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COSUBNORMAL DILATION SEMIGROUPS ON BERGMAN SPACES

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Let Ω be a Jordan domain in the complex plane whose boundary $\partial\Omega$ contains 0 and such that $\Omega \cup \{0\}$ is starlike about 0. For $t \geqslant 0$ we consider the operators $D_t : f(z) \to f(\mathrm{e}^{-t}z)$ acting on the Bergman space $A^2(\Omega)$, the Hilbert space of all functions f analytic on Ω with

$$\|f\|^2 \equiv \frac{1}{\pi}\int_{\Omega} |f(x + \mathrm{i}y)|^2 \mathrm{d}x\,\mathrm{d}y < \infty.$$

One easily checks that $\{\mathrm{e}^{-t}D_t\}_{t\geqslant 0}$ is a strongly continuous semigroup of contraction operators on $A^2(\Omega)$. We will be interested in when $\{D_t^*\}_{t\geqslant 0}$ is a subnormal semigroup, that is, when there exists a strongly continuous semigroup of normal operators on a larger Hilbert space whose restriction to $A^2(\Omega)$ is $\{D_t^*\}_{t\geqslant 0}$. A theorem of Ito [3] asserts that this is equivalent to each D_t^* being separately subnormal.

THEOREM 1. *Let Ω be as above. Suppose that $\partial\Omega$ is analytic near* 0 *and that the interior unit normal vector to $\partial\Omega$ at* 0 *is in the positive real direction. Then $\{D_t^*\}_{t\geqslant 0}$ is subnormal if and only if there exists non-negative numbers A and $\varkappa$, not both zero, such that Ω is the set of points $r\mathrm{e}^{\mathrm{i}\theta}$ with $-\pi/2 < \theta < \pi/2$ and*

$$r < \frac{2}{\varkappa}\cos\theta \quad \textit{if} \quad A = 0,\ \varkappa > 0 \ \textit{(a disk)},$$

$$r < \frac{1}{\sqrt{A}} \quad \textit{if} \quad A > 0,\ \varkappa = 0 \ \textit{(a half-disk)},$$

$$r < \frac{-\varkappa + \sqrt{\varkappa^2 + 16A\cos^2\theta}}{4A\cos\theta} \quad \textit{if} \quad A > 0,\ \varkappa > 0.$$

A few remarks are in order. First, the parameter $\varkappa$ in Theorem 1 is the curvature of $\partial\Omega$ at 0. Second, note that if $c \neq 0$ and $\Omega_1 = \{cz : z \in \Omega\}$, then $(Xf)(z) = cf(cz)$ defines a unitary operator X from $A^2(\Omega_1)$ onto $A^2(\Omega)$ satisfying $XD_t = D_tX$. Thus, given that $\partial\Omega$ is smooth at 0, we may as well assume that the interior unit normal points along the positive real axis. By the same token, we may and do assume that $(0, 1] \subset \Omega$; when Ω is as described in Theorem 1, this says that $2A + \varkappa < 2$.

Third, the results here are an application of the general theory developed in [6], and I will draw freely on that paper. Recall that the *cogenerator* [8] of $\{e^{-t}D_t\}_{t \geq 0}$ is a contraction L for which 1 is not an eigenvalue and with $e^{-t}D_t = E_t(L)$, where

$$E_t(z) = e^{t\frac{z+1}{z-1}};$$

we have

$$(Lf)(z) = f(z) - \frac{2}{z^2}\int_0^z f(\zeta)\zeta \, d\zeta, \quad f \in A^2(\Omega),$$

see [6, § 4]. Moreover, every kernel function in $A^2(\Omega)$ is a cyclic vector for L^* [6, Theorem 1], so Bram's Theorem [1] implies that if L^* is subnormal, it is unitarily equivalent to the operator $M_\mu : f(z) \to zf(z)$ acting on $P^2(\mu)$, the $L^2(\mu)$ closure of the polynomials, for some finite positive Borel measure μ on $\{z : |z| \leqslant 1\}$. The eigenvalue condition on L implies that $\mu(\{1\}) = 0$, and $e^{-t}D_t$ corresponds to the operator $V_t^\mu : f(z) \to E_t(z)f(z)$ on $P^2(\mu)$. The point spectrum of L is $\mathbf{D} = \{z : |z| < 1\}$; indeed the eigenvectors of L are precisely $\{z^u : \operatorname{Re} u > -1\}$, and $Lz^{\rho(w)} = wz^{\rho(w)}$ where $\rho(w) = 2w(1-w)^{-1}$, a conformal map of $\mathbf{D}$ onto $\{u : \operatorname{Re} u > -1\}$. It follows that every point in $\mathbf{D}$ induces a bounded point evaluation on $P^2(\mu)$ [2, p. 169]; the elements of $P^2(\mu)$ are actually analytic on $\mathbf{D}$.

The reproducing kernel for $A^2(\Omega)$ is

$$k(w, z) = \frac{\overline{\varphi'(w)}\varphi'(z)}{(1 - \overline{\varphi(w)}\varphi(z))^2}$$

where φ is a conformal map from Ω to $\mathbf{D}$. The analyticity requirement on $\partial\Omega$ at 0 says that φ is analytic at 0 and $\varphi'(0) \neq 0$, hence $k(\bar{z}, z)$ has a pole of order 2 at $z = 0$. This allows us to specialize Proposition 8 of [6] to the present setting, where it will form the cornerstone of the proof:

SUBNORMALITY CRITERION. *$\{D_t^*\}_{t \geq 0}$ is subnormal if and only if there exists a sequence $\{\beta_n\}_{n=-2}^{\infty}$ of finite positive Borel measures on $(-\infty, \infty)$ such that*

$$k(re^{-iv/2}, re^{iv/2}) = \sum_{n=-2}^{\infty} \hat{\beta}_n(v) r^n, \tag{1}$$

$0 < r \leqslant 1$, $2\log r \leqslant v \leqslant -2\log r$, *where*

$$\hat{\beta}_n(v) = \int_{-\infty}^{\infty} e^{-ivy}\, d\beta_n(y).$$

In the event that $\{D_t^*\}$ is subnormal, we can take $\mu = \beta \circ \rho$, where ρ (as above) maps $\bar{\mathbf{D}} \setminus \{1\}$ onto $\{x + iy : x \geqslant -1\}$,

$$\text{(2)} \qquad d\beta(x + iy) = \sum_{n=-2}^{\infty} d\beta_n(y)\, d\delta_{n/2}(x),$$

and $\delta_{n/2}$ is a unit point mass at $n/2$, see [6]. Formulas for the β_n will occur in the proof of Theorem 1, thus yielding a description of μ.

THEOREM 2. *Suppose that $\{D_t^*\}_{t\geqslant 0}$ is subnormal on $A^2(\Omega)$, where Ω is as described in Theorem 1. Then $\{e^{-t}D_t^*\}_{t\geqslant 0}$ and L^* are unitarily equivalent to $\{V_t^\mu\}_{t\geqslant 0}$ and M_μ respectively, where $\mu = \beta \circ \rho$ and β is given by* (2) *with*

$$d\beta_{-2}(y) = \frac{y}{2\sinh(\pi y)}\, dy, \quad d\beta_{-1}(y) = \frac{\varkappa\left(y^2 + \frac{1}{4}\right)}{2\cosh(\pi y)}\, dy,$$

$$d\beta_n(y) = \left(\frac{n}{2} + 1\right) A^{\frac{n}{2}+1}\, d\delta_0(y) + \sum_{j=2}^{n+4} q(n, j) \left|\Gamma\left(\frac{j}{2} + iy\right)\right|^2 dy$$

for even $n \geqslant 0$, and

$$d\beta_n(y) = \sum_{j=1}^{n+4} q(n, j) \left|\Gamma\left(\frac{j}{2} + iy\right)\right|^2 dy$$

for odd $n \geqslant 1$; here $q(n, j)$ is a non-negative polynomial in $\varkappa$ and A with $q(n, j) = 0$ if n is even and j odd, or if n is odd and j even.

TWO SPECIAL CASES. (i) $\varkappa = 0$, $0 < A < 1$; Ω is a half-disk of radius $A^{-1/2}$. Then $\beta_n = 0$ if n is odd and

$$d\beta_n(y) = \left(\frac{n}{2} + 1\right) A^{\frac{n}{2}+1}\, d\delta_0(y), \quad n = 0, 2, 4, \ldots .$$

It follows that

$$d\mu = W(e^{i\theta})\, d\theta + \sum_{n=0}^{\infty} (n + 1) A^{n+1}\, d\delta_{z_n}$$

where δ_{z_n} is a unit point mass at $z_n = n(n+2)^{-1}$ and $W\,\mathrm{d}\theta$ is the pullback of $\mathrm{d}\beta_{-2}(y)\,\mathrm{d}\delta_{-1}(x)$ to $\partial\mathbf{D}$ via ρ. Since $\log W \notin L^1(\mathrm{d}\theta)$, $P^2(W\mathrm{d}\theta) = L^2(W\mathrm{d}\theta)$ in contrast to $P^2(\mu)$, which admits many bounded point evaluations (all of $\mathbf{D}$).

(ii) $A = 0$, $\varkappa = 1$; Ω is a disk of radius 1 centered at 1. Then

$$\mathrm{d}\beta_n(y) = \frac{1}{2\pi(n+3)!}\left|\Gamma\left(\frac{n+4}{2} + \mathrm{i}y\right)\right|^2 \mathrm{d}y, \quad n = -2, -1, 0, \ldots,$$

and so μ closely resembles the measure associated with the discrete Cesàro operator C_0 on l^2 [7]. This is not surprising, because for this Ω, L^* is unitarily equivalent to $g(I - C_0)$, where $g(z) = (3z-1)(3-z)^{-1}$, see [6, § 4].

Proofs. We may assume that $\varphi(0) = 1$ so that for $|z|$ small,

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

with $c_1 < 0$. Let us suppose that $\{D_t^*\}_{t \geq 0}$ is subnormal, so that (1) holds. Using the form of $k(w, z)$, we will solve for the $\hat{\beta}_n(v)$ and see that the requirement that they be Fourier transforms determines φ. We introduce the abbreviations $\hat{\beta}_n = \hat{\beta}_n(v)$, $t = e^{v/2}$ and $\lambda = t + t^{-1}$, and record for future reference the equation

$$\frac{1}{\lambda^k} = \frac{1}{2\pi(k-1)!}\int_{-\infty}^{\infty} e^{ivy}\left|\Gamma\left(\frac{k}{2} + \mathrm{i}y\right)\right|^2 \mathrm{d}y, \quad k = 1,2,\ldots, \tag{3}$$

see [5, p. 39]. From (1) and the form of $k(w, z)$ we have

$$\overline{\varphi'(r/t)}\varphi'(rt) = (1 - \overline{\varphi(r/t)}\varphi(rt))^2 \sum_{n=-2}^{\infty} \hat{\beta}_n r^n. \tag{4}$$

Now we calculate that

$$\overline{\varphi'(r/t)}\varphi'(rt) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(j+1)(n-j+1)c_{j+1}\overline{c_{n-j+1}}\,t^{2j-n}\right)r^n$$

and that

$$(1 - \overline{\varphi(r/t)}\varphi(rt))^2 = \sum_{n=2}^{\infty} B_n r^n,$$

where

$$B_n = \sum_{j=1}^{n-1} A_j A_{n-j} \quad \text{and} \quad A_k = \sum_{j=0}^{k} c_j \overline{c_{k-j}}\, t^{2j-k}.$$

If we equate the constant terms and the coefficients of r in (4) we may solve for $\hat{\beta}_{-2}$ and $\hat{\beta}_{-1}$, yielding

$$\hat{\beta}_{-2} = \frac{1}{\lambda^2} \quad \text{and} \quad \hat{\beta}_{-1} = 2\varkappa\frac{1}{\lambda^3},$$

where $\varkappa \equiv (2\operatorname{Re} c_2 - c_1^2)/c_1$; $\varkappa$ turns out to be the curvature of $\partial\Omega$ at 0, see [6, § 4.1.1]. So far so good: equation (3) and Γ-function identities [4, p. 3,4] yield the desired expressions for β_{-2} and β_{-1}. In general, if we equate coefficients of r^n in (4) and solve for $\hat{\beta}_{n-2}$ we have, using $B_2 = c_1^2\lambda^2$,

$$\text{(5)} \qquad \hat{\beta}_{n-2} = \frac{1}{c_1^2\lambda^2}\left(\sum_{j=0}^{n}(j+1)(n-j+1)c_{j+1}\overline{c_{n-j+1}}t^{2j-n} - \sum_{i=-2}^{n-3}\hat{\beta}_i B_{n-i}\right).$$

Now the definition of B_k tells us that

$$B_k = \left(\sum_{p=1}^{k-1} c_p c_{k-p}\right)t^k + O(t^{k-2}) \quad \text{as } v \to \infty.$$

Let us use this in conjunction with (5) (with $n = 2$) and our formulas for $\hat{\beta}_{-2}$ and $\hat{\beta}_{-1}$ to estimate $\hat{\beta}_0$. We find $\hat{\beta}_0 = A + O(t^{-2})$ as $v \to \infty$, where

$$A = \frac{c_1c_3 - c_2^2}{c_1^2}.$$

Since β_0 is a real measure we have $\hat{\beta}_0(-v) = \overline{\hat{\beta}_0(v)}$, hence $\hat{\beta}_0 = \bar{A} + O(t^2)$ as $v \to -\infty$. An argument in the proof of Proposition 10 in [6] shows that $A = \bar{A}$ and $d\beta_0(y) = A\,d\delta_0(y) + w(y)\,dy$ where $w \in L^1(-\infty, \infty)$. Since $\beta_0 \geqslant 0$ we have $A \geqslant 0$.

Next we consider $\hat{\beta}_{n-2}$ for $n \geqslant 3$. From (5), the fact that all $\hat{\beta}_j$ are bounded, and our formulas for $\hat{\beta}_{-2}$, $\hat{\beta}_{-1}$, $\hat{\beta}_0$ and B_k, we compute

$$\hat{\beta}_{n-2} = \frac{1}{c_1^2}\left\{(n+1)c_{n+1}c_1 - \left(\sum_{p=1}^{n+1} c_p c_{n+2-p}\right) - A\left(\sum_{p=1}^{n-1} c_p c_{n-p}\right)\right\} t^{n-2} + O(t^{n-3})$$

as $v \to \infty$. Since $\hat{\beta}_{n-2}$ is a bounded function, the coefficient of t^{n-2} vanishes, or replacing $n + 1$ by n and rearranging,

$$c_n = \frac{1}{(n-2)c_1}\left(\sum_{p=2}^{n-1} c_p c_{n+1-p}\right) + \frac{A}{(n-2)c_1}\left(\sum_{p=1}^{n-2} c_p c_{n-1-p}\right)$$

for $n \geqslant 4$; this holds for $n = 3$ as well. The first term on the right is the n^{th} Taylor coefficient of the function

$$\frac{z^2}{c_1}\int_0^z \frac{(\varphi(w) - 1 - c_1 w)^2}{w^4}\,\mathrm{d}w$$

while the second term is the n^{th} coefficient of

$$\frac{Az^2}{c_1}\int_0^z \left(\frac{\varphi(w) - 1}{w}\right)^2 \mathrm{d}w.$$

Hence $\sum_{n=3}^{\infty} c_n z^n$ is the sum of these functions, a fact which we may convert (by rearranging and differentiating) into the Ricatti equation

$$c_1 \frac{\mathrm{d}U}{\mathrm{d}z} = U^2 + A(zU + c_1)^2$$

for the function

$$U = \frac{\varphi - 1 - c_1 z}{z^2}.$$

The appropriate initial condition is $U(0) = c_2$. The solution is

$$U(z) = -c_1 \frac{Az + \frac{c_2}{c_1}}{Az^2 + \frac{c_2}{c_1} z - 1},$$

whence

(6)
$$\varphi(z) = 1 - \frac{c_1 z}{Az^2 + \frac{c_2}{c_1} z - 1}.$$

One checks that if $\varphi(z) = \varphi(w)$, then $wz = -1/4$ and thus φ is univalent on $\{z : \operatorname{Re} z > 0\}$. Now $\Omega \subset \{z : \operatorname{Re} z > 0\}$ is defined by the condition $|\varphi(z)|^2 < 1$, which is equivalent to

$$(2 \operatorname{Re} z)(A|z|^2 - 1) + \varkappa|z|^2 < 0;$$

the desired description of Ω appears on inserting polar coordinates. Since $|\varphi|^2 = 1$ an $\partial\Omega$, we see that φ maps Ω onto $\mathbf{D}$.

Conversely, if Ω is described in terms of A and $\varkappa$ as in Theorem 1, we can define φ by (6) with $c_1 = -1$ and $c_2 = (1 - \varkappa)/2$. The above discussion implies that φ maps Ω onto $\mathbf{D}$. We find that

$$k(w, z) = \frac{A^2\bar{w}^2z^2 + A(\bar{w}^2 + z^2) + 1}{[(A\bar{w}z - 1)(\bar{w} + z) + \varkappa\bar{w}z]^2}$$

whence

$$k(r/t, rt) = \frac{A^2r^4 + A(\lambda^2 - 2)r^2 + 1}{\lambda^2 r^2\left[1 - r\left(Ar + \frac{\varkappa}{\lambda}\right)\right]^2} =$$

$$= \left[\frac{1}{\lambda^2} r^{-2} + A\left(1 - \frac{2}{\lambda^2}\right) + \frac{A^2}{\lambda^2} r^2\right] \sum_{n=0}^{\infty} F_n r^n, \tag{7}$$

with

$$F_n = \sum_{k=(n/2)}^{n} (k+1)\binom{k}{n-k}\left(\frac{\varkappa}{\lambda}\right)^{2k-n} A^{n-k};$$

here $(n/2)$ is the least integer not exceeded by $n/2$. We therefore have

$$k(r/t, rt) = \frac{F_0}{\lambda^2} r^{-2} + \frac{F_1}{\lambda^2} r^{-1} + \left[\frac{F_2}{\lambda^2} + A\left(1 - \frac{2}{\lambda^2}\right)F_0\right] +$$

$$+ \left[\frac{F_3}{\lambda^2} + A\left(1 - \frac{2}{\lambda^2}\right)F_1\right] r + \sum_{n=2}^{\infty}\left[\frac{A}{\lambda^2} F_{n-2} + A\left(1 - \frac{2}{\lambda^2}\right)F_n + \frac{1}{\lambda^2} F_{n+2}\right] r^n. \tag{8}$$

The coefficient of r^n in this expansion is our purported Fourier transform $\hat{\beta}_n$. We already know that β_{-2} and β_{-1} exist (and have the stated form). Let us check $\hat{\beta}_0$ and $\hat{\beta}_1$. We have

$$F_0 = 1, \quad F_1 = 2\varkappa\frac{1}{\lambda},$$

$$F_2 = 2A + 3\varkappa^2\frac{1}{\lambda^2}, \quad F_3 = 6\varkappa A\cdot\frac{1}{\lambda} + 4\varkappa^3\frac{1}{\lambda^3},$$

from which we find

$$\hat{\beta}_0 = A + 3\varkappa^2\frac{1}{\lambda^4},$$

and

$$\hat{\beta}_1 = 2\varkappa A \frac{1}{\lambda} + 2\varkappa A \frac{1}{\lambda^3} + 4\varkappa^3 \frac{1}{\lambda^5},$$

so that β_0 and β_1 both exist and are positive measures.

Consider now $\hat{\beta}_n$ for even $n \geqslant 2$. It is convenient to take $n = 2p$ and write F_n as

$$F_{2p} = \sum_{j=0}^{p} (j + p + 1) \binom{p+j}{p-j} \varkappa^{2j} A^{p-j} \left(\frac{1}{\lambda^2}\right)^j.$$

It is clear from (8) that the coefficient of r^{2p} is a polynomial in $\frac{1}{\lambda^2}$, of degree $p + 2$. To verify the existence of β_{2p} it is sufficient, in view of (3), to check that this polynomial has non-negative coefficients. It is easy to see that the constant term is $(p + 1)A^{p+1}$ and the coefficients of $\left(\frac{1}{\lambda^2}\right)^{p+2}$ and $\left(\frac{1}{\lambda^2}\right)^{p+1}$ are $(2p + 3)\varkappa^{2p+2}$ and $2p(2p + 1)\varkappa^{2p}A$, respectively. A somewhat more tedious calculation reveals that for $j = 1, 2, \ldots, p$, the coefficient of $\left(\frac{1}{\lambda^2}\right)^j$ is

$$\frac{(2j-2)(2j-1)}{(p-j+2)} \binom{p+j-1}{p-j+1} \varkappa^{2j-2} A^{p-j+2} + (j + p + 2) \binom{p+j}{p-j} \varkappa^{2j} A^{p-j+1} \geqslant 0.$$

We may then write, for n even,

$$\hat{\beta}_n = \left(\frac{n}{2} + 1\right) A^{\frac{n}{2}+1} + \sum_{j=2}^{n+4} q(n, j) \frac{1}{\lambda^j}$$

where $q(n, j) \geqslant 0$ with $q(n, j) = 0$ for j odd. An entirely similar analysis shows that if n is odd, $\hat{\beta}_n$ is given by

$$\hat{\beta}_n = \sum_{j=1}^{n+4} q(n, j) \frac{1}{\lambda^j}$$

with coefficients $q(n, j) \geqslant 0$ which vanish for j even. The equation (3) now implies that the measures β_n all exist (proving Theorem 1) and that they have the form stated in Theorem 2, provided we absorb the factor $\frac{1}{2\pi(j-1)!}$ from (3) into $q(n,j)$.

The two special cases can of course be deduced from the general form for $q(n, j)$, but it is easier to directly expand $k(r/t, rt)$ from the first identity in (7). The results are

$$\text{Case (i):} \qquad k(r/t, rt) = \frac{1}{\lambda^2} r^{-2} + \sum_{n=0}^{\infty} (n+1) A^{n+1} r^{2n},$$

$$\text{Case (ii):} \qquad k(r/t, rt) = \sum_{n=-2}^{\infty} \frac{n+3}{\lambda^{n+4}} r^n,$$

and the desired conclusions follow from (3).

CONCLUDING REMARKS. Given a Jordan domain Ω satisfying our hypotheses, the domain $\Lambda = \{-\log z : z \in \Omega\}$ is right-translation invariant, and the map $W: f(z) \to e^{-\zeta} f(e^{-\zeta})$ is a unitary operator from $A^2(\Omega)$ onto $A^2(\Lambda)$ satisfying $W(e^{-t} D_t) = S_t W$, where $(S_t g)(\zeta) = g(\zeta + t)$. Our conditions on Ω say that for some $c > 0$

$$\Lambda \subset \{x + iy : x > -c, -\pi/2 < y < \pi/2\},$$

that the top and bottom legs of $\partial\Lambda$ are asymptotic to the lines $y = \pm\pi/2$ respectively, and that "$\partial\Lambda$ is analytic at ∞". Therefore, within this class of Λ we have characterized those for which $\{S_t^*\}_{t\geq 0}$ is subnormal on $A^2(\Lambda)$. Note that in special case (i), Λ is a half-strip. On the other hand, it was shown in [6, § 4.2] that if Λ is a right--translation invariant sector with interior angle π/n, $n = 2,3, \ldots$, then $\{S_t^*\}_{t\geq 0}$ is subnormal on $A^2(\Lambda)$, though the measure μ has a rather different character. The problems of finding all Λ for which $\{S_t^*\}_{t\geq 0}$ is subnormal, and more generally, of describing the structure of $\{S_t\}_{t\geq 0}$ for any Λ, seem to me worthy of further investigation.

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