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TENSOR PRODUCTS OF LINEAR OPERATORS IN BANACH SPACES AND TAYLOR'S JOINT SPECTRUM

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1. INTRODUCTION

Let A denote an algebra over the complex numbers $\mathbf{C}$ with unit element e. If $B \subset A$ is a unital subalgebra and $b \in B$, we let

$$\mathrm{Sp}(b; B) := \{z \in \mathbf{C} : ze - b \text{ has no inverse in } B\}$$

denote the spectrum of b with respect to the algebra B. If $b = (b_1, \ldots, b_n)$ is an n-tuple of elements from B, we consider as a joint spectrum of b with respect to B the set

$$\mathrm{Sp}(b;B) := \mathrm{Sp}_{\mathrm{left}}(b;B) \cup \mathrm{Sp}_{\mathrm{right}}(b; B)$$

where

$$\mathrm{Sp}_{\mathrm{left}}(b; B) := \{z \in \mathbf{C}^n : e \notin \sum_j B(b_j - z_j e)\}$$

and

$$\mathrm{Sp}_{\mathrm{right}}(b; B) := \{z \in \mathbf{C}^n : e \notin \sum_j (b_j - z_j e)B\}$$

denote the left and right spectrum, respectively (Harte [4]). Throughout this paper we shall restrict our attention to the algebra $A = L(X)$ of all bounded linear operators on a Banach space X. For a given n-tuple $a = (a_1, \ldots, a_n)$ of mutually commuting operators, B will either be the commutant

$$\{a\}^{\mathrm{c}} := \{b \in L(X) : ba_j = a_j b \text{ for } 1 \leqslant j \leqslant n\}$$

or the bicommutant

$$\{a\}^{\mathrm{cc}} := \{b \in L(X) : bc = cb \text{ for all } c \in \{a\}^{\mathrm{c}}\}.$$

In the situation under consideration, we have $\mathrm{Sp}(a; B) = \mathrm{Sp}_{\mathrm{left}}(a; B) = \mathrm{Sp}_{\mathrm{right}}(a; B)$.

Of course these spectra heavily depend on the algebra B and the question arises, whether the joint commutant spectrum $\mathrm{Sp}(a;\{a\}^c)$ or the joint bicommutant spectrum $\mathrm{Sp}(a;\{a\}^{cc})$ are natural choices for n-tuple of commuting operators at all.

However, J. L. Taylor [9] gave another notion of a joint spectrum, which is a subset of the joint commutant spectrum and gives rise to an analytic functional calculus, which in general admits more functions. The purpose of this paper is to show that these concepts coincide in the case of tensor products of several operators acting on Banach spaces. This is a complete answer to a problem raised in the paper [2] of Ceauşescu and Vasilescu.

We recall J. L. Taylor's concept of a joint spectrum for an n-tuple of mutually commuting operators as given in [9] and [10]. Let

$$\wedge\,[s_1, \ldots, s_n] = \bigoplus_{p=0}^{\infty} \wedge^p [s_1, \ldots, s_n]$$

denote the exterior algebra over $\mathbf{C}$ generated by the indeterminates $s_1, \ldots, s_n$ and $\wedge^p[s_1, \ldots, s_n]$ the space of elements $z \cdot s_{j_1} \wedge s_{j_2} \wedge \ldots \wedge s_{j_p}$ of degree p, where $\wedge$ denotes exterior multiplication. Given a Banach space X, we let $\wedge[s_1, \ldots, s_n; X]$ and $\wedge^p[s_1, \ldots, s_n; X]$ denote the tensor products $X \otimes \wedge[s_1, \ldots, s_n]$ and $X \otimes \wedge^p[s_1, \ldots, s_n]$, respectively. If $a = (a_1, \ldots, a_n)$ is an n-tuple of mutually commuting operators $a_i \in L(X)$, we consider $\tilde{a} = a_1 s_1 + \ldots + a_n s_n$ acting as a graded module homomorphism of degree 1:

$$\tilde{a}\colon \wedge^p[s_1, \ldots, s_n; X] \to \wedge^{p+1}[s_1, \ldots, s_n; X] \quad \text{for each } p,$$

$$\tilde{a}(x_{j_1 \ldots j_p} \otimes s_{j_1} \wedge \ldots \wedge s_{j_p}) := \sum_{k=1}^{n} a_k(x_{j_1 \ldots j_p}) s_k \wedge s_{j_1} \wedge \ldots \wedge s_{j_p}.$$

As $a_i a_j = a_j a_i$, we have $\tilde{a} \wedge \tilde{a} := \sum_{i<j} (a_i a_j - a_j a_i) \otimes s_i \wedge s_j = 0$. So $\tilde{a}$ acts as a coboundary operator on $\wedge[s_1, \ldots, s_n; X]$.

We say that the n-tuple a is non-singular, if the associated sequence $F(X, a)$:

$$0 \to X = \wedge^0[s_1, \ldots, s_n; X] \xrightarrow{\tilde{a}} \wedge^1[s_1, \ldots, s_n; X] \xrightarrow{\tilde{a}} \ldots$$

$$\ldots \to \wedge^{n-1}[s_1, \ldots, s_n; X] \xrightarrow{\tilde{a}} \wedge^n[s_1 \wedge, \ldots, s_n; X] = X \to 0$$

is exact.

1.1. Definition. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of mutually commuting operators $a_i \in L(X)$ and $\tilde{a} = a_1 s_1 + \ldots + a_n s_n$. An element $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ belongs to the *Taylor spectrum* $\sigma(a;X)$ if $z - a := (z_1 e - a_1, z_2 e - a_2, \ldots, z_n e - a_n)$ is not non-singular.

We recall that (see Taylor [9])

$$\sigma(a; X) \subseteq \mathrm{Sp}(a; \{a_1, \ldots, a_n\}^c) \subseteq \mathrm{Sp}(a; \{a_1, \ldots, a_n\}^{cc})$$

and each of these inclusions is strict in general.

We now formally restate an observation of Taylor, which he used in proving half of the projection property of $\sigma(a; X)$. Given $k \in \{1, \ldots, n\}$, let $\tilde{a}(k) := a_1 s_1 + \ldots + a_k \check{s}_k + \ldots + a_n s_n$, ("$\vee$" means omission of the quantity underneath). There is a direct decomposition of $\wedge^p[s_1, \ldots, s_n; X]$ into a sum of p-forms not involving s_k, i.e. $\wedge^p[s_1, \ldots, \check{s}_k, \ldots, s_n; X]$ and those which all have s_k as a factor. This latter summand may be written as $\tilde{s}_k(\wedge^{p-1}[s_1, \ldots, \check{s}_k, \ldots, s_n; X])$, where $\tilde{s}_k(x \otimes s_{j_1} \wedge \ldots \wedge s_{j_{p-1}}) = x \otimes s_k \wedge s_{j_1} \wedge \ldots \wedge s_{j_{p-1}}$. Consequently, $\wedge^p[s_1, \ldots, s_n; X] = \wedge^p[s_1, \ldots, \check{s}_k, \ldots, s_n; X] \oplus \tilde{s}_k(\wedge^{p-1}[s_1, \ldots, \check{s}_k, \ldots, s_n; X])$. According to this decomposition $\tilde{a}$ may be rewritten as

$$(*) \qquad \tilde{a} = \begin{pmatrix} \tilde{a}(k) & 0 \\ a_k s_k & \tilde{a}(k) \end{pmatrix}.$$

From this representation of $\tilde{a}$ one not only concludes that

$$\pi_k \sigma(a; X) \subset \sigma(a(k); X),$$

where $\pi_k(z_1, \ldots, z_n) := (z_1, \ldots, \check{z}_k, \ldots, z_n)$ denotes the canonical projection, but moreover it will allow us to prove the announced result by induction.

We are now going to consider tensor products of Banach spaces and linear operators.

1.2. Definition. Let X, Y denote two complex Banach spaces, and $X \otimes Y$ their algebraic tensor product. A norm α on $X \otimes Y$ is called

(i) *cross-norm*, provided $\alpha(x \otimes y) = \|x\| \cdot \|y\|$ for all $x \otimes y \in X \otimes Y$;

(ii) *quasi-uniform cross-norm*, if α is a cross-norm and if there exists a constant c such that

$$\alpha((T \otimes S)z) \leqslant c\|T\| \cdot \|S\| \cdot \alpha(z) \quad \text{for all } T \in L(X),\ S \in L(Y),\ z \in X \otimes Y.$$

Remarks. (i) Quasi-uniform cross-norms as introduced by Ichinose [6] are a slight generalization of Schatten's uniform cross-norms [7], where $c = 1$. So $L^2(\mathbf{R}, (\mathbf{C}^n, \|\cdot\|_\infty))$ induces a quasi-uniform but not a uniform cross-norm onto $L^2(\mathbf{R}) \otimes (\mathbf{C}^n, \|\cdot\|_\infty)$. This is easily seen by considering the vector-valued Fourier-transform, which is not an isometry in this situation. Quasi-uniform cross-norms represent some sort of minimal assumptions upon cross-norms, when tensor products of operators are involved. So only quasi-uniform cross-norms shall be considered. We let $X \hat{\otimes}_\alpha Y$

denote the completion of $X\otimes Y$ with respect to α, and $T\hat{\otimes}S$ the unique continuous extension of $T\otimes S$ upon $X\hat{\otimes}_\alpha Y$. Many (quasi-)uniform cross-norms are globally defined in the category of Banach spaces in that they represent a rule how to associate with each pair of Banach spaces (X, Y) a completed tensor product $X\hat{\otimes}_\alpha Y$ with respect to a quasi-uniform cross-norm. If more than two Banach spaces are involved the notion "quasi-uniform cross-norm" will be used only in this more restrictive global sense. Moreover, if X, Y, Z denote three Banach spaces and α a quasi-uniform cross-norm, then $(X\hat{\otimes}_\alpha Y)\hat{\otimes}_\alpha Z$ and $X\hat{\otimes}_\alpha(Y\hat{\otimes}_\alpha Z)$ will represent non-isomorphic Banach spaces in general. Therefore our globally defined quasi--uniform cross-norms are assumed to be associative. Consequently, writing $\widehat{X_1\otimes \ldots \otimes X_n}_\alpha$ means two things: if $n = 2$, then $X_1\hat{\otimes}_\alpha X_2$ is the completion of the tensor product with respect to an individual quasi-uniform cross-norm α which need not be defined on any other tensor product $X\otimes Y$ of Banach spaces X and Y. For $n > 2$, $\widehat{X_1\otimes \ldots \otimes X_n}_\alpha$ is the completion of $X_1\otimes \ldots \otimes X_n$ with respect to a (globally defined) associative quasi-uniform cross-norm α.

(ii) If $T\in L(X)$, and I denotes the identity operator (on Y), then

$$\sigma(T\hat{\otimes}I; X\hat{\otimes}_\alpha Y) = \sigma(T; X).$$

2. MAIN RESULT

2.1. Theorem. *Let $X_1, \ldots, X_n$ denote complex Banach spaces, $T_i \in L(X_i)$ $(1 \leqslant i \leqslant n)$. If $n = 2$ let α denote an (individual) quasi-uniform cross-norm on $X_1\otimes X_2$; if $n > 2$, let α denote an associative (globally defined) quasi-uniform cross--norm. Let $a_i := \widehat{I_1\otimes \ldots \otimes I_{i-1}}\hat{\otimes}T_i\hat{\otimes}\widehat{I_{i+1}\otimes \ldots \otimes I_n} \in L(\widehat{X_1\otimes \ldots \otimes X_n}_\alpha)$, I_i denoting the identity on X_i $(1 \leqslant i \leqslant n)$. Then*

$$(**)\qquad \sigma((a_1, \ldots, a_n); \widehat{X_1\otimes \ldots \otimes X_n}_\alpha) = \prod_{i=1}^{n} \sigma(T_i; X_i).$$

Proof. The inclusion "$\subset$" is obvious. The reverse inclusion will be proved by induction, where the case $n = 2$ (which is true for a strictly larger class of cross--norms than the case $n > 2$) is of course contained in the induction step.

For $n = 1$ nothing has to e shown. So let us assume that $(**)$ holds true for any tensor product of at most $n - 1$ Banach spaces. Let $z = (z_1, \ldots, z_n) \notin$

$\notin \sigma((a_1, \ldots, a_n);\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n)$. We are going to prove that $z \notin \prod_{i=1}^{n} \sigma(T_i; X_i)$. Without loss of generality we may assume that $z = 0$.

We may assume that for $1 \leqslant p \leqslant n-2$ and $1 \leqslant k \leqslant n-p$ we have

$$\underbrace{(0, \ldots, 0)}_{p+1} \in \sigma((a_k, \ldots, a_{k+p});\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n)$$

for otherwise we are done by induction hypotheses.

By the exactness of $F(X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n, (a_1, \ldots, a_n))$ we especially know that

$$(***) \qquad \tilde{a} : \Lambda^{n-1}[s_1, \ldots, s_n;\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n] \to \Lambda^n[s_1, \ldots, s_n;\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n]$$

is a surjection. Consequently at least one T_k is a surjection, for otherwise we find $\varphi_j^{(i)} \in X_i'$ such that $\|\varphi_j^{(i)}\| = 1$, $\|\varphi_j^{(i)} T_i\| \to 0$ as $j \to \infty$ for $1 \leqslant i \leqslant n$, and therefore

$$\varphi_j^{(1)} \widehat{\otimes \ldots \otimes} \varphi_j^{(n)} (a_1 s_1 + \ldots + a_n s_n) \to 0 \quad \text{as } j \to \infty,$$

whereas

$$\alpha'(\varphi_j^{(1)} \widehat{\otimes \ldots} \otimes \varphi_j^{(n)}) \geqslant \|\varphi_j^{(1)}\| \ldots \|\varphi_j^{(n)}\| = 1$$

(α' denoting the dual norm of the cross-norm α). But this contradicts the surjectivity of $(***)$. So suppose that T_k is surjective but not injective. Moreover

$$\underbrace{(0, \ldots, 0)}_{k-1} \in \sigma((a_1, \ldots, a_{k-1});\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n) =$$

$$= \sigma((b_1, \ldots, b_{k-1});\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_{k-1})$$

and

$$\underbrace{(0, \ldots, 0)}_{n-k} \in \sigma((a_{k+1}, \ldots, a_n);\ X_1 \widehat{\otimes_\alpha \ldots} \otimes X_n) =$$

$$= \sigma((c_{k+1}, \ldots, c_n);\ X_{k+1} \widehat{\otimes_\alpha \ldots} \otimes X_n)$$

by induction hypotheses, where b_j and c_j denote the restrictions of a_j to $X_1 \widehat{\otimes \ldots \otimes}_{\alpha} X_{k-1}$ and $X_{k+1} \widehat{\otimes \ldots \otimes}_{\alpha} X_n$, respectively. Let $\tilde{b} = b_1 s_1 + \ldots + b_{k-1} s_{k-1}$ and $\tilde{c} = c_{k+1} s_{k+1} + \ldots + c_n s_n$ be the associated coboundary operators on $\wedge[s_1, \ldots, s_{k-1}; X_1 \widehat{\otimes \ldots \otimes}_{\alpha} X_{k-1}]$ and $\wedge[s_{k+1}, \ldots, s_n; X_{k+1} \widehat{\otimes \ldots \otimes}_{\alpha} X_n]$, respectively. Since α is not symmetric in general, we have to consider two cases:

1° If $k = 1$, then $F(X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n; \tilde{c})$ is not exact and hence there exists a $p \in \mathbf{N} \cup \{0\}$ such that

$$\operatorname{Im}(\tilde{c} : \wedge^p[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n] \to \wedge^{p+1}[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n]) \neq$$

$$\neq \operatorname{Ker}(\tilde{c} : \wedge^{p+1}[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n] \to \wedge^{p+2}[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n])$$

or

$$\neq \operatorname{Ker}(\tilde{c} : \wedge^0[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n] \to \wedge^1[s_2, \ldots, s_n; X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n])$$

is nontrivial. If the latter is true, we find $z \in X_2 \widehat{\otimes \ldots \otimes}_{\alpha} X_n$ such that

$$\sum_{i=2}^{n} c_i(z) s_i = 0, \quad \text{i.e. } c_i(z) = 0 \quad \text{for } 2 \leqslant i \leqslant n.$$

But since T_1 is not one-to-one, let $0 \neq x_1 \in \operatorname{Ker} T_1$. Then

$$\sum_{i=1}^{n} a_i(x_1 \otimes z) \otimes s_i = T_1 x_1 \otimes z \otimes s_1 + \sum_{i=2}^{n} x_1 \otimes c_i(z) \otimes s_i = 0$$

contradicts the exactness of $F(X_1 \widehat{\otimes \ldots \otimes}_{\alpha} X_n; \tilde{a})$. So let $\xi = \sum z_{j_1 \ldots j_{p+1}} \otimes s_{j_1} \wedge \ldots \wedge s_{j_{p+1}} \in \operatorname{Ker} \tilde{c} \setminus \operatorname{Im} \tilde{c}$, and x_1 as above. Then

$$x_1 \otimes \xi \in \operatorname{Ker}(a_2 s_2 + \ldots + a_n s_n) \cap \operatorname{Ker} \tilde{a}.$$

Keeping in mind the direct decomposition of $\wedge^p[s_1, \ldots, s_n; X_1 \widehat{\otimes \ldots \otimes}_{\alpha} X_n]$ and the exactness of $F(\widehat{X_1\otimes \underset{\alpha}{\ldots} \otimes X_n}; \tilde{a})$, we find an element $\xi_0 \in \Lambda^p[s_2, \ldots, s_n; \widehat{X_1\otimes \underset{\alpha}{\ldots} \otimes X_n}]$ such that by means of $(*)$ we have

$$\tilde{a}(1)(\xi_0) = x_1 \otimes \xi.$$

Finally take $\varphi_1 \in X_1'$ such that $\varphi_1(x_1) = 1$, and observe that

$$\varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n} \circ a_i = c_i \circ \varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n} \quad \text{for } 2 \leqslant i \leqslant n.$$

Consequently,

$$\xi = \varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n}(x_1 \otimes \xi) = \varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n} \circ \tilde{a}(1)(\xi_0) =$$

$$= \varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n}(a_2 s_2 + \ldots + a_n s_n)\xi_0 = \tilde{c}\xi_0',$$

where

$$\xi_0' = \varphi_1 \hat{\otimes} \widehat{I_2 \otimes \ldots \otimes I_n}(\xi_0) \in \Lambda^p[s_2, \ldots, s_n; \widehat{X_2 \otimes \underset{\alpha}{\ldots} \otimes X_n}].$$

But this contradicts our assumption $\xi \in \operatorname{Ker} \tilde{c} \setminus \operatorname{Im} \tilde{c}$. We deal analogously with $k = n$, which proves the theorem for $n = 2$.

2° If $1 < k < n$, then $F(\widehat{X_1 \otimes \underset{\alpha}{\ldots} \otimes X_{k-1}}; b)$ and $F(\widehat{X_{k+1} \otimes \underset{\alpha}{\ldots} \otimes X_n}; c)$ are not exact. Let $0 \neq x_k \in \operatorname{Ker} T_k$ and $\varphi_k \in X_k'$ such that $\varphi_k(x_k) = 1$. If there exist $0 \neq z_1 \in \widehat{X_1 \otimes \underset{\alpha}{\ldots} \otimes X_{k-1}}$ and $0 \neq z_2 \in \widehat{X_{k+1} \otimes \underset{\alpha}{\ldots} \otimes X_n}$ such that $\tilde{b}(z_1) = 0$ and $\tilde{c}(z_2) = 0$, then $\tilde{a}(z_1 \otimes x_k \otimes z_2) = 0$, contradicting the exactness of $F(\widehat{X_1 \otimes \underset{\alpha}{\ldots} \otimes X_n}; a)$. Therefore take the forms $[\xi \in \operatorname{Ker} \tilde{b} \setminus \operatorname{Im} \tilde{b}$ and $\eta \in \operatorname{Ker} \tilde{c} \setminus \operatorname{Im} \tilde{c}$ such that $\xi \otimes x_k \otimes \eta \in \Lambda^{p+1}[s_1, \ldots, s_n; \widehat{X_1 \otimes \underset{\alpha}{\ldots} \otimes X_n}]$. Then $\xi \otimes x_k \otimes \eta \in \operatorname{Ker} \tilde{a}(k) \cap \operatorname{Ker} \tilde{a}$.

By the exactness of $F(\widehat{X_1 \otimes \underset{\alpha}{\ldots} \otimes X_n}; a)$ and the decomposition $(*)$ we find $\xi_0 = \xi_{01} + \tilde{s}_k(\xi_{02})$ such that $\xi \otimes x_k \otimes \eta = \tilde{a}(\xi_0) = \tilde{a}(k)(\xi_{01})$. If $Y_{k+1} := \operatorname{Im} \tilde{c} \cap$ $\cap \Lambda^q[s_{k+1}, \ldots, s_n; \widehat{X_{k+1} \otimes \underset{\alpha}{\ldots} \otimes X_n}]$, where $q :=$ degree(η), is not dense in $\operatorname{Ker} \tilde{c} \cap$

$\cap \Lambda^q[s_{k+1}, \ldots, s_n; X_{k+1} \widehat{\otimes}_\alpha \ldots \otimes X_n]$, let $\psi \in Z'_{k+1} \cap Y^\perp_{k+1}$ such that $\psi(\eta) = 1$. As $\xi \otimes x_k \otimes \eta$ is a form of bidegree (r, q) in $\{s_1, \ldots, s_{k-1}\}$ and $\{s_{k+1}, \ldots, s_n\}$, we may assume that $\xi_{01} = \xi_{01}^{(r-1,q)} + \xi_{01}^{(r,q-1)}$ where $\xi_{01}^{(l,m)}$ is a form of bidegree (l, m). Consequently,

$$\tilde{a}(k)(\xi_{01}) = \tilde{b} \otimes I(\xi_{01}^{(r-1,q)}) + I \otimes \tilde{c}(\xi_{01}^{(r,q-1)})$$

and therefore

$$\xi = \xi \varphi_k(x_k) \psi(\eta) = I \otimes \varphi_k \otimes \psi \circ \tilde{a}(k)(\xi_{01}) =$$

$$= \tilde{b}(I \otimes \varphi_k \otimes \psi(\xi_{01}^{(r-1,q)})) + I \otimes \varphi_k \otimes (\psi \circ \tilde{c})(\xi_{01}^{(r,q-1)}) =$$

$$= \tilde{b}(I \otimes \varphi_k \otimes \psi(\xi_{01}^{(r-1,q)})),$$

a contradiction since $\xi \in \operatorname{Ker} \tilde{b} \setminus \operatorname{Im} \tilde{b}$. We proceed in the same way if $Y_{k-1} := \operatorname{Im} \tilde{b} \cap \Lambda^{p+1-q}[s_1, \ldots, s_{k-1}; X_1 \widehat{\otimes}_\alpha \ldots \otimes X_{k-1}]$ is not dense in $Z_{k-1} := \operatorname{Ker} \tilde{b} \cap \Lambda^{p+1-q}[s_1, \ldots, s_{k-1}; X_1 \widehat{\otimes}_\alpha \ldots X_{k-1}]$. Therefore we may assume that there exist sequences $(\xi_j)_{j \in \mathbf{N}}$ and $(\eta_j)_{j \in \mathbf{N}}$ in $\Lambda^{p-q}[s_1, \ldots, s_{k-1}; X_1 \widehat{\otimes}_\alpha \ldots \otimes X_{k-1}]$ and $\Lambda^{q-1}[s_{k+1}, \ldots, s_n; X_{k+1} \widehat{\otimes}_\alpha \ldots \otimes X_n]$, respectively, such that $b(\xi_j) \to \xi$ in Z_{k-1} and $c(\eta_j) \to \eta$ in Z_{k+1} as $j \to \infty$. But then we have

$$(+) \quad \begin{aligned} &\tilde{a}\left(\frac{1}{2} \xi_j \otimes x_k \otimes \tilde{c}(\eta_j) + \frac{1}{2} b(\xi_j) \otimes x_k \otimes \eta_j - \xi_0\right) = \\ &= b(\xi_j) \otimes x_k \otimes c(\eta_j) - \xi \otimes x_k \otimes \eta \to 0 \quad \text{as } j \to \infty. \end{aligned}$$

As $F(X_1 \widehat{\otimes}_\alpha \ldots \otimes X_n; a)$ is exact,

$$\tilde{a}\colon \Lambda^p[s_1, \ldots, s_n; X_1 \widehat{\otimes}_\alpha \ldots \otimes X_n] \to \Lambda^{p+1}[\ldots]$$

is a topological homomorphism onto its image. Therefore (+) and the exactness of $F(X_1 \widehat{\otimes}_\alpha \ldots \otimes X_n; a)$ give the existence of a sequence (γ_j) in $\Lambda^{p-1}[s_1, \ldots, s_n;$

$X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n]$ such that

$$\tilde{a}\left(\frac{1}{2}\xi_j\otimes x_k\otimes\eta_j - \gamma_j\right) - \xi_0 = \frac{1}{2}b(\xi_j)\otimes x_k\otimes\eta_j + \frac{1}{2}\xi_j\otimes x_k\otimes c(\eta_j) -$$

$$- \tilde{a}(\gamma_j) - \xi_0 \to 0 \quad \text{as } j \to \infty.$$

Hence ξ_0 is in the closure of $\operatorname{Im}\tilde{a} \cap \wedge^p[\ldots]$. Hence $0 = \tilde{a}(\xi_0) = \xi\otimes x_k\otimes\eta \neq 0$, a contradiction which proves the theorem.

From the general property of Taylor's joint spectrum

$$\sigma(a; X) \subseteq \mathrm{Sp}(a; \{a_1, \ldots, a_n\}^c) \subseteq \mathrm{Sp}(a; \{a_1, \ldots, a_n\}^{cc})$$

we especially obtain the result of Dash and Schechter [3] that

$$\mathrm{Sp}((a_1, \ldots, a_n); \{a_1, \ldots, a_n\}^{cc}) = \prod_{i=1}^{n} \sigma(T_i; X_i),$$

where we used the notation of Theorem 2.1.

In the special framework of Hilbert spaces and the Hilbert space tensor product norm Theorem 2.1 is due to Vasilescu [12] ($n = 2$) and Ceaușescu and Vasilescu [2] ($n \geqslant 2$). Their proofs heavily rely on the Hilbert space structure, because they use a characterization of Taylor's joint spectrum for Hilbert space operators due to Vasilescu [11].

Moreover we have the following

2.2. COROLLARY. *Using conditions and notation of Theorem 2.1, let $\mathcal{O}(a_1, \ldots, a_n)$ denote the algebra of germs of functions which are analytic in an open neighborhood of $\sigma((a_1, \ldots, a_n); X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n)$ provided with its inductive limit topology. Then*

$$(****) \quad \hat{f}(a_1, \ldots, a_n) = (2\pi i)^{-n} \int_{\Gamma_1} \ldots \int_{\Gamma_n} f(z_1, \ldots, z_n)(z_1 - a_1)^{-1} \widehat{\otimes \ldots \otimes} (z_n - a_n)^{-1} dz_1 \ldots dz_n$$

where f is a representative of $\hat{f}$ analytic in an open neighborhood $G_1 \times \ldots \times G_n$ of $\sigma((a_1, \ldots, a_n); X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n)$ and $\Gamma_i \subset G_i$ $(1 \leqslant i \leqslant n)$ denote Jordan curves *surrounding $\sigma(T_i\,;X_i)$ counterclockwise, defines a continuous algebra homomorphism from $\mathcal{O}(a_1,\ldots,a_n)$ into $\{a_1,\ldots,a_n\}^{cc}$. Moreover,*

$$\sigma(\hat{f}(a_1,\ldots,a_n): X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n) = f\left(\prod_{i=1}^{n} \sigma(T_i\,;X_i)\right).$$

REMARK. Corollary 2.2 is due to Ichinose [5, Theorems 2.1, 2.4]. The spectral mapping theorem was proved by Schechter [8] for polynomials, and by Brown and Pearcy [1] for the special case $f(z_1, z_2) = z_1 z_2$ in the Hilbert space framework. All these proofs of the spectral mapping theorem did not use Theorem 2.1 or the result of Dash and Schechter [3]. So in a certain sense they hide the reason, why they are true. On the other hand using Theorem 2.1 or the Dash and Schechter result and Gelfand theory in the commutative Banach algebra $\{a_1,\ldots,a_n\}^{cc}$ one obtains an elegant proof of the spectral mapping theorem part of Corollary 2.2 via the representation (∗∗∗∗).

Finally, 2.2 also is an immediate consequence of Taylor's functional calculus [10]. But these proofs both do not seem to be simple. For that reason we offer an alternative proof which is simple.

Proof of 2.2. It is simple that (∗∗∗∗) defines a continuous algebra homomorphism and thus

$$\sigma(\hat{f}(a_1,\ldots,a_n); X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n) \subseteq \hat{f}\left(\prod_{i=1}^{n} \sigma(T_i\,;X_i)\right)$$

is true. In order to prove the other inclusion, we use the fact that given $(t_1,\ldots,t_n) \in \prod_{i=1}^{n} \sigma(T_i\,;X_i)$, and f analytic in an open neighborhood $G_1 \times \ldots \times G_n$ of this set we have $f(z_1,\ldots,z_n) = f(t_1,\ldots,t_n) + (z_1 - t_1)g_1(z_1,\ldots,z_n) + \ldots + (z_n - t_n)g_n(z_1,\ldots,z_n)$ with suitable functions $g_1,\ldots,g_n$ being analytic in $G_1 \times \ldots \times G_n$, and being defined in a rather obvious way as "difference quotients". So if $f(t_1,\ldots,t_n)$ were not in $\sigma(f(a_1,\ldots,a_n); X_1 \widehat{\otimes \ldots \otimes}_\alpha X_n)$, there exists the inverse $b := (f(t_1,\ldots,t_n) - f(a_1,\ldots,a_n))^{-1}$ and consequently

$$I_1 \widehat{\otimes \ldots \otimes} I_n = (t_1 - a_1)g_1(a_1,\ldots,a_n)b + \ldots + (t_n - a_n)g_n(a_1,\ldots,a_n)b$$

means that $(t_1,\ldots,t_n) \notin \mathrm{Sp}((a_1,\ldots,a_n); \{a_1,\ldots,a_n\}^{cc})$. Using 2.1 or the Dash and Schechter result we have a contradiction. This proves the spectral mapping theorem.

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Note added in proof. In a forthcoming paper "Tensor products of linear operators in Banach spaces and Taylor's joint spectrum. II" we succeeded in proving Theorem 2.1 for general (non-associative) quasi-uniform cross-norms.