, $\tilde{v} = (v_1, \ldots, v_m) \in (M \vee JAJ)^m$, then by (2) we have

$$\|\tilde{a} - \tilde{v}\tilde{a}\tilde{v}^*\|_{2,\sim}^2 < (\varepsilon/2m^2)\|\tilde{a}\|_{2,\sim}^2 .$$

By 1.1 in [1] we may assume $\tilde{a}$ and $\tilde{v}\tilde{a}\tilde{v}^*$ commute so that by the classical Namioka trick (see e.g. [2]) there is a $t>0$ so that the spectral projection $E_{[t,\infty)}(\tilde{a})$ of $\tilde{a}$ corresponding to the interval $[t,\infty)$ satisfies

$$\|E_{[t,\infty)}(\tilde{a})-E_{[t,\infty)}(\tilde{v}\tilde{a}\tilde{v}^*)\|_{2,\sim}^2<(\varepsilon/2m^2)\|E_{[t,\infty)}(\tilde{a})\|_{2,\sim}^2.$$

Indeed if $\tilde{a}=h$, $\tilde{v}\tilde{a}\tilde{v}^*=k\in L^\infty(X,\mu)$ we have:

$$\int_0^\infty\|E_{[s^{1/2},\infty)}(h)-E_{[s^{1/2},\infty)}(k)\|_2^2\,\mathrm{d}s=$$

$$=\int_0^\infty\|E_{[s,\infty)}(h^2)-E_{[s,\infty)}(k^2)\|_2^2\,\mathrm{d}s=$$

$$=\int_0^\infty\int_X|\chi_{[s,\infty)}(h^2(x))-\chi_{[s,\infty)}(k^2(x))|^2\,\mathrm{d}\mu(x)\,\mathrm{d}s=$$

$$=\int_X\int_0^\infty|\chi_{[s,\infty)}(h^2(x))-\chi_{[s,\infty)}(k^2(x))|\,\mathrm{d}s\,\mathrm{d}\mu(x)=$$

$$=\int_X|h^2(x)-k^2(x)|\,\mathrm{d}\mu(x)=\|h^2-k^2\|_1\leqslant\|h-k\|_2\|h+k\|_2\leqslant$$

$$\leqslant 2\|h-k\|_2<2(\varepsilon/2m)\|h\|_2=(\varepsilon/m)\|h\|_2^2=(\varepsilon/m)\int_0^\infty\|E_{[s^{1/2},\infty)}(h)\|_2^2\,\mathrm{d}s$$

so that a certain $t=s^{1/2}$ will do.

But $E_{[t,\infty)}(\tilde{a})=(E_{[t,\infty)}(a),E_{[t,\infty)}(a),\ldots,E_{[t,\infty)}(a))$ and $E_{[t,\infty)}(\tilde{v}\tilde{a}\tilde{v}^*)=\tilde{v}E_{[t,\infty)}(\tilde{a})\tilde{v}^*$ so that by the preceding inequality we get

$$\sum_k\|E_{[t,\infty)}(a)-v_kE_{[t,\infty)}(a)v_k^*\|_{2,\varphi}^2<\varepsilon\|E_{[t,\infty)}(a)\|_{2,\varphi}^2.$$

Thus there exists a finite projection $a_0=E_{[t,\infty)}(a)$ in $M\vee JAJ$ so that

$$\|a_0-v_ka_0v_k^*\|_{2,\varphi}^2<\varepsilon\|a_0\|_{2,\varphi}^2,\quad m\geqslant k\geqslant 1. \tag{2'}$$

By 1.8 we may assume the projection a_0 is of the form $a_0=\sum_{j=1}^n p_{x_j}$ with $E_A(x_j^*x_i)=\delta_{ij}f_i$, $1\leqslant i\leqslant n$, where f_i are projections in A.

Since for $y \in (M \vee JAJ)_+$ we have

$$\|y\|_{2,\varphi}^2 \geqslant \left\|\left(\sum_{j=1}^n p_{x_j}\right)y\right\|_{2,\varphi}^2 = \sum_{j=1}^n \|p_{x_j}y\|_{2,\varphi}^2 \geqslant \sum_{j=1}^n \|p_{x_j}yp_{x_j}\|_{2,\varphi}^2,$$

by 1.6 and 1.7 we get

$$\|a_0 - v_k a_0 v_k^*\|_{2,\varphi}^2 \geqslant \|a_0 - \sum_j p_{x_j} v_k a_0 v_k^* p_{x_j}\|_{2,\varphi}^2 = \sum_{j=1}^n \|p_{x_j} - p_{x_j} v_k a_0 v_k p_{x_j}\|_{2,\varphi}^2 =$$

$$= \sum_{j=1}^n \left\| f_j - \sum_{i=1}^n E_A(x_j^* v_k x_i) E_A(x_i^* v_k^* x_j) f_j f_i \right\|_2^2 .$$

Thus (2′) becomes

$$\sum_{j=1}^n \tau\left(\left(f_j - \sum_{i=1}^n E_A(x_j^* v_k x_i) E_A(x_i^* v_k^* x_j) f_j f_i\right)^2\right) < \varepsilon \sum_{j=1}^n \tau(f_j). \tag{3}$$

But A is abelian, so it has to exist a nonzero projection $f \in A$ for which we have the operator inequality:

$$\sum_{j=1}^n \left(f_j f - \sum_{i=1}^n E_A(x_j^* v_k x_i) E_A(x_i^* v_k^* x_j) f_j f_i f\right) < \varepsilon \sum_{j=1}^n f_j f. \tag{4}$$

Since any projection $f' \in A$, $0 \neq f' \leqslant f$ still satisfies (4), using that A is abelian we can replace f by a smaller nonzero projection, also denoted by f, which satisfies for each j, $ff_j = f$ or $ff_j = 0$. Moreover, because $ff_j \neq 0$ at least for one j, by reiterating if necessary, we may suppose $f_j = f_j f = f$ for all j and (4) becomes:

$$\sum_{j=1}^n \left(f - \sum_{i=1}^n E_A(x_j^* v_k x_i) E_A(x_i^* v_k^* x_j) f\right)^2 < \varepsilon n f. \tag{5}$$

At this stage we need the following local Rohlin type lemma.

2.3. Lemma. *If $0 \neq f \in A$ is a projection, $y_1, \ldots, y_t \in M$ and $\varepsilon' > 0$ then there exists a projection $e \in A$, $e \leqslant f$ such that, for $\lambda_i = \tau(ey_ie)/\tau(e)$,*

$$\|ey_ie - \lambda_i e\|_2 < \varepsilon' \|e\|_2 .$$

Proof. Since A is abelian, by spectral decomposition it follows that there exists $f' \in A$, $0 \neq f' \leqslant f$, such that $\|E_A(y_i)f' - \lambda_i' f'\| < \varepsilon'/2$ for some scalars $\lambda_i' \in \mathbf{C}$. By [9] there is a partition of f' in Af', $e_1, e_2, \ldots, e_s$, such that $\sum_i \sum_j \|e_j y_i e_j - E_A(y_i)e_j\|_2^2 < (\varepsilon'/2)^2 \sum_j \|e_j\|_2^2$. It follows that for some j, $\|e_j y_i e_j - E_A(y_i)e_j\|_2^2 < (\varepsilon'/2)^2 \|e_j\|_2^2$ for all i and thus $\|e_j y_i e_j - \lambda_i' e_j\|_2 < \varepsilon' \|e_j\|_2$. But by the definition of λ_i, $\|e_j y_i e_j - \lambda_i e_j\|_2 \leqslant \|e_j y_i e_j - \lambda_i' e_j\|_2$ for any $\lambda_i' \in \mathbf{C}$ so that the projection $e = e_j$ satisfies the statement. Q.E.D.

By (5) and the preceding lemma for any $\delta > 0$ there is a projection $e \in A$, $e \leqslant f$ such that

$$(6)\qquad \begin{aligned} &1^\circ.\quad \sum_{j=1}^{n}\Big(e - \sum_{i=1}^{n} E_A(x_j^* v_k x_i)E_A(x_i^* v_k^* x_j)e\Big)^2 < \varepsilon n e;\\ &2^\circ.\quad \|e x_j^* x_i e - \delta_{ij} e\|_2 < \delta\|e\|_2;\\ &3^\circ.\quad \|e x_j^* v_k x_i e - \lambda_{ji} e\|_2 < \delta\|e\|_2,\end{aligned}$$

for some scalars $\lambda_{ji} \in \mathbf{C}$, $n \geqslant i,\ j \geqslant 1,\ m \geqslant k \geqslant 1$.

Since in 2° and 3° δ is arbitrary small, independently of the norms of x_i, it follows from 2° and some standard polar decomposition and functional calculus arguments (see e.g. Chapter III, § 7.3 in [4]) that there exist partial isometries $u_j \in M$ such that for a certain projection $e' \leqslant e$ (e' is not necessary in A), $u_i^* u_j = \delta_{ij} e'$ and $\|e - e'\| < \delta'\|e'\|$, $\|x_i e - u_i\|_2 < \delta'\|e'\|_2$, with δ' depending on δ but tending to zero as δ does. Thus, for δ small enough, in (6) we can approximate $E_A(x_j^* v_k x_i)E_A(x_i^* v_k^* x_j)e = E_A(e x_j^* v_k x_i e)E_A(e x_i^* v_k^* x_j e)$ by $e x_j^* v_k x_i e x_i^* v_k^* x_j e$ and this in turn by $u_j^* v_k u_i u_i^* v_k^* u_j$ so that to have

$$(7)\qquad \begin{aligned} &1^\circ.\quad \sum_j \tau\Big(\Big(e' - \sum_{i=1}^{n} u_j^* v_k u_i u_i^* v_k^* u_j\Big)^2\Big) < \varepsilon n\, \tau(e')\\ &2^\circ.\quad \|u_j^* v_k u_i - \lambda_{ji} e'\|_2^2 < (\varepsilon/n)\|e'\|_2^2 .\end{aligned}$$

We denote by $e_{ij} = u_i u_j^*$ and $s = \sum_i e_{ii}$. Since

$$\|e' - \sum_i u_j^* v_k u_i u_i^* v_k^* u_j\|_2 = \|e' - u_j^* v_k s v_k^* u_j\|_2 = \|u_j(e' - v_k s v_k^*)u_j^*\|_2 =$$

$$= \|e_{jj} - e_{jj} v_k s v_k^* e_{jj}\|_2,$$

$$\|u_j^* v_k u_i - \lambda_{ji} e'\|_2 = \|u_j(u_j^* v_k u_i - \lambda_{ji} e')u_i^*\|_2 = \|e_{jj} v_k e_{ii} - \lambda_{ji} e_{ji}\|_2$$

and $n\tau(e') = n\|e'\|_2^2 = \|s\|_2^2$, it follows that

$$(8)\qquad \begin{aligned} &1^\circ.\quad \|s - \sum_j e_{jj} v_k s v_k^* e_{jj}\|_2^2 < \varepsilon\|s\|_2^2\\ &2^\circ.\quad \|s v_k s - \sum_{j,i} \lambda_{ji} e_{ji}\|_2^2 = \|\sum_{i,j} e_{jj} v_k e_{ii} - \sum_{i,j} \lambda_{ji} e_{ji}\|_2^2 =\\ &\qquad = \sum_{j,i} \|e_{jj} v_k e_{ii} - \lambda_{ji} e_{ji}\|_2^2 < \varepsilon\|s\|_2^2 .\end{aligned}$$

By the paralelogram low we have

$$\|v_k s v_k^* - \sum_j e_{jj} v_k s v_k^* e_{jj}\|_2^2 + \|\sum_j e_{jj} v_k s v_k^* e_{jj}\|_2^2 = \|v_k s v_k^*\|_2^2 = \|s\|_2^2$$

so that by (8), 1° we get

$$\|v_k s v_k^* - \sum_j e_{jj} v_k s v_k^* e_{jj}\|_2^2 = (\|s\|_2 + \|\sum_j e_{jj} v_k s v_k^* e_{jj}\|_2)(\|s\|_2 - \|\sum_j e_{jj} v_k s v_k^* e_{jj}\|_2) <$$

$$< (2\|s\|_2)(\varepsilon^{1/2}\|s\|_2) = 2\varepsilon^{1/2}\|s\|_2^2.$$

This gives together with (8):

$$\text{(9)} \qquad \begin{aligned} &1^\circ.\ \|s - v_k s v_k^*\|_2 < (\varepsilon^{1/2} + 2^{1/2}\varepsilon^{1/4})\|s\|_2 \\ &2^\circ.\ \|s v_k s - \sum_{j,i} \lambda_{ji} e_{ji}\|_2 < \varepsilon^{1/2}\|s\|_2. \end{aligned}$$

The Følner type result follows now easily:

2.2. PROPOSITION. *Let $\varepsilon > 0$ and $y_1, \ldots, y_m \in M$. There exists a projection $s \in M$ and a matrix algebra $M_0 \subset M$ supported on s such that*

$$\|E_{M_0}(s y_i s) - (y_i - (1 - s) y_i (1 - s))\|_2 < \varepsilon\|s\|_2.$$

Proof. Since any y_i is a linear combination of four unitary elements we may suppose y_i are unitaries and by extending the set $\{y_i\}_i$ we may also suppose it is a selfadjoint set. Then by (9) it follows that there exist s and M_0 so that

$$\|E_{M_0}(s y_i s) - y_i s\|_2 < (\varepsilon/2)\|s\|_2, \quad \text{for all } i.$$

Since $\{y_i\}_i$ is selfadjoint $\|E_{M_0}(s y_i s) - s y_i\|_2 < (\varepsilon/2)\|s\|_2$, for all i, so that $\|s y_i (1 - s)\|_2 < (\varepsilon/2)\|s\|_2$ and thus

$$\|E_{M_0}(s y_i s) - (y_i - (1 - s) y_i (1 - s))\|_2 \leqslant \|E_{M_0}(s y_i s) - y_i s\|_2 +$$

$$+ \|s y_i (1 - s)\|_2 < \varepsilon\|s\|_2. \qquad \text{Q.E.D.}$$

2.3. REMARK. In this section we used several technical devices that we discuss below.

1°. First we used the Powers-Stormer inequality: if $a, b \in M \vee JAJ$ are finite positive elements then $\|a - b\|_{2,\varphi}^2 \leqslant \|a^2 - b^2\|_{1,\varphi}$. A simple proof of it is as follows. Let $p - q$ be the selfadjoint element in the polar decomposition of $a - b$, where p, q are projections with $pq = 0$. Then $\|a^2 - b^2\|_{1,\varphi} \geqslant \varphi((a^2 - b^2)(p - q)) = \varphi(p(a^2 - b^2)p) + \varphi(q(b^2 - a^2)q)$. But $\varphi(p(a^2 - b^2)p) - \varphi(p(a - b)^2 p) = \varphi(pb(a - b)p) + \varphi(p(a - b)bp) = 2\varphi(b(a - b)_+) \geqslant 0$ and similarly $\varphi(q(b^2 - a^2)q) \geqslant \varphi(q(b - a)^2 q)$ so that $\|a^2 - b^2\|_{1,\varphi} \geqslant \varphi(p(a - b)^2 p) + \varphi(q(a - b)^2 q) = \varphi((a - b)^2(p + q)) = \varphi((a - b)^2) = \|a - b\|_{2,\varphi}^2$.

2°. Instead of 1.1 in [1] and the argument before (3) we could use directly 1.2 in [1]. We did like this to make the proof more elementary and self contained.

3°. The result in [9] used in the proof of 2.1 is as follows: if $A \subset M$ is maximal abelian and $y_1, \ldots, y_m \in M$, $\varepsilon > 0$ then there exists a partition of the unity $e_1, \ldots, e_s \in A$ such that if $A_0 = \text{span}\{e_j\}_j$ then $\|E_{A_0}(y_i) - E_{A_0' \cap M}(y_i)\|_2 < \varepsilon$.

4°. The deformation argument needed at (8) can be done as an exercise. If M would be a factor we could take $e' = e$.

3. END OF THE PROOF OF "INJECTIVITY IMPLIES HYPERFINITENESS"

We can now complete the proof of Connes' theorem. Recall that a finite von Neumann algebra M is said to be hyperfinite (or approximately finite dimensional) if given any normal finite trace τ on M, $\varepsilon > 0$ and $y_1, \ldots, y_n \in M$, there exists a finite dimensional subalgebra $N \subset M$ and $y_1', \ldots, y_n' \in N$ such that $\|y_i - y_i'\|_2 < \varepsilon$ (where $\|y\|_2 = \tau(y^*y)^{1/2}$, $y \in M$) (see [4], [5]).

THEOREM. *The injective finite von Neumann algebras are hyperfinite.*

Proof. Since a direct sum of hyperfinite algebras is hyperfinite it is sufficient to consider the case when M has a faithful normal trace τ with $\tau(1) = 1$. Let $\varepsilon > 0$, $y_1, \ldots, y_m \in M$ and consider the set of all families of matrix subalgebras $\{M_i\}_{i \in I}$ of M having mutually orthogonal supports $\{s_i\}_{i \in I}$ and such that if $s = \sum_i s_i$ then

$$\|E_{\oplus_i M_i}(sy_ks) - (y_k - (1 - s)y_k(1 - s))\|_2^2 \leqslant \varepsilon^2 \|s\|_2^2, \quad m \geqslant k \geqslant 1.$$

This set is clearly inductively ordered by inclusion so we can take a maximal family $\{M_i^0\}_i$ with corresponding supports $\{s_i^0\}_i$. Suppose $s = \sum_i s_i^0 \neq 1$ and let $f = 1 - s$. By 2.2 applied to $\{fy_kf\} \subset fMf$, there exists a matrix algebra $M_0 \subset fMf$, supported on some $s_0 \neq 0$ with

$$\|E_{M_0}(s_0y_ks_0) - (fy_kf - (f - s_0)y_k(f - s_0))\|_2^2 < \varepsilon^2\|s_0\|_2^2.$$

It follows that for all k we have:

$$\|E_{\oplus_i M_i^0 \oplus M_0}((s + s_0)y_k(s + s_0)) - (y_k - (1 - (s + s_0))y_i(1 - (s + s_0)))\|_2^2 =$$

$$= \|E_{\oplus_i M_i^0}(sy_ks) - (y_k - (1 - s)y_k(1 - s))\|_2^2 +$$

$$+ \|E_{M_0}(s_0y_ks_0) - (fy_kf - (f - s_0)y_k(f - s_0))\|_2^2 <$$

$$< \varepsilon^2\|s\|_2^2 + \varepsilon^2\|s_0\|_2^2 = \varepsilon^2\|s + s_0\|_2^2,$$

which is in contradiction with the maximality of the family $\{M_i^0\}_i$. Thus $\sum_i s_i^0 = 1$. Since M is countably decomposable (as it has a faithful trace), the set $\{s_i^0\}_i$ is countable. Taking a subset $\{s_1, s_2, \ldots, s_n\} \subset \{s_i^0\}$ with $\|y_k - (\sum_j s_j)y_k(\sum_j s_j)\|_2 < \varepsilon$ and denoting by $M_1, M_2, \ldots, M_n \in \{M_i^0\}_i$ the corresponding matrix algebras it follows that $N = (\bigoplus_{j=1}^{n} M_j) \oplus \mathbf{C}$ is a finite dimensional subalgebra of M with

$$\|E_N(y_k) - y_k\|_2 < 2\varepsilon, \quad \text{for all } k.$$

Q.E.D.

FINAL REMARK. If we choose in §1 the algebra A to be a Cartan subalgebra of M and the elements x_j in 1.8 to be partial isometries in the normalizing grupoid of M (as is easily seen to be possible, see e.g. § 2 in [10]) then we obtain by the above arguments a proof of the Connes, Feldman, Weiss theorem, showing that when M is separable injective the normalizer of A is single generated as a full group. In fact even if M is nonseparable we obtain a nonseparable version of this theorem, namely that given any countable generated subgroup in the normalizer of A, the corresponding full group is single generated.

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