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TWISTED COBOUNDARY OPERATOR ON A TREE AND THE SELBERG PRINCIPLE

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0. INTRODUCTION

In this paper, we show that the methods of non commutative differential geometry [4] can provide a very simple proof of the Selberg Principle for supercuspidal representations of split-rank 1 simple algebraic groups over non-archimedian local fields. Namely, if G is such a group, e an idempotent of the convolution algebra $\mathscr{A} = C_c^\infty(G)$ of locally constant compactly supported functions on G (e.g. e coefficient of a supercuspidal representation), and if C is a hyperbolic conjugacy class in G, then:

$$\int_C e(\mathring{g})\,d\mathring{g} = 0.$$

The main tool is the construction of 1-summable Fredholm modules (cf. [4]) associated to any simplicial action of a locally compact group G on a *tree*. In [10], [11] we introduced such a Fredholm module γ over $\mathscr{A} = C_c^\infty(G)$ by a construction involving elementary geometry of trees. As noted in [11], p. 214, the Chern character of γ is the trace on $\mathscr{A}$ given by the central function on G which is 0 on the hyperbolic elements (i.e. without fixed points on the tree) and 1 on the others. In order to reach the Selberg Principle, we clearly need more sophisticated Fredholm modules, whose characters will involve the hyperbolic conjugacy classes as well.

To do this, we construct in § 2 a one-parameter family $(\gamma_t)_{t\in]t_0,\infty[}$ of 1-summable Fredholm modules over $\mathscr{A}$ by making use of a *discrete analogue of Witten's idea* in his proof of the Morse inequalities [18]. This idea is as follows. Let Δ^0 (resp. Δ^1) be the set of vertices (resp. edges) of the tree. The orientation allows one to define the simplicial co-boundary operator $d\colon \ell^2(\Delta^0) \to \ell^2(\Delta^1)$, which commutes with the natural representations of G on $\ell^2(\Delta^0)$ and $\ell^2(\Delta^1)$. If the tree is uniformly locally finite (we will assume that), d is a bounded operator, but it is not in general a Fredholm operator (as the example of the tree of $\mathbf{Z}$ already shows). However, it is possible as in [18], to conjugate d by $e^{t\rho}$, where ρ is a suitable function on Δ^0, and t is a non-negative parameter. We will take for ρ the distance to some fixed origin x_0; we can view ρ as a "Morse function" on a tree, with a unique critical point at x_0. So we will work with the twisted operator $d_t = e^{-t\rho}de^{t\rho}$; making t big enough, one creates a hole in the spectrum of the twisted laplacian $d_t^* d_t$, and 0 becomes an isolated eigenvalue. In particular, d_t is a Fredholm operator. Of course, d_t does not commute anymore with the representations of G on $\ell^2(\Delta^0)$ and $\ell^2(\Delta^1)$, but it commutes modulo finite rank operators, defining a 1-summable Fredholm module γ_t over $\mathscr{A}$; letting t tend to infinity, we will obtain an operatorial homotopy between γ_t and γ, thus defining a unique class in $KK_G(\mathbf{C}, \mathbf{C})$ (cf. [12]).

In § 3, we compute the Chern character of γ_t (in the sense of [4], I, §1). It is very simply given by

$$\tau_t(a) = \int_G a(g) e^{-tp(g)} \, dg \quad (a \in \mathscr{A})$$

where $p(g)$ is the minimal distance between a vertex x and its image gx. The τ_t's define the same class in the periodized cyclic cohomology group $H^*(\mathscr{A})$, hence the same map $K_0(\mathscr{A}) \to \mathbf{Z}$.

More generally, one can also consider, for any central distribution χ on G, the trace $\tau_t\chi$ on $\mathscr{A}$ given by:

$$\tau_t\chi(a) = \int_G a(g) e^{-tp(g)} \chi(g) \, dg \quad (a \in \mathscr{A}).$$

We show in § 4 that the class of $\tau_t\chi$ in the periodized cyclic cohomology of $\mathscr{A}$ is independent of t. If χ is the characteristic distribution of a hyperbolic conjugacy class in G, we have $\tau_t\chi = e^{-tn}\chi$ for some $n \geqslant 1$. Hence $\chi = 0$ in $H^*(\mathscr{A})$, which implies the Selberg Principle.

Our § 5 has a slightly different aim. We show how our familly $(\gamma_t)_{t \in]t_0, \infty[}$ makes now clear the meaning of the homotopy of [10], [11] which was the technical part of our proof of the K-amenability of G: the idea is that the use of a family of kernels of positive type on Δ^0 allows us to continue the γ_t's for all $t \in [0, \infty]$.

Note that the above formula for τ_t clearly defines a trace on $\mathscr{A}$, not only for t big enough, but for any non-negative t. In § 5, we use one of the constructions in [10] to show that, for any t, the trace τ_t is the Chern character of a 1-summable Fredholm module over $\mathscr{A}$, which, for t big, is very simply related to our family γ_t. In particular, any τ_t pairs with K_0 of the full C^*-algebra $C^*(G)$ and they all define the same map $K_0(C^*(G)) \to \mathbf{Z}$. Actually, it was already observed in [11] that τ_0 (which corresponds to the trivial Fredholm module $1_G = (\mathbf{C}, 0, 0)$) and τ_∞ (which corresponds to our element γ) define the same pairing with $K_0(C^*(G))$. However the proof given here is more natural, and more in the spirit of cyclic cohomology; it was P. Baum who suggested us to look for such a direct proof.

Finally, we notice that, for *discrete groups*, a computation of $H^*(\mathscr{A})$ is now available (see [1]; also [14] for free groups); in many cases the equality $[\tau_t] = [\tau_s]$ in $H^*(\mathscr{A})$ follows easily from these computations.

The origin of our paper is as follows: soon after the publication of [10], looking for a better understanding of the proofs in it, and for possible generalizations, and having in mind the analogy with Kasparov's element γ [12], we made use of Witten's idea to construct our γ_t's. Later, we computed their Chern character and found the link with the homotopy of [10]. It is only after discussions with A. Connes that we understood that our results implied the Selberg Principle; he had been long aware of the link between the Selberg Principle and bivariant cyclic cohomology. We would like to thank him for having communicated to us these ideas, and also for having contributed, together with G. Skandalis, to the simplifications of some proofs in §§ 2 and 3. We are also very grateful to P. Cartier who pointed out to us a mistake in a preliminary version of this work.

1. NOTATIONS AND TERMINOLOGY

A *tree* is a connected, simply connected, oriented simplicial complex of dimension 1. The orientation allows one to define the *extremity* e(b) (resp. the *origin* o(b)) of an edge b. The natural *distance* on Δ^0 will be denoted by $\delta(\cdot,\cdot)$. We will assume that the tree is *uniformly locally finite*, meaning that the degrees of the vertices are uniformly bounded by some integer N.

We denote by $(\delta_x)_{x\in\Delta^0}$ (resp. $(\delta_b)_{b\in\Delta^1}$) the canonical basis of $\ell^2(\Delta^0)$ (resp. $\ell^2(\Delta^1)$).

For any function f on Δ^0, and any edge b, we define

$$Rf(b) = f(\mathrm{e}(b)) \qquad Sf(b) = f(\mathrm{o}(b)).$$

The simplicial coboundary operator d is defined by $d = R - S$.

LEMMA 1. *R, S and b are bounded as operators $\ell^2(\Delta^0) \to \ell^2(\Delta^1)$.*

Proof. For $f \in \ell^2(\Delta^0)$, we have:

$$\|Rf\|^2 = \sum_{b\in\Delta^1} |f(\mathrm{e}(b))|^2 = \sum_{x\in\Delta^0}\ \sum_{b:e(b)=x} |f(x)|^2 \leqslant N\|f\|^2.$$

A similar estimate holds for S.

If f (resp. ξ) is a function on Δ^0 (resp. Δ^1), we define a function $f\cdot\xi$ on Δ^1 by

$$(f\cdot\xi)(b) = f(\mathrm{e}(b))\cdot\xi(b).$$

Functions on Δ^1 form a discrete analogue of 1-forms on a manifold. Note the Leibniz' rule:

$$d(fg)(b) = (f\cdot dg)(b) + df(b)g(o(b)).$$

In particular, for any function φ on Δ^0

$$\text{(1)} \qquad e^{-\varphi}\cdot d(e^{\varphi}f)(b) = df(b) + (1 - e^{-d\varphi(b)})f(o(b)) = Rf(b) - e^{-d\varphi(b)}Sf(b)$$

which shows that the operator $e^{-\varphi}\cdot d(e^{\varphi}\cdot): \ell^2(\Delta^0) \to \ell^2(\Delta^1)$ is bounded as soon as $d\varphi$ is bounded.

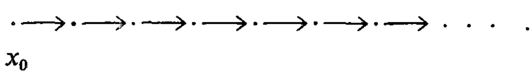
Let x_0 be some origin on Δ^0. We define

$$\rho = \delta(x_0, \cdot); \quad \varepsilon = d\rho.$$

Notice that ε only takes values ± 1. We denote by $[x, y]$ the unique geodesic between the vertices x and y. The operator q of *integration on geodesics* is defined on a function ξ on Δ^1 by

$$q\xi(x) = \sum_{b \subset [x_0, x]} \varepsilon(b)\,\xi(b) \quad (x \in \Delta^0).$$

As an operator $\ell^2(\Delta^1) \to \ell^2(\Delta^0)$, q is in general unbounded; consider for instance the tree



An elementary computation yields:

LEMMA 2. i) *For any function* ξ *on* Δ^1: $(dq)\xi = \xi$.
ii) *For any function* f *on* Δ^0, *and any vertex* x:

$$(qd)f(x) = f(x) - f(x_0).$$

These formulae are a discrete analogue of Poincaré's lemma.

2. THE 1-SUMMABLE FREDHOLM MODULES γ_t

Let t be a real, non-negative parameter. We define the *twisted coboundary operator* d_t:

$$d_t = e^{-t\rho}\cdot d(e^{t\rho}\cdot) = R - e^{-t\varepsilon}S.$$

By the remark following (1), $d_t: \ell^2(\Delta^0) \to \ell^2(\Delta^1)$ is a bounded operator. We also define

$$q_t = e^{-t\rho} q(e^{t\rho} \cdot).$$

Note the explicit formula on a function ξ on Δ^1:

$$(2) \qquad q_t\xi(x) = \sum_{b \subset [x_0, x]} \varepsilon(b) e^{-t\delta(e(b), x)} \xi(b) \quad (x \in \Delta^0).$$

By Lemma 2, we have

$$(3) \qquad d_t q_t = 1$$

and, for any function f on Δ^0:

$$(q_t d_t) f(x) = f(x) - e^{-t\rho(x)} f(x_0)$$

hence

$$(4) \qquad q_t d_t = 1 - e^{-t\rho} \langle \cdot, \delta_{x_0} \rangle.$$

Our aim now is to prove that, for t big enough, d_t is a Fredholm operator of index 1. Assume we know that q_t is bounded for t big enough; then formulae (3) and (4) show that q_t is a parametrix for d_t, and that d_t has the right index. We first make sure that $e^{-t\rho}$ is a ℓ^2-function for t big enough.

LEMMA 3. *Define* $C(t) = \dfrac{1 + e^{-t}}{1 - e^{-t}(N-1)}$ *; for* $t > \log(N-1)$, *one has*

$$\sum_{x \in \Delta^0} e^{-t\rho(x)} \leqslant C(t).$$

Proof.

$$\sum_{x \in \Delta^0} e^{-t\rho(x)} = 1 + \sum_{n=1}^{\infty} \sum_{x:\, \rho(x) = n} e^{-tn}.$$

Since there are at most $N(N-1)^{n-1}$ vertices at distance n from x_0, this is bounded above by

$$1 + \sum_{n=1}^{\infty} N(N-1)^{n-1} e^{-tn}$$

and this geometric series converges to $C(t)$ for $t > \log(N-1)$. This proof was inspired by Corollary 3.2 in [8].

PROPOSITION 1. *For* $t > \log(N-1)$, *the operator* $d_t: \ell^2(\Delta^0) \to \ell^2(\Delta^1)$ *is Fredholm with index* 1.

Proof. The remarks preceding Lemma 3 show that the point is to prove that, for $t > \log(N-1)$, the operator q_t is bounded. Now q_t is given by a kernel $k(\cdot,\cdot)$ on $\Delta^0 \times \Delta^1$, with

$$k(x,b) = \begin{cases} \varepsilon(b)e^{-t\delta(x,e(b))} & \text{if } b \subset [x_0, x] \\ 0 & \text{if not.} \end{cases}$$

Then

$$\sup_{x\in\Delta^0} \sum_{b\in\Delta^1} |k(x,b)| = \sup_{x\in\Delta^0} \sum_{b\subset[x_0,x]} e^{-t\delta(x,e(b))} \leqslant$$

$$\leqslant \sup_{x\in\Delta^0} \frac{1-e^{-t\rho(x)}}{1-e^{-t}} \leqslant (1-e^{-t})^{-1}.$$

This shows that $q_t : \ell^\infty(\Delta^1) \to \ell^\infty(\Delta^0)$ is a bounded operator. Moreover:

$$\sup_{b\in\Delta^1} \sum_{x\in\Delta^0} |k(x,b)| \leqslant \sup_{b\in\Delta^1} \sum_{x:\, b\subset[x_0,x]} e^{-t\delta(x,e(b))} \leqslant (1 - e^{-t}(N-1))^{-1}$$

as in Lemma 3.

So $q_t : \ell^1(\Delta^1) \to \ell^1(\Delta^0)$ is also a bounded operator. Applying the Riesz-Thorin interpolation theorem (see [15], Theorem IX. 17), we see that q_t is bounded as an operator $\ell^2(\Delta^1) \to \ell^2(\Delta^0)$, with explicit bound:

$$\|q_t\| \leqslant [(1-e^{-t})(1-e^{-t}(N-1))]^{1/2}.$$

COROLLARY 1. *Let G be a locally compact group acting simplicially on the tree. For* $t > \log(N-1)$, *the element* $\gamma_t = (\ell^2(\Delta^0), \ell^2(\Delta^1), d_t)$ *is a 1-summable Fredholm module on* $\mathscr{A}$.

Proof. Let R_0 (resp. R_1) be the natural representation of G on $\ell^2(\Delta^0)$ (resp. $\ell^2(\Delta^1)$). For f in $\ell^2(\Delta^0)$, g in G, b in Δ^1, we have by (1):

$$(d_t R_0(g) - R_1(g)d_t)f(b) = (e^{-t\varepsilon(g^{-1}b)} - e^{-t\varepsilon(b)})f(o(g^{-1}b)). \tag{5}$$

But Lemma 1.4 in [11] shows that $\varepsilon(g^{-1}b) = \varepsilon(b)$ unless b lies on $[x_0, gx_0]$. So $d_t R_0(g) - R_1(g)d_t$ has rank at most $\rho(gx_0)$. On the other hand, it is clear that the map

$$g \to \mathrm{Tr}(R_0(g) - q_t R_1(g) d_t)$$

is locally constant, hence continuous on G. So, for $a \in \mathscr{A}$, the operator $R_0(a) - q_t R_1(a) d_t$ is trace class. This concludes the proof.

Since $t \to d_t$ is a norm continuous map, all γ_t's are operatorially homotopic; so it is tempting to let t tend to ∞. However, we have in general

$$\lim_{t\to\infty} \|d_t\| = \infty$$

but it is possible to modify d_t by some operator $D^t: \ell^2(\Delta^1) \to \ell^2(\Delta^1)$ behaving approximately like $(d_t d_t^*)^{1/2}$ (positive square root of the twisted Laplacian on Δ^1); then $D^{-t} d_t$ will tend to the 1-summable Fredholm module γ of [10], [11].

First of all, we recall the definition of γ. A bijection $\beta: \Delta^0 \setminus \{x_0\} \to \Delta^1$ is defined by mapping a vertex x to the unique edge through x lying on $[x_0, x]$. We define $U: \ell^2(\Delta^0) \to \ell^2(\Delta^1)$ by:

$$U\delta_x = \varepsilon(\beta(x))\delta_{\beta(x)} \quad \text{if } x \neq x_0$$

$$U\delta_{x_0} = 0.$$

For a function f in $\ell^2(\Delta^0)$, this amounts to:

$$Uf(b) = \begin{cases} f(\mathrm{e}(b)) & \text{if } \varepsilon(b) = 1 \\ -f(\mathrm{o}(b)) & \text{if } \varepsilon(b) = -1. \end{cases}$$

By Corollary 1.5 in [11], U is a coisometry of index 1, and $\gamma = (\ell^2(\Delta^0), \ell^2(\Delta^1), U)$ is a 1-summable Fredholm module over $\mathscr{A}$. Define now a strictly positive diagonal operator $D: \ell^2(\Delta^1) \to \ell^2(\Delta^1)$ by

$$D\xi(b) = \mathrm{e}^{\frac{1}{2}(1-\varepsilon(b))}\xi(b).$$

LEMMA 4. $\lim_{t\to\infty} \|D^{-t}d_t - U\| = 0.$

Proof. Fix some f in $\ell^2(\Delta^0)$. Using (1), we have:

$$\|(D^{-t}d_t - U)f\|^2 = \sum_{b\in\Delta^1} |\mathrm{e}^{-\frac{t}{2}(1-\varepsilon(b))} Rf(b) - \mathrm{e}^{-\frac{t}{2}(1+\varepsilon(b))} Sf(b) - Uf(b)|^2 =$$

$$= \sum_{b:\,\varepsilon(b)=1} \mathrm{e}^{-2t}|Sf(b)|^2 + \sum_{b:\,\varepsilon(b)=-1} \mathrm{e}^{-2t}|Rf(b)|^2 \leqslant$$

$$\leqslant \mathrm{e}^{-2t}(\|Rf\|^2 + \|Sf\|^2) \leqslant 2N\mathrm{e}^{-2t}\|f\|^2$$

by the proof of Lemma 1. This concludes the proof.

PROPOSITION 2. *The 1-summable Fredholm module γ_t $(t > \log(N-1))$ is operatorially homotopic to γ.*

Proof. The almost-G-invariance of ε (already used in the proof of Corollary 1) shows that $D^{-t}d_t$ defines a 1-summable Fredholm module over $\mathscr{A}$, which is operatorially homotopic to γ_t by $s \to D^{-st}d_t$ $(s \in [0,1])$. Then Lemma 4 shows that $(\ell^2(\Delta^0), \ell^2(\Delta^1), D^{-t}d_t)$ is operatorially homotopic to γ.

3. COMPUTATION OF TRACES

In this section we compute the Chern character (in the sense of [4], §1) of the 1-summable Fredholm module γ_t $(t > \log(N-1))$. It is defined by

$$\tau_t(a) = \mathrm{Tr}(R_0(a) - q_t R_1(a) d_t)$$

where Tr denotes the usual trace. Let us introduce the measurable function on G:

$$\tau_t(g) = \mathrm{Tr}(R_0(g) - q_t R_1(g) d_t)$$

(which is well-defined since $R_0(g) - q_t R_1(g) d_t$ has finite rank). Clearly

$$\tau_t(a) = \int_G a(g) \tau_t(g) \, \mathrm{d}g.$$

Now the functional τ_t on $\mathscr{A}$ is a trace ([4], Lemma 1), meaning that the function τ_t on G is central. This shows that τ_t, which a priori depends on the choice of the origin, does not change when x_0 is replaced by gx_0. Actually much more is true, as the following lemma indicates:

LEMMA 5. *τ_t does not depend on the choice of the origin x_0.*

Proof. Choose another vertex x_0', and denote by d_t', q_t' the analogues of d_t, q_t, defined with respect to x_0'. We have to show that, for $a \in \mathscr{A}$:

$$\mathrm{Tr}(q_t' R_1(a) d_t' - q_t R_1(a) d_t) = 0.$$

Introduce the operators

$$X = d_t' - d_t, \quad Y = q_t' - q_t.$$

As in §2, Corollary 1, we show that X is finite rank. Indeed, for $f \in \ell^2(\Delta^0)$, one has

$$Xf(b) = (\mathrm{e}^{-t\varepsilon(b)} - \mathrm{e}^{-t\varepsilon'(b)}) f(\mathrm{o}(b)).$$

Then by (3) and (4) one has $d_t'q_t' = d_tq_t = 1$ and $q_td_t - 1$, $q_t'd_t' - 1$ are rank one. From that we see that $Y = q_t' - q_t$ is also finite rank and that one has:

$$d_tY + Xq_t + XY = 0.$$

On the other hand:

$$q_t'R_1(a)d_t' - q_tR_1(a)d_t = YR_1(a)d_t + q_tR_1(a)X + YR_1(a)X.$$

Then by the trace property (since X and Y are trace class)

$$\mathrm{Tr}(q_t'R_1(a)d_t' - q_tR_1(a)d_t) = \mathrm{Tr}((d_tY + Xq_t + XY)R_1(a)) = 0.$$

This concludes the proof.

It remains to compute the function $g \to \tau_t(g)$. Remember that we defined

$$p(g) = \inf_{x \in \Delta^0} \delta(x, gx).$$

Obviously p is a central function on G, moreover it is a locally constant function; indeed, for $g \in G$, let $x \in \Delta^0$ be such that

$$p(g) = \delta(x, gx).$$

Let H be the stabilizer of the geodesic $[x, gx]$; it is an open subgroup of G; then, for $h \in H$: $p(gh) = p(g)$. We say that $g \in G$ is *hyperbolic* if $p(g) \neq 0$.

PROPOSITION 3. *For any* $g \in G$, *we have* $\tau_t(g) = \mathrm{e}^{-tp(g)}$.

Proof. Let P be set of vertices x such that $\delta(x, gx) = p(g)$. If $p(g) = 0$, then P is the fixed point set of g. If g is hiperbolic, by [16], Proposition 24, P is a subtree isomorphic to the tree of $\mathbf{Z}$, and g acts by translation of $p(g)$ along P; in this case we shall say that P is the *axis* of g. In any case, we may assume, by Lemma 5, that x_0 belongs to P. By formula (4), we have:

$$R_0(g) - q_tR_1(g)d_t = (\mathrm{e}^{-tp}\langle \cdot, \delta_{x_0}\rangle)R_0(g) + q_t(d_tR_0(g) - R_1(g)d_t)$$

so that

$$\mathrm{Tr}(R_0(g) - q_tR_1(g)d_t) = \mathrm{e}^{-tp(g)} + \mathrm{Tr}(q_t(d_tR_0(g) - R_1(g)d_t))$$

and we have to show that the final term is zero. But, by (5) and the proof of Corollary 1, we have for any $x \in \Delta^0$:

$$(d_tR_0(g) - R_1(g)d_t)\delta_x = \sum_{\substack{b \subset [x_0, gx_0] \\ gx = o(b)}} (\mathrm{e}^{-t\varepsilon(g^{-1}b)} - \mathrm{e}^{-t\varepsilon(b)})\delta_b$$

hence

$$q_t(d_tR_0(g) - R_1(g)d_t)\delta_x = \sum_{\substack{b \subset [x_0, gx_0] \\ gx = o(b)}} \sum_{y:\, b \subset [x_0, y]} \varepsilon(b)(\mathrm{e}^{-t\varepsilon(g^{-1}b)} - \mathrm{e}^{-t\varepsilon(b)})\mathrm{e}^{-t\delta(e(b), y)}\,\delta_y.$$

So, for the scalar product $\langle q_t(d_t R_0(g) - R_1(g)d_t)\delta_x, \delta_x\rangle$ to be non-zero, we must have simultaneously $b \subset [x_0, gx_0] \cap [x_0, x]$ and $gx = o(b)$. Since x_0 belongs to P, this is impossible, and we get the desired conclusion.

REMARK. The Chern character of the 1-summable Fredholm module $(\ell^2(\Delta^0), \ell^2(\Delta^1), D^{-t}d_t)$ is equal to τ_t. Indeed, by (3), we have

$$\mathrm{Tr}(R_0(g) - q_t D^t R_1(g) D^{-t} d_t) = \mathrm{Tr}(R_0(g) - q_t R_1(g) d_t) + \mathrm{Tr}(R_1(g) - D^t R_1(g) D^{-t})$$

but since

$$(R_1(g) - D^t R_1(g) D^{-t})\delta_b = (1 - e^{-\frac{t}{2}(\varepsilon(gb)-\varepsilon(b))})\delta_{gb}$$

it is clear that

$$\mathrm{Tr}(R_1(g) - D^t R_1(g) D^{-t}) = 0.$$

This remark and Lemma 4 show that the Chern character τ_∞ of the 1-summable Fredholm module γ is given on $a \in \mathscr{A}$ by the formula:

$$\tau_\infty(a) = \mathrm{Tr}(R_0(a) - U^* R_1(a) U) = \int_{p^{-1}(0)} a(g)\,\mathrm{d}g\,.$$

This can also be checked by direct computation (cf. [11]).

COROLLARY 2. *The family of traces τ_t $(t \in]\log(N-1), \infty])$ defines a unique pairing with the K-theory group $K_0(\mathscr{A})$. This pairing maps $K_0(\mathscr{A})$ to the integers.*

Proof. Since τ_t is the Chern character of γ_t, the pairing of τ_t with $K_0(\mathscr{A})$ maps $K_0(\mathscr{A})$ to the integers ([4], § 1, Lemma 1). Moreover, since the family $(\ell^2(\Delta^0), \ell^2(\Delta^1), D^{-t}d_t)$ is norm continuous, it defines the same class in the cyclic cohomology group $H^0(\mathscr{A})$, hence the same map $K_0(\mathscr{A}) \to \mathbf{C}$ (see [4], § 5, Corollary 3). A more direct but less general argument is the following one: if $[e]$ is the K-theory class of some idempotent in some matrix algebra over $\mathscr{A}$, the map $t \to \tau_t^*[e]$ is integer-valued and continuous in t; so it is constant.

4. THE SELBERG PRINCIPLE

In order to reach the Selberg principle, we will need to treat τ_t not only as a trace on $\mathscr{A}$, but also as a locally constant central function on G, acting by multiplication on central distributions. If χ is a central distribution on G (i.e. a linear functional on $\mathscr{A}$ invariant under conjugacy or, to put it more simply, a trace on $\mathscr{A}$), we define another central distribution $\tau_t\chi$ by

$$\tau_t\chi(a) = \int_G \tau_t(g)a(g)\chi(g)\,\mathrm{d}g \quad (a \in \mathscr{A}).$$

A central distribution χ defines an element $[\chi]$ in the cyclic cohomology group $H^0_\lambda(\mathscr{A})$. Our aim in this section is to prove the following

THEOREM. *The family* $[\tau_t\chi]$ *(*$t > \log(N-1)$*) defines a constant element in the periodized cyclic cohomology* $H^0(\mathscr{A}) = \varinjlim_S H^{2n}_\lambda(\mathscr{A})$ *(where S is Connes' suspension operator* [4], Chapter I, §4*).*

The idea underlying the proof of this result is that the central function τ_t defines an element of the bivariant cyclic cohomology $\mathrm{Ext}^0_\Lambda(\mathscr{A}^\natural, \mathscr{A}^\natural)$ (in the sense of Connes [4]); moreover, τ_t should be the "bivariant Chern character" of the Kasparov element $\gamma_t \in \mathrm{KK}_G(\mathbf{C}, \mathbf{C})$ or, more precisely, of the element $j_G(\gamma_t) \in \mathrm{KK}(C^*(G), C^*(G))$ (see [12] for definitions and notations); this idea will appear clearly in Lemma 6.

Before embarking upon the proof, we give corollaries of our theorem.

COROLLARY 3. *Assume that the support of* χ *is contained in the set of hyperbolic elements of G. Then* $[\chi] = 0$ *in* $H^0(\mathscr{A})$.

Proof. For $n \in \mathbf{N}$, denote by χ_n the central distribution on G defined by

$$\chi_n(a) = \int_{p^{-1}(n)} a(g)\chi(g)\,\mathrm{d}g.$$

Since

$$\chi = \sum_{n=1}^{\infty} \chi_n$$

it suffices to show that $[\chi_n] = 0$ for $n \geq 1$. But

$$\tau_t\chi_n = \mathrm{e}^{-tn}\chi_n$$

and its cohomology class is constant, by the theorem. The result follows.

Since the pairing with $\mathrm{K}_0(\mathscr{A})$ is left unchanged by S ([4], Chapter II, Corollary 17) we immediately get the following result, which deserves to be called the *abstract Selberg principle*:

COROLLARY 4. *Let χ be a central distribution on G, supported in the hyperbolic elements, and let e be an idempotent in $\mathscr{A}$. Then*

$$\int_G e(g)\chi(g)\,dg = 0.$$

Now, let G be (the group of rational points of) a split-rank 1 simple algebraic group defined over some non-archimedean local field F (e.g. $G = SL_2(F)$). We recall from [17] that G acts on a certain tree, special case of Bruhat-Tits building, such that the stabilizers of the vertices are precisely the maximal compact subgroups of G. In particular, a conjugacy class C is hyperbolic if and only if it meets no compact subgroup. Now, orbital integrals over such a C do converge ([9], Lemma 19), hence $a \to \int_C a(\mathring{g})\,d\mathring{g}$ defines a central distribution on $\mathscr{A}$. On the other hand, coefficient-functions of supercuspidal representations of G define idempotents in $\mathscr{A}$, up to a multiplicative constant ([2], Theorem 1.1). From Corollary 4, we get the *Selberg principle for G:*

COROLLARY 5. *Let G, C be as above, and let R be a supercuspidal representation of G. For any $\xi \in V_R$, $\tilde{\xi} \in V_R^*$ (the dual of V_R):*

$$\int_C \langle R(\mathring{g})\xi, \tilde{\xi}\rangle\,d\mathring{g} = 0.$$

To the best of our knowledge, the first proof of the Selberg principle for general reductive p-adic groups was given by Harish-Chandra ([9], Lemma 45).

We now turn to the proof of the theorem. Let G be a locally compact group acting on a tree, let χ be a central distribution on G, and let $\mathscr{H}$ be a Hilbert space. We consider the right $\mathscr{A}$-module

$$\mathscr{E} = \mathscr{H} \otimes \mathscr{A}$$

(algebraic tensor product) which we endow with the $\mathscr{A}$-valued scalar product

$$\langle \xi, \eta\rangle(h) = \int_G \langle \xi(g), \eta(gh)\rangle\,dg \quad (\xi, \eta \in \mathscr{E})$$

(in this way $\mathscr{E}$ becomes a pre-C^*-module over the pre-C^*-algebra $\mathscr{A}$). The rank 1 operators $\theta_{\xi,\eta}$ $(\xi, \eta \in \mathscr{E})$ are defined by

$$\theta_{\xi,\eta}(x) = \xi\langle \eta, x\rangle \quad (x \in \mathscr{E})$$

and finite rank operators are linear combinations of rank 1 operators. We define the *trace* $\mathrm{Tr}_\chi \theta_{\xi,\eta}$ by

$$\mathrm{Tr}_\chi \theta_{\xi,\eta} = \int_G \langle \eta, \xi \rangle(g)\chi(g)\,\mathrm{d}g$$

and we extend by linearity to finite-rank operators (it is easy to check that this is well-defined). Since χ is central, Tr_χ is a trace on the algebra of finite rank operators on $\mathscr{E}$. Note that, if S is a finite rank operator on $\mathscr{H}$ and $a \in \mathscr{A}$, then $S \otimes a$ is a finite rank operator on $\mathscr{E}$ and

$$\mathrm{Tr}_\chi(S \otimes a) = \mathrm{Tr}\, S \cdot \chi(a). \tag{7}$$

We will apply this with $\mathscr{E}^i = \ell^2(\Delta^i) \otimes \mathscr{A}$ $(i = 0,1)$. We have a diagonal action of G (on the left) on $\mathscr{E}^i$ and, by integration, we get a representation $\tilde{R}_i$ of $\mathscr{A}$ on $\mathscr{E}^i$. We also have operators

$$\tilde{d}_t = d_t \otimes 1 : \mathscr{E}^0 \to \mathscr{E}^1,$$

$$\tilde{q}_t = q_t \otimes 1 : \mathscr{E}^1 \to \mathscr{E}^0.$$

LEMMA 6. *For $a \in \mathscr{A}$, the operator $\tilde{R}_0(a) - \tilde{q}_t \tilde{R}_1(a) \tilde{d}_t$ has finite rank, and $\mathrm{Tr}_\chi(\tilde{R}_0(a) - \tilde{q}_t \tilde{R}_1(a) \tilde{d}_t) = (\tau_t \chi)(a)$.*

Proof. We may assume that a is the characteristic function 1_{gK} of some left coset gK of a compact open subgroup K stabilizing x_0. Let L be the intersection of K with the subgroup of G leaving the ball $\{x \in \Delta^0 : \rho(x) \leqslant \rho(gx_0)\}$ pointwise fixed; L is compact open. It follows from (4), (5) that, for $k \in K$, $l \in L$, we have

$$R_0(gkl) - q_t R_1(gkl) d_t = R_0(gk) - q_t R_1(gk)\, d_t.$$

So, if $k_1, \ldots, k_n$ is a set of representatives for the left cosets of L in K, we have

$$\tilde{R}_0(a) - \tilde{q}_t \tilde{R}_1(a) \tilde{d}_t = \sum_{i=1}^{n} \int_L [(R_0(gk_i l) - q_t R_1(gk_i l) d_t) \otimes gk_i l]\,\mathrm{d}l =$$

$$= \sum_{i=1}^{n} (R_0(gk_i) - q_t R_1(gk_i) d_t) \otimes 1_{gk_i L}.$$

This is a finite rank operator and, by (7) and the definition of τ_t, its trace is given by:

$$\mathrm{Tr}_\chi(\tilde{R}_0(a) - \tilde{q}_t \tilde{R}_1(a) \tilde{d}_t) = \sum_{i=1}^{n} \tau_t(gk_i) \int_G 1_{gk_i L}(h)\chi(h)\,\mathrm{d}h = \int_G a(h)(\tau_t \chi)(h)\,\mathrm{d}h.$$

This proves the lemma.

To proceed, we have to adopt the formalism of [4], Chapter I. Let $\mathscr{H}$ be the $\mathbf{Z}/2$-graded Hilbert space whose even part is $\ell^2(\Delta^0)$, and odd part is $\ell^2(\Delta^1) \oplus \mathbf{C}$ where $\mathbf{C}$ carries the *zero* action of G. On $\mathscr{H}$, we have the degree 1 operator F_t defined by

$$F_t f = (d_t f, \langle f, \delta_{x_0} \rangle) \quad (f \in \ell^2(\Delta^0)),$$

$$F_t(\xi, \lambda) = q_t \xi + \lambda e^{-t\rho} \quad ((\xi, \lambda) \in \ell^2(\Delta^1) \oplus \mathbf{C}).$$

Then, for $t > \log(N-1)$, F_t is bounded and $F_t^2 = 1$. Moreover

$$\frac{1}{2} \operatorname{Tr} \varepsilon F_t [F_t, a] = \operatorname{Tr}(R_0(a) - q_t R_1(a) d_t)$$

where ε is the grading operator. Similarly, if we define $\mathscr{E} = \mathscr{H} \otimes \mathscr{A}$, $\tilde{F}_t = F_t \otimes 1$, we have:

$$\frac{1}{2} \operatorname{Tr}_\chi \varepsilon \tilde{F}_t [\tilde{F}_t, a] = \operatorname{Tr}_\chi (\tilde{R}_0(a) - \tilde{q}_t \tilde{R}_1(a) \tilde{d}_t) = \int_G a(g)(\tau_t \chi)(g) \, dg$$

so that our Lemma 6 generalizes Lemma 1 in [4]. Define now a trilinear form on $\mathscr{A}$ by

$$\varphi_t(a^0, a^1, a^2) = - \operatorname{Tr}_\chi \varepsilon a^0 [\tilde{F}_t, a^1] [\tilde{F}_t, a^2].$$

LEMMA 7. i) *φ_t is a cyclic 2-cocycle on $\mathscr{A}$, hence defines an element $[\varphi_t]$ in $H_\lambda^2(\mathscr{A})$.*
ii) *One has $[S(\tau_t \chi)] = [\varphi_t]$ in $H_\lambda^2(\mathscr{A})$.*

Proof. i) is proved exactly like Proposition 1 in [4], Chapter II.
ii) is proved like Theorem 1 in [4], Chapter I: if

$$\psi(a^0, a^1) = \frac{1}{2} \operatorname{Tr}_\chi \varepsilon \tilde{F}_t (a^1 [\tilde{F}_t, a^0] - a^0 [\tilde{F}_t, a^1])$$

then the Hochschild boundary of ψ is $S(\tau_t \chi) - \varphi_t$.

LEMMA 8. *For $s, s' > \log(N-1)$, there exists a Hochschild cocycle ψ such that $\varphi_{s'} - \varphi_s = B\psi$ (where B is Connes' operator* [4], Chapter II, § 3*).*

Proof. We imitate the proof of [4], Chapter I, § 5. Notice that one has

$$\varphi_t(a^0, a^1, a^2) = - \operatorname{Tr}_\chi \varepsilon \rho_t(a^0) [\tilde{F}, \rho_t(a^1)] [\tilde{F}, \rho_t(a^2)]$$

where the graded module is now two copies of $\mathcal{E}^+ = \mathcal{H}^+ \otimes \mathcal{A}$, the representation ρ_t is unchanged on the even part, and transported from $\mathcal{E}^-$ by $\tilde{F}_t$ on the odd part, and $\tilde{F} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (cf. [4], p. 65).

Let $\delta_t(a) = \dfrac{\mathrm{d}}{\mathrm{d}t} \rho_t(a)$ for $a \in \mathcal{A}$, which makes sense since $a - \tilde{F}_t a \tilde{F}_t$ is finite rank with fixed range.

Define

$$\psi_t(a^0, a^1, a^2, a^3) = \sum_{k=1}^{3} (-1)^k \mathrm{Tr}_\chi \varepsilon \rho_t(a^0)[\tilde{F}, \rho_t(a^1)] \ldots \delta_t(a^k) \ldots [\tilde{F}, \rho_t(a^3)].$$

Then as in [4], p. 64 one checks that $b\psi_t = 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(a^0, a^1, a^2) = \psi_t(1, a^0, a^1, a^2) = B_0 \psi_t(a^0, a^1, a^2) = \frac{1}{3} B\psi_t(a^0, a^1, a^2).$$

We get the lemma by integration over $[s, s']$.

Let us now complete the proof of the theorem: by Lemma 8, the cyclic cocycle $\varphi_s - \varphi_{s'} \in H^2_\lambda(\mathcal{A})$ belongs to $\mathrm{Im}\, B = \mathrm{Ker}\, S$ (cf. [4], Chapter II, §4). Thus by Lemma 7,

$$S^2(\tau_s \chi - \tau_{s'} \chi) = S(\varphi_s - \varphi_{s'}) = 0 \quad \text{in } H^4_\lambda(\mathcal{A}).$$

5. τ_t AS A CHERN CHARACTER

In one previous section, we showed that the Chern character τ_t of γ_t was given by

$$\tau_t(a) = \int_G a(g) \mathrm{e}^{-t p(g)} \mathrm{d}g \quad (a \in \mathcal{A}). \tag{8}$$

Of course, this formula defines a trace on $\mathcal{A}$ not only for $t > \log(N-1)$, but for any non-negative t. So a natural question arises: is it possible to realize this trace as a Chern character for any value of t? We will see that this question has a positive answer; for that, we have to appeal to the constructions of [10], [11]. There we obtained an homotopy (in Kasparov's sense [12]) between γ and the trivial Fredholm module 1_G. More precisely, we constructed a family $(\mathcal{H}_t, \ell^2(\Delta^0), U_t)_{t \in [0,\infty]}$ of 1-summable Fredholm modules such that:

i) $\mathcal{H}_\infty = \ell^2(\Delta^0)$, $U_\infty = 1$ (so the Fredholm module $(\mathcal{H}_\infty, \ell^2(\Delta^0), U_\infty)$ is degenerate, hence trivially homotopic to zero).

ii) $\mathscr{H}_0 = \mathbf{C} \oplus \ell^2(\Delta^1)$, and for $\lambda \in \mathbf{C}$, $\xi \in \ell^2(\Delta^1)$:

$$U_0(\lambda, \xi) = \lambda\delta_{x_0} + U^*\xi$$

$(\mathscr{H}_0, \ell^2(\Delta^0), U_0)$ corresponds to the 1-summable Fredholm module $1_G - \gamma$.

iii) For $t \in]0, \infty[$, $\mathscr{H}_t$ is a Hilbert space topologically spanned by vectors ξ_x^t $(x \in \Delta^0)$ such that

$$\langle \xi_x^t, \xi_y^t \rangle = e^{-t\delta(x,y)}$$

and carrying a unitary representation ρ_t of G such that

$$\rho_t(g)\xi_x^t = \xi_{gx}^t.$$

Moreover, U_t is a unitary operator defined by

$$U_t\xi_x^t = e^{-t\delta(x,x_0)}\delta_{x_0} + (1 - e^{-2t})^{1/2} \sum_{y \in]x_0, x]} e^{-t\delta(y,x)}\delta_y.$$

The technical part of [11] shows that this family of triples actually satisfies the definition of [12] for an homotopy. In particular, defining $V_t = UU_t$, we see that the family $(\mathscr{H}_t, \ell^2(\Delta^1), V_t)_{t\in[0,\infty]}$ is still an homotopy, where this time we get γ for $t = \infty$, and for $t = 0$, we get the triple $(\mathbf{C}\oplus\ell^2(\Delta^1), 0 \oplus \ell^2(\Delta^1), 0 \oplus 1)$ which is 1_G up to a trivial homotopy. (The homotopy obtained in this way is more direct, and perhaps simpler, than those of [10], [11], but the authors confess with some remorse that they did not think of it at the time.) We shall see that this homotopy, which could seem rather mysterious, can in fact be understood as a continuation to all t's of the family d_t.

Our purpose is to show that the Chern character of $(\mathscr{H}_t, \ell^2(\Delta^1), V_t)$ is given by (8) for any t. We begin by relating this family of 1-summable Fredholm modules to our family γ_t. For any t in $]0, \infty[$, we define a densely defined operator $W_t : \mathscr{H}_t \to \ell^2(\Delta^0)$ by

$$W_t\xi_x^t = \delta_x \quad (x \in \Delta^0).$$

Lemma 9. *W_t extends linearly and continuously to a bounded operator $\mathscr{H}_t \to \ell^2(\Delta^0)$; moreover:*

$$W_t = (1 - e^{-2t})^{-1/2} d_t^* D^{-t} V_t + \langle \cdot, \xi_{x_0}^t \rangle \delta_{x_0}.$$

Proof. Since the right hand side of this formula defines a bounded operator for any t, it suffices to check the equality on any ξ_x^t. Since $V_t\xi_{x_0}^t = 0$, this is clear for $x = x_0$, so we may assume $x \neq x_0$. Then

$$V_t\xi_x^t = (1 - e^{-2t})^{1/2} \sum_{y \in]x_0, x]} \varepsilon(\beta(y)) e^{-t\delta(x,y)} \delta_{\beta(y)}$$

and

$$(1 - e^{-2t})^{-1/2} d_t^* D^{-t} V_t \xi_x^t =$$

$$= \sum_{y \in]x_0, x]} \varepsilon(\beta(y)) e^{-t\delta(x,y)} e^{-(t/2)(1-\varepsilon(\beta(y)))} (\delta_{e(\beta(y))} - e^{-t\varepsilon(\beta(y))} \delta_{o(\beta(y))}).$$

Evaluate this function on Δ^0 at a vertex z; it yields

$$\sum_{y \in]x_0, x]} \varepsilon(\beta(y)) e^{-t\delta(x,y)} e^{-(t/2)(1-\varepsilon(\beta(y)))} [\delta_z(e(\beta(y))) - e^{-t\varepsilon(\beta(y))} \delta_z(o(\beta(y)))] =$$

$$= \sum_{b \subset [x_0, x]} \varepsilon(b) e^{-t\delta(x, e(b))} (\delta_z(e(b)) - e^{-t\varepsilon(b)} \delta_z(o(b))) =$$

by an easy check

$$= \langle q_t d_t \delta_z, \delta_x \rangle = \langle \delta_z, \delta_x \rangle - e^{-t\rho(x)} \langle \delta_z', \delta_{x_0} \rangle \quad \text{by formula (4).}$$

Consequently

$$(1 - e^{-2t})^{-1/2} d_t^* D^{-t} V_t \xi_x^t = \delta_x - e^{-t\rho(x)} \delta_{x_0}$$

which is the desired result.

LEMMA 10. *For $t > \log(N-1)$, the operator W_t is invertible and*

$$W_t^{-1} = (1 - e^{-2t})^{1/2} V_t^* D^t q_t^* + \langle \cdot, e^{-t\rho} \rangle \, \xi_{x_0}^t .$$

Proof. The algebraic verification of the equality is routine, and is left to the reader. By the proof of Proposition 1, the right hand side is a bounded operator for $t > \log(N-1)$, so the result follows. An explicit bound for the norm of W_t^{-1} can be obtained via Lemma 3 and the interpolation argument used in Proposition 1; one gets $\| W_t^{-1} \| \leqslant C(t)^{1/2}$.

PROPOSITION 4. *For any $t \in]0, \infty[$, the Chern character of the 1-summable Fredholm module $(\mathcal{H}_t, \ell^2(\Delta^1), V_t)$ is given by formula* (8).

Proof. For any $g \in G$, the operator $\rho_t(g) - V_t^* R_1(g) V_t$ has finite rank, by formula $(*)$ in [11]. So it is enough to prove that

$$\text{(9)} \qquad \operatorname{Tr}(\rho_t(g) - V_t^* R_1(g) V_t) = e^{-t\rho(g)} .$$

First assume that $t > \log(N-1)$; then by Lemmas 9 and 10

$$(1-e^{-2t})^{1/2} V_t W_t^* = D^{-t} d_t$$

$$(1-e^{-2t})^{-1/2} (W_t^{-1})^* V_t^* = q_t D^t.$$

Since W_t^* intertwines R_0 and ρ_t, by its very definition, we have

$$\mathrm{Tr}(\rho_t(g) - V_t^* R_1(g) V_t) = \mathrm{Tr}(R_0(g) - q_t D^t R_1(g) D^{-t} d_t)$$

and this equals $e^{-tp(g)}$ by Proposition 3 and the remark following it. To get (9) for a general t, one simply notices that the left hand side of (9) depends analytically on t; so the result follows by analytic continuation.

From this, one obtains a generalization of Corollary 2, with $\mathscr{A}$ replaced by the full group C^*-algebra $C^*(G)$.

COROLLARY 6. *There exists a unique map* $\varphi : \mathrm{K}_0(C^*(G)) \to \mathbf{Z}$ *such that, for any* t *in* $[0,\infty]$, *and any idempotent* e *in* $M_k(\mathrm{Dom}\,\tau_t)$:

$$\varphi[e] = (\tau_t \otimes \mathrm{Tr})(e).$$

Proof. Define

$$\mathscr{B}_t = \{a \in C^*(G) : \rho_t(a) - V_t^* R_1(a) V_t \text{ is trace-class}\}.$$

By the results of [4], Appendix 3, $\mathscr{B}_t$ is a dense subalgebra of $C^*(G)$ and the inclusion $\mathscr{B}_t \hookrightarrow C^*(G)$ induces isomorphisms in K-theory. By Proposition 4, the trace τ_t on $\mathscr{A}$ extends to a trace on $\mathscr{B}_t$, so we get a map $\varphi_t : \mathrm{K}_0(C^*(G)) \to \mathbf{Z}$ such that, for any idempotent e in $M_k(\mathscr{B}_t)$:

$$\varphi_t[e] = (\tau_t \otimes \mathrm{Tr})(e).$$

The fact that this map φ_t does not depend on t can be proved as in Corollary 2.

REMARK. In our papers [10], [11], the main consequence of the existence of an homotopy between γ and 1_G was the K-amenability of G (in the sense of Cuntz [6]), provided the action of G on the tree is proper. Connes has communicated to us the following conjecture: if a locally compact group G admits a continuous function φ of negative type such that $e^{-t\varphi}$ belongs to $L^1(G)$ for t big enough, then G is K-amenable. Because of the results in [7], this conjecture would imply the K-amenability of $\mathrm{SO}(n,1)$ amd $\mathrm{SU}(n,1)$. (Kasparov [12] has proved the K-amenability of $\mathrm{SO}(n,1)$ by a different method.)

On the other hand, Chiswell studied in [3] a very peculiar class of functions of negative type, called integer-valued length functions. One result in [3] is that φ is an integer-valued length function on G if and only if there exists a tree on which G acts, such that for some vertex x_0:

$$\varphi(g) = \delta(gx_0, x_0).$$

Bearing Connes' conjecture in mind, we may rephrase the main corollary of [11]: if a group G admits an integer-valued length function φ such that $e^{-t\varphi}$ belongs to $L^1(G)$ for t big enough, then G is K-amenable. The fact that φ has negative type was of fundamental importance for the proof of this result, but the L^1--condition on $e^{-t\varphi}$ was not really needed. The present paper shows how this L^1--condition can be used to construct a family of 1-summable Fredholm modules over $\mathscr{A}$.

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