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# JOINT QUASITRIANGULARITY OF A SOLID TORUS SUPPORTED PAIR AND A QUESTION OF FIALKOW

SUDHIR GOKHALE and NORBERTO SALINAS

## 1. INTRODUCTION

An essentially commuting pair $(U, E)$ of essentially normal operators acting on a separable infinite dimensional complex Hilbert space $\mathscr{H}$ is called a *solid torus supported pair* if its joint essential spectrum $\sigma_e(U, E)$ is contained in the solid torus $S^1 \times \mathbf{D}$, where $S^1$ is the unit circle in the complex plane $\mathbf{C}$ and $\mathbf{D}$ is the closed unit disc in $\mathbf{C}$. In this paper we give a spectral characterization for the joint quasitriangularity of a special solid torus supported pair $(U, E)$ of operators acting on $\mathscr{H}$ (see Theorem 1.1). As a particular consequence of this characterization we construct for a given integer $n \geqslant 2$ a simple example of a commuting pair $(U, E)$ of essentially normal operators which is not jointly quasitriangular and for which every polynomial in $(U, E)$ of degree at most $n-1$ in $E$ is quasitriangular (see Example 2.4). This answers a recent question raised by Professor L. Fialkow (see [7]). For the history and general background of the problem of joint quasitriangularity of operators acting on $\mathscr{H}$ we refer the reader to [9] and especially to Remark 6.8.

In order to state our spectral characterization we need to introduce the following terminology after formally defining the joint quasitriangularity of a pair of operators acting on $\mathscr{H}$.

A pair $(U, E)$ of operators acting on $\mathscr{H}$ is said to be *jointly quasitriangular* if there exists an increasing sequence $(P_m)_{m=1}^{\infty}$ of finite rank projections that converges strongly to the identity operator $1_{\mathscr{H}}$ on $\mathscr{H}$ and for which both $\|(1-P_m)UP_m\|$ and $\|(1-P_m)EP_m\|$ tend to zero as $m$ tends to infinity.

Throughout, let $n$ denote a positive integer and let $k$ be in $\{1, 2, \ldots\} \cup \{\infty\}$. Let $\mathbf{Z}_k^+ = \{1, \ldots, k\}$ if $k$ is a positive integer and let $\mathbf{Z}_k^+$ denote the set $\mathbf{Z}^+$ of all positive integers if $k = \infty$. Let $K$ be a $2\pi/n$-rotation invariant (i.e. $e^{2\pi i/n}K = K$) compact and connected subset of $\mathbf{D}$ and assume that $S^1$ is contained in $K$. Further let $\sim$ denote the equivalence relation in the family of all holes in $K$ (i.e. in the family of all bounded components of $\mathbf{C}\setminus K$) defined by $\Omega \sim \Omega'$ if $\Omega' = e^{2\pi ij/n}\Omega$ for some $1 \leqslant j \leqslant n$. Let $\{\Omega_l \mid l \in \mathbf{Z}_k^+\}$ denote a family of holes in $K$ obtained by choosing one element from each of the equivalence class of holes in $K$. Choose an element $c_l$ from $\Omega_l$ for each $l$ in $\mathbf{Z}_k^+$ and observe that $K = \mathbf{D}\setminus \bigcup_{(l,j)} e^{2\pi ij/n}\Omega_l$. Now rotate the top of the cylinder $[0, 1]\times K$ by $2\pi/n$ radians and then identify its lower and upper faces. The resulting nonempty compact and connected subset $X'$ of $\mathbf{R}^3$ is then homeomorphic to the subset $X = \{(e^{2\pi it}, ze^{2\pi it/n}) \mid (t, z) \in [0, 1]\times K\}$ of the solid torus $S^1\times\mathbf{D}$. We observe that each hole $\Omega_l$ in $K$ generates a hole in $X'$ which winds around $n$-times inside $X'$ before joining itself. Furthermore, we note that the family of all these holes is pairwise disjoint although the boundaries of any two of its members may be touching each other. If $K$ does not contain zero then, throughout, we assume without loss of generality that $\Omega_1$ contains zero and $c_1 = 0$.

The main result of this paper is:

THEOREM 1.1. *A solid torus supported pair $(U, E)$ of operators acting on $\mathscr{H}$ with $\sigma_e(U, E) = X$ is jointly quasitriangular if and only if the indices of the Fredholm operators $U$ and $E^n - (c_l)^n U$ are nonnegative for each $l$ in $\mathbf{Z}_k^+$.*

Since in its proof we have used well known definitions and facts from both K-theory and Ext-theory for $C^*$-algebras we refer the reader to [10], [4], [5] for all such definitions and facts.

## 2. THE FINITE CASE AND FIALKOW'S QUESTION

In this section we prove Theorem 1.1 when $\mathbf{Z}_k^+ = \{1, \ldots, k\}$ and $k$ is a positive integer. We then answer a recent question of Fialkow as mentioned in the introduction.

Let $\nu(t, z) = (e^{2\pi it}, ze^{2\pi it/n})$ whenever $(t, z)$ belongs to $[0, 1]\times K$ and let $\sim$ be the natural equivalence relation in $[0,1]\times K$ induced by $\nu$. Then the quotient map $\tilde{\nu}$ from $[0, 1]\times K/\sim$ onto $X$ is a homeomorphism and induces an isomorphism $\tilde{\nu}^*$ from the $C^*$-algebra $C(X)$ of complex valued continuous functions defined on $X$ onto $C([0, 1]\times K/\sim)$. The function $\rho\colon C(K)\to C(K)$ defined by $\rho(a)(z) = a(ze^{2\pi i/n})$ is an automorphism on $C(K)$. Since for $\chi$ in the mapping torus $C^*$-algebra $T_\rho(C(K))$ of $C(K)$ by $\rho$, $\chi(1) = \rho(\chi(0))$, the rule $v$ defined from $C([0,1]\times K/\sim)$ to $T_\rho(C(K))$ by taking $v(y)(t)(z) = y(\overline{(t,z)})$ is an onto isomorphism of unital $C^*$-algebras. It follows that $(v\circ\tilde{\nu}^*)_* = v_*\circ(\tilde{\nu}^*)_*$ is an abelian group isomorphism induced by $v\circ\tilde{\nu}^*$ from the $\mathrm{K}_1$-group $\mathrm{K}_1(C(X))$ onto $\mathrm{K}_1(T_\rho(C(K)))$. Let $Z_1, Z_2$ be the coordinate functions from $X$ into $\mathbf{C}$ while $u_1$, $u_2$ be the maps from $[0, 1]$ to $C(K)$ respectively defined by the rules $u_1(t)(z) = e^{2\pi it}$ and $u_2(t)(z) = ze^{2\pi it/n}$. We then have:

LEMMA 2.1. *If $K$ contains zero then the $\mathrm{K}_1$-group, $\mathrm{K}_1(T_\rho(\mathscr{A}))$, of the mapping torus $C^*$-algebra of $\mathscr{A} = C(K)$ by the automorphism $\rho$ on $\mathscr{A}$ is freely generated by the set $\{[u_1]\}\cup\{[u_2^n - (c_l)^n u_1] \mid l\in\mathbf{Z}_k^+\}$. Further, $\mathrm{K}_1(C(X))$ is freely gener-*

*ated by the set* $\{[Z_1]\} \cup \{[Z_2^n - (c_l)^n Z_1] \mid l \in \mathbf{Z}_k^+\}$. *However, if* $K$ *does not contain zero then* $K_1(T_\rho(\mathscr{A}))$ *and* $K_1(C(X))$ *are freely generated respectively by the sets* $\{[u_1], [u_2]\} \cup \{[u_2^n - (c_l)^n u_1] \mid l \in \mathbf{Z}_k^+ \setminus \{1\}\}$ *and* $\{[Z_1], [Z_2]\} \cup \{[Z_2^n - (c_l)^n Z_1 \mid l \in \in \mathbf{Z}_k^+ \setminus \{1\}\}$.

*Proof.* Let $e\colon T_\rho(\mathscr{A}) \to \mathscr{A}$ be the morphism defined by $e(\chi) = \chi(0)$ and let $\sigma$ be the inclusion of the loop algebra $\Omega\mathscr{A}$ of $\mathscr{A}$ into $T_\rho(\mathscr{A})$. If $\beta$ is the Bott isomorphism from the $K_0$-group $K_0(\mathscr{A})$ onto $K_1(\Omega\mathscr{A})$ then the cyclic six term sequence in the diagram below

$$(*) \qquad \begin{array}{ccccc} K_1(\mathscr{A}) & \longrightarrow & K_0(T_\rho(\mathscr{A})) & \xrightarrow{e_*} & K_0(\mathscr{A}) \\ {\scriptstyle \rho_* - \mathrm{id}} \uparrow & & & & \downarrow {\scriptstyle \rho_* - \mathrm{id}} \\ K_1(\mathscr{A}) & \xleftarrow[e_*]{} & K_1(T_\rho(\mathscr{A})) & \xleftarrow[\sigma_* \circ \beta]{} & K_0(\mathscr{A}) \end{array}$$

is exact (see Lemma 1 in [8]). Hence $K_0(\mathscr{A})$ is a free abelian group generated by $[1]$, where $1\colon K \to \mathbf{C}$ is the unit idempotent element of $\mathscr{A}$ and the invertible map $\rho_* - \mathrm{id}$ at the right of the diagram $(*)$ is the zero map. In particular $\ker(\sigma_* \circ \beta) = 0$ and so the following sequence

$$0 \to K_0(\mathscr{A}) \xrightarrow{\sigma_* \circ \beta} K_1(T_\rho(\mathscr{A})) \xrightarrow{e_*} \operatorname{img}(e_*) \to 0$$

is exact. We now compute $\operatorname{img}(e_*) = \ker(\rho_* - \mathrm{id})$ where $\rho_* - \mathrm{id}$ is the vertical map at the left of the diagram $(*)$. For $l$ in $\mathbf{Z}_k^+$ and $j$ in $\{1, \dots, n\}$ define the invertible elements $a_{lj}$ of $\mathscr{A}$ by taking $a_{lj}(z) = z - e^{2\pi i j/n} c_l$. Now suppose $K$ contains zero. Then the abelian group $K_1(\mathscr{A})$ is freely generated and the set given by $\{[a_{lj}] \mid (l, j) \in \mathbf{Z}_k^+ \times \{1, \dots, n\}\}$ generates it. Hence an element of $K_1(\mathscr{A})$ is of the form $\prod_{l \in \mathbf{Z}_k^+} [a_{l1}]^{m_{l1}} \dots [a_{ln}]^{m_{ln}}$ where $m_{l1}, \dots, m_{ln}$ are some integers. It belongs to $\ker(\rho_* - \mathrm{id})$ if and only if

$$(**) \qquad \prod_{l \in \mathbf{Z}_k^+} [\rho(a_{l1})]^{m_{l1}} \cdot \ldots \cdot [\rho(a_{ln})]^{m_{ln}} \cdot [a_{l1}]^{-m_{l1}} \cdot \ldots \cdot [a_{ln}]^{-m_{ln}} = [1].$$

However,

$$\rho(a_{lj})(z) = a_{lj}(z e^{2\pi i/n}) = z e^{2\pi i/n} - e^{2\pi i j/n} c_l = (z - e^{2\pi i(j-1)/n} c_l) e^{2\pi i/n} = a_{l(j-1)}(z) e^{2\pi i/n}$$

for each $l$ in $\mathbf{Z}_k^+$ and $j$ in $\{1, \dots, n\}$ where by definition $a_{l0} = a_{ln}$. This shows that as elements of the $K_1$-group of $\mathscr{A}$, the elements $[\rho(a_{lj})]$ and $[a_{l(j-1)}]$ are equal for each $l$ in $\mathbf{Z}_k^+$ and $j$ in $\{1, \dots, n\}$. In particular it follows from $(**)$ that $\prod_{l \in \mathbf{Z}_k^+} [a_{l1}]^{m_{l1}} \cdot \ldots \cdot$

$\cdot\, [a_{ln}]^{m_{ln}}$ is in $\ker(\rho_* - \mathrm{id})$ if and only if $m_{l1} = \ldots = m_{ln}$ for each $l$ in $\mathbf{Z}_k^+$. Thus $\mathrm{img}(e_*) = \ker(\rho_* - \mathrm{id})$ is a free abelian group and the set given by $\{[a_{l1}\cdot \ldots \cdot a_{ln}] \mid l \in \mathbf{Z}_k^+\}$ generates it. We note that

$$(a_{l1}\cdot \ldots \cdot a_{ln})(z) = \prod_{j=1}^{n} (z - c_l e^{2\pi i j/n}) = z^n - (c_l)^n$$

for each $l$ in $\mathbf{Z}_k^+$. Thus the short sequence

$$0 \to \mathrm{K}_0(\mathscr{A}) \xrightarrow{\sigma_*\circ\beta} \mathrm{K}_1(T_p(\mathscr{A})) \xrightarrow{e_*} \mathrm{img}(e_*) \to 0$$

is exact and both $\mathrm{K}_0(\mathscr{A})$ and $\mathrm{img}(e_*)$ are free. From this it follows that $\mathrm{K}_1(T_p(\mathscr{A}))$ is free and is generated by the union of the two sets $\{(\sigma_*\circ\beta)([1])\}$ and $\{y_l \mid l \in \mathbf{Z}_k^+\}$ where for each $l$ in $\mathbf{Z}_k^+$ the element $y_l$ of $\mathrm{K}_1(T_p(\mathscr{A}))$ is chosen from the nonempty set $e_*^{-1}(\{[a_{l1}\cdot \ldots \cdot a_{ln}]\})$. Since $(\sigma_*\circ\beta)([1]) = [u_1]$ and $e_*([u_2^n - (c_l)^n u_1]) = [a_{l1}\cdot \ldots \cdot a_{ln}]$ it follows that $\mathrm{K}_1(T_p(\mathscr{A}))$ is freely generated by $\{[u_1]\} \cup \{[u_2^n - (c_l)^n u_1] \mid l \in \mathbf{Z}_k^+\}$. Now

$$v\circ\bar{v}^*(Z_1)(t)(z) = v(\bar{v}^*(Z_1))(t)(z) = \bar{v}^*(Z_1)(\overline{(t, z)}) = Z_1\circ\bar{v}(\overline{(t, z)}) =$$

$$= Z_1(\bar{v}(\overline{t, z})) = Z_1(v(t, z)) = Z_1((e^{2\pi i t}, ze^{2\pi i t/n})) = e^{2\pi i t} = u_1(t)(z).$$

Analogously $v\circ\bar{v}^*(Z_2)(t)(z) = ze^{2\pi i t/n} = u_2(t)(z)$. So $v\circ\bar{v}^*(Z_1) = u_1$ and $v\circ\bar{v}^*(Z_2^n - (c_l)^n Z_1) = u_2^n - (c_l)^n u_1$. This shows that $\mathrm{K}_1(C(X))$ is freely generated by the union of the two sets $\{[Z_1]\}$ and $\{[Z_2^n - (c_l)^n Z_1] \mid l \in \mathbf{Z}_k^+\}$.

In the case when $K$ does not contain zero we need only to go through analogous arguments and observe that the sets given by $\{[a_{11}]\} \cup \{[a_{lj}] \mid (l, j) \in (\mathbf{Z}_k^+ \setminus \{1\}) \times \{1, \ldots, n\}\}$ and $\{[a_{11}]\} \cup \{[a_{l1}\cdot \ldots \cdot a_{ln}] \mid l \in \mathbf{Z}_k^+ \setminus \{1\}\}$ respectively generate $\mathrm{K}_1(\mathscr{A})$ and $\mathrm{img}(e_*) = \ker(\rho_* - \mathrm{id})$.

In order to trace a joint quasitriangulating set of generators for the Ext-group $\mathrm{Ext}(C(X))$ of $C(X)$ we look to the boundaries of the components of $\mathbf{C}^2 \setminus X$. The boundary of the unbounded component of $\mathbf{C}^2 \setminus X$ can be conceived as being generated by $S^1 \times \{1\}$. On the other hand, for each $l$ in $\mathbf{Z}_k^+$, the boundary of the hole in $X$ corresponding to $\Omega_l$ can be thought of as generated by $\{1\} \times \partial\Omega_l$. Note the set $Z = S^1 \times \{1\} \cup \bigcup_{l \in \mathbf{Z}_k^+} (\{1\} \times \partial\Omega_l)$ is contained in $X$. Let $\gamma\colon Z \hookrightarrow X$ be the inclusion map.

In the following theorem we show that the image of the natural generating system of $\mathrm{Ext}(C(Z))$ under the homomorphism $\gamma_*$ at the Ext level is a desired set of generators for $\mathrm{Ext}(C(X))$. We note that $\mathrm{Ext}(C(Z))$ is naturally isomorphic to

$\mathrm{Ext}(S^1\times\{1\}) \oplus (\bigoplus_{l\in Z_k^+} \mathrm{Ext}(\{1\}\times\partial\Omega_l)) \cong \mathbf{Z} \oplus (\bigoplus_{l\in Z_k^+} \mathbf{Z})$ because each $\Omega_l$ is simply connected. For a given bounded simply connected open subset $\Omega$ of $\mathbf{C}$ let $T_\Omega$ be an essentially normal operator which is quasitriangular and for which $\sigma_e(T_\Omega) \subseteq \partial\Omega$, $\sigma(T_\Omega) = \bar{\Omega}$, $\mathrm{ind}_\Omega(T_\Omega)$, that is, the index of $T_\Omega - \lambda_\Omega$ for a $\lambda_\Omega$ in $\Omega$ is 1 and $\|T_\Omega\| \leqslant \max\{|z| \mid z \in \bar{\Omega}\}$ (see page 122 and 125 in [4]). Let $\tau_0 = \Pi\circ\mu_0$ be a trivial extension by $C(Z)$ and let $\tau_z$ be the extension by $C(Z)$ induced by the pair $(U_+^* \oplus \mu_0(\chi_1), 1 \oplus \mu_0(\chi_2))$ where $\chi_1, \chi_2$ are the coordinate maps from $Z$ into $\mathbf{C}$. For each $l$ in $\mathbf{Z}_k^+$, let $T_{\Omega_l}$ be denoted by $T_l$ and let $\tau_l$ be the extension by $C(Z)$ induced by the pair $(1 \oplus \mu_0(\chi_1), T_l \oplus \mu_0(\chi_2))$. We observe that the pairs inducing $\tau_z, \tau_1, \ldots, \tau_k$ are actually commuting pairs of essentially normal operators and are jointly quasi-triangular. Thus not only $\mathrm{Ext}(C(Z))$ is isomorphic to $\mathbf{Z} \oplus (\bigoplus_{l\in Z_k^+} \mathbf{Z})$ but has the natural generating system given by $\{[\tau_z]\} \cup \{[\tau_l] \mid l \in \mathbf{Z}_k^+\}$. Using these notations we have:

THEOREM 2.2. *Let $\gamma: Z \hookrightarrow X$ be the inclusion map. Then the abelian group* $\mathrm{Ext}(C(X))$ *is freely generated by* $\{\gamma_*([\tau_z])\} \cup \{\gamma_*([\tau_l]) \mid l \in \mathbf{Z}_k^+\}$. *In particular the natural homomorphism $\gamma_*$ from* $\mathrm{Ext}(C(Z))$ *to* $\mathrm{Ext}(C(X))$ *is an onto isomorphism.*

*Proof.* We know that $X$ is homeomorphic to a subset of $\mathbf{R}^3$. So by a result of L. Brown (see [3]) the index induced map $\gamma_\infty: \mathrm{Ext}(C(X)) \to \mathrm{Hom}(\mathrm{K}_1(C(X)), \mathbf{Z})$ is an onto isomorphism. If $K$ contains zero then by Lemma 2.1 the $\mathrm{K}_1$-group of $C(X)$ is freely generated by $\{[Z_1]\} \cup \{[Z_2^n - (c_m)^n Z_1] \mid m \in \mathbf{Z}_k^+\}$. Let $h_z$ be the homomorphism from $\mathrm{K}_1(C(X))$ to $\mathbf{Z}$ determined by $h_z([Z_1]) = 1$ and $h_z([Z_2^n - (c_m)^n Z_1]) = 0$ for each $m$ in $\mathbf{Z}_k^+$. Also for each $l$ in $\mathbf{Z}_k^+$ let $h_l$ denote the homomorphism from $\mathrm{K}_1(C(X))$ to $\mathbf{Z}$ determined by $h_l([Z_1]) = 0$ and $h_l([Z_2^n - (c_m)^n Z_1]) = \delta_{lm}$ for each $m$ in the set $\mathbf{Z}_k^+$. Since $\mathbf{Z}_k^+$ is a finite set it is easy to verify that $\mathrm{Hom}(\mathrm{K}_1(C(X)), \mathbf{Z})$ is a free abelian group and is generated by $\{h_z\} \cup \{h_l \mid l \in \mathbf{Z}_k^+\}$. Thus to prove the theorem it suffices to observe that $\gamma_\infty(\gamma_*([\tau_z])) = h_z$ and $\gamma_\infty(\gamma_*([\tau_l])) = h_l$ for each $l$ in $\mathbf{Z}_k^+$. Let $\tau_0'$ be a trivial extension by $C(X)$ and denote $\gamma_\infty(\gamma_*([\tau_z]))$ and $\gamma_\infty(\gamma_*([\tau_l]))$ respectively by $h_z'$ and $h_l'$. Now

$$h_z'([Z_1]) = \mathrm{ind}(((\tau_z\circ\gamma^*)(Z_1) \oplus \tau_0'(Z_1))) =$$

$$= \mathrm{ind}(\tau_z(\chi_1)) = \mathrm{ind}(U_+^* \oplus \mu_0(\chi_1)) = 1$$

and for $m$ in $\mathbf{Z}_k^+$ we have

$$h_z'([Z_2^n - (c_m)^n Z_1]) = \mathrm{ind}(\tau_z(\chi_2^n - (c_m)^n \chi_1)) =$$

$$= \mathrm{ind}((1 \oplus \mu_0(\chi_2))^n - (c_m)^n (U_+^* \oplus \mu_0(\chi_1))) =$$

$$= \mathrm{ind}(1 - (c_m)^n U_+^*) + \mathrm{ind}(\mu_0(\chi_2^n - (c_m)^n \chi_1)) = 0,$$

after observing $|c_m| < 1$ and $\mu_0(\chi_2^n - (c_m)^n\chi_1)$ is a Fredholm normal operator. For $l, m$ in $\mathbf{Z}_k^+$ we can compute analogously to get:

$$h_l'([Z_1]) = \operatorname{ind}(\tau_l(\chi_1)) = \operatorname{ind}(1 \oplus \mu_0(\chi_1)) = 0$$

and

$$h_l'([Z_2^n - (c_m)^n Z_1]) = \operatorname{ind}(\tau_l(\chi_2^n - (c_m)^n\chi_1)) =$$

$$= \operatorname{ind}((T_l \oplus \mu_0(\chi_2))^n - (c_m)^n (1 \oplus \mu_0(\chi_1))) =$$

$$= \operatorname{ind}((T_l^n - (c_m)^n) \oplus \mu_0(\chi_2^n - (c_m)^n\chi_1)) =$$

$$= \operatorname{ind}(T_l^n - (c_m)^n) = \operatorname{ind}\left(\prod_{j=1}^{n} (T_l - c_m e^{2\pi i j/n})\right) =$$

$$= \operatorname{ind}((T_l - c_m)) + \sum_{j=1}^{n-1} \operatorname{ind}(T_l - c_m e^{2\pi i j/n}).$$

Since $\sigma_e(T_l) \subseteq \partial\Omega_l$, $\sigma(T_l) = \overline{\Omega}_l$,

$$\{e^{2\pi i j/n}\Omega_l \mid (l,j) \in \mathbf{Z}_k^+ \times \{1, \ldots, n\}\}$$

is a pairwise disjoint family, each $\Omega_m$ does not contain zero and $\operatorname{ind}(T_l - c_m) = \delta_{lm}$ it follows that $h_l'([Z_2^n - (c_m)^n Z_1]) = \delta_{lm}$. These computations establish the theorem in the present case.

If $K$ does not contain zero then for analogous computations we need only to use the corresponding set of generators of the free abelian group $K_1(C(X))$ as stated in Lemma 2.1 and note the obvious changes in the definitions of $h_z$, $h_1$ and $h_l$ for $l$ in $\mathbf{Z}_k^+\setminus\{1\}$.

The next corollary is Theorem 1.1 in the case when $X$ has finitely many holes.

COROLLARY 2.3. *A solid torus supported pair $(U, E)$ of operators acting on $\mathscr{H}$ with $\sigma_e(U, E) = X$ is jointly quasitriangular if and only if the indices of the Fredholm operators $U$ and $E^n - (c_l)^n U$, $l \in \mathbf{Z}_k^+$ are nonnegative.*

*Proof.* Recall that $X = \{(e^{2\pi i t}, z e^{2\pi i t/n}) \mid (t, z) \in [0, 1] \times K\}$, and that we have denoted the coordinate functions from $X$ into $\mathbf{C}$ by $Z_1$, $Z_2$. Since $\sigma_e(U) = \operatorname{Ran} Z_1 = S^1$ and

$$\sigma_e(E^n - (c_l)^n U) = \operatorname{Ran}(Z_2^n - (c_l)^n Z_1) =$$

$$= \{(z^n - (c_l)^n)e^{2\pi i t} \mid (t, z) \in [0, 1] \times K\},$$

zero does not belong to both $\sigma_e(U)$ and $\sigma_e(E^n-(c_l)^n U)$ for each $l$ in $\mathbf{Z}_k^+$. In particular $U$ and $E^n-(c_l)^n U$ are Fredholm whenever $(U, E)$ is a solid torus supported pair of operators with $\sigma_e(U, E)=X$. If $(U, E)$ is jointy quasitriangular then it is known that any polynomial operator in $U$, $E$ is quasitriangular. In particular $U, E^n-(c_l)^n U$, $l\in\mathbf{Z}_k^+$ are quasitriangular. It now follows that their indices are nonnegative (see [6]). Now suppose the Fredholm operators $U$ and $E^n-(c_l)^n U$, $l\in\mathbf{Z}_k^+$ have nonnegative indices. Let $\tau$ be the extension by $C(X)$ induced by the pair $(U, E)$. Then by Theorem 2.2 there exists unique integers $a_z$, $a_l$, $l\in\mathbf{Z}_k^+$ such that $[\tau]=a_z\gamma_*([\tau_z])+ +\sum_{l\in\mathbf{Z}_k^+} a_l\gamma_*([\tau_l])$. Now

$$\operatorname{ind}(U)=\operatorname{ind}(\tau(Z_1))=\gamma_\infty([\tau])\,([Z_1])=$$

$$=a_z\gamma_\infty(\gamma_*([\tau_z]))\,([Z_1])+\sum_{l\in\mathbf{Z}_k^+} a_l\gamma_\infty(\gamma_*([\tau_l]))\,([Z_1])=$$

$$=a_z h_z'([Z_1])+\sum_{l\in\mathbf{Z}_k^+} a_l h_l'([Z_1])=a_z \qquad \text{(see Theorem 2.2).}$$

Also, for $m$ in $\mathbf{Z}_k^+$, it follows that $\operatorname{ind}(E^n-(c_m)^n U)=a_m$. Thus $a_z, a_1, \ldots, a_k$ are all nonnegative. This shows that

$$[\tau]=\gamma_*\left(\left(\left(\bigoplus_1^{a_z}\tau_z\right)\oplus\left(\bigoplus_{l\in\mathbf{Z}_k^+}\left(\bigoplus_1^{a_l}\tau_l\right)\right)\right)\right)=$$

$$=\left[\left(\left(\left(\bigoplus_1^{a_z}\tau_z\right)\oplus\left(\bigoplus_{l\in\mathbf{Z}_k^+}\left(\bigoplus_1^{a_l}\tau_l\right)\right)\right)\circ\gamma^*\right)\oplus\tau_0'\right]$$

where $\tau_0'=\Pi\circ\mu_0'$ is a trivial extension by $C(X)$. Hence there exists a unitary transformation $L$ from $\mathscr{H}$ onto $\left(\left(\bigoplus_1^{a_z}\mathscr{H}\right)\oplus\left(\bigoplus_{l\in\mathbf{Z}_k^+}\left(\bigoplus_1^{a_l}\mathscr{H}\right)\right)\right)\oplus\mathscr{H}$ such that

$$U=L^*\left(\left(\bigoplus_1^{a_z}(U_+^*\oplus\mu_0(\chi_1))\right)\oplus\left(\left(\bigoplus_{l\in\mathbf{Z}_k^+}\left(\bigoplus_1^{a_l}(1\oplus\mu_0(\chi_1))\right)\right)\oplus\mu_0'(Z_1)\right)\right)L+C_1$$

and

$$E=L^*\left(\left(\bigoplus_1^{a_z}(1\oplus\mu_0(\chi_2))\right)\oplus\left(\left(\bigoplus_{l\in\mathbf{Z}_k^+}\left(\bigoplus_1^{a_l}(T_l\oplus\mu_0(\chi_2))\right)\right)\oplus\mu_0'(Z_2)\right)\right)L+C_2$$

for some compact operators $C_1$, $C_2$ on $\mathscr{H}$. Now since the pairs $(U_+^*\oplus\mu_0(\chi_1)$, $1\oplus\mu_0(\chi_2))$, $(1\oplus\mu_0(\chi_1)$, $T_l\oplus\mu_0(\chi_2))$, $l\in\mathbf{Z}_k^+$ and $(\mu_0'(Z_1), \mu_0'(Z_2))$ are jointly quasitriangular, the pair $(U, E)$ is jointly quasitriangular.

In [7, Question 4.12] Fialkow asked if $A_1, \ldots, A_m$ are commuting operators in $L(\mathscr{H})$ and every linear combination of $A_1, \ldots, A_m$ is quasitriangular then is $(A_1, \ldots, A_m)$ jointly quasitriangular? In the following example we shall show that this question has a negative answer. In fact, we shall construct a pair of operators that is not jointly quasitriangular and that satisfies even stronger properties than the ones stipulated in the question. In this construction we freely use the notations introduced just prior to Theorem 1.1 and Theorem 2.2. However, we assume $n \geqslant 2$, $0 \in K$ and the finitely many chosen holes $\Omega_l$, $l \in \mathbf{Z}_k^+$ of $K$ are invariant under conjugation.

EXAMPLE 2.4. Let $\tau_0' = \Pi \circ \mu_0'$ be a trivial extension of $\mathscr{K}(\mathscr{H})$ by $C(X)$, $\mu_0'(Z_1) = U_0$ and $\mu_0'(Z_2) = E_0$. If $U = U_0 \oplus (\bigoplus_{l \in \mathbf{Z}_k^+} 1_{\mathscr{H}})$ and $E = E_0 \oplus (\bigoplus_{l \in \mathbf{Z}_k^+} T_l^*)$ then $(U, E)$ is a commuting pair of essentially normal operators which is not jointly quasitriangular and for which every polynomial operator in $U$, $E$ of degree at most $n-1$ in $E$ is quasitriangular.

Due to our conjugation hypothesis we observe that $(U, E)$ is a solid torus supported pair with $\sigma_e(U, E) = X$. For $m$ in $\mathbf{Z}_k^+$ the operator $E^n - (\bar{c}_m)^n U$ is the direct sum of the normal Fredholm operator $E_0^n - (\bar{c}_m)^n U_0$ and the Fredholm operator $\bigoplus_{l \in \mathbf{Z}_k^+} (T_l^{*n} - (\bar{c}_m)^n 1_{\mathscr{H}})$ of index $-1$ (use similar computations done in Theorem 2.2 against $h_l'([Z_2^n - (c_m)^n Z_1])$). It follows that $(U, E)$ is not jointly quasitriangular. However, we claim below that even if $(U, E)$ is any solid torus supported pair with $\sigma_e(U, E) = X$ (as mentioned earlier $0 \in K$) and $U$ is a unitary operator then every polynomial operator $T$ in $U$, $E$ of degree at most $n-1$ in $E$ is quasitriangular (in fact quasidiagonal). It suffices to show that $\operatorname{ind}(p(U, E)) = 0$ whenever $p(Z_1, Z_2)$ is an invertible polynomial in $Z_1$, $Z_2$ from $C(X)$ of degree at most $n-1$ in $Z_2$. Let $\tau$ from $C(X)$ to $Q(\mathscr{H})$ be the extension induced by the pair $(U, E)$. Then as in the proof of Corollary 2.3 the element $[\tau]$ of $\operatorname{Ext} C(X)$ is given by $a_z \gamma_*([\tau_z]) + \sum_{l \in \mathbf{Z}_k^+} a_l \gamma_*([\tau_l])$ where $a_z = \operatorname{ind}(U)$ and $a_l = \operatorname{ind}(E^n - (c_l)^n U)$.

Since $K_1(C(X))$ is freely generated by $\{[Z_1], [Z_2^n - (c_1)^n Z_1], \ldots, [Z_2^n - (c_k)^n Z_1]\}$ and $[p(Z_1, Z_2)]$ is in $K_1(C(X))$ there exist integers $b_z, b_1, \ldots, b_k$ such that

$$[p(Z_1, Z_2)] = [Z_1]^{b_z} \cdot [Z_2^n - (c_1)^n Z_1]^{b_1} \cdot \ldots \cdot [Z_2^n - (c_k)^n Z_1]^{b_k}.$$

Now

$$\operatorname{ind}(p(U, E)) = \operatorname{ind}(\tau(p(Z_1, Z_2))) =$$

$$= \gamma_\infty([\tau])([p(Z_1, Z_2)]) = a_z b_z + \sum_{l \in \mathbf{Z}_k^+} a_l b_l.$$

Since $U$ is unitary $a_z = 0$. Hence it is sufficient to show that each $b_l = 0$. However, applying $e_* \circ (v \circ v^*)_*$ (for these notations see Lemma 2.1) to both sides of the above mentioned equality of the elements of $K_1(C(X))$ we get the equality

$$[p(u_1, u_2)(0)] = [u_1(0)]^{b_z} \cdot [(u_2^n - (c_1)^n u_1)(0)]^{b_1} \cdot \ldots \cdot [(u_2^n - (c_k)^n u_1)(0)]^{b_k}$$

of elements in $K_1(C(K))$. Since $u_1(0)$ and $u_2(0)$ are respectively the unit map and the inclusion map from $K$ into $\mathbf{C}$ we get $[q] = \prod_{l=1}^{k} [q_l]^{b_l}$ where $q$ is an invertible polynomial element of $C(K)$ of degree $r \leqslant n-1$ and $q_l(z) = z^n - (c_l)^n = \prod_{j=1}^{n} (z - c_l e^{2\pi i j/n})$, $z \in K$. Since $q(z) = \alpha(z - \alpha_1) \cdot \ldots \cdot (z - \alpha_r)$ for some $\alpha$ in $\mathbf{C}$ and $\alpha_1, \ldots, \alpha_r$ in the components of $\mathbf{C} \setminus K$ and since the $K_1$-group of $C(K)$ is freely generated by $\{[z - c_l e^{2\pi i j/n}] \mid (l, j) \in \mathbf{Z}_k^+ \times \{1, \ldots, n\}\}$ each $b_l = 0$. This establishes not only the quasidiagonality (so quasitriangularity) of $T$ but also the equality $[p(Z_1, Z_2)] = [Z_1]^{b_z}$ of elements in $K_1(C(X))$ whenever $p(Z_1, Z_2)$ is an invertible polynomial in $C(X)$ of degree at most $n-1$ in $Z_2$.

More explicitly for $n \geqslant 2$ and $0 < c < 1$ let $K$ denote the set obtained from the closed unit disc $\mathbf{D}$ by removing the $n$ open discs $\Omega$, $e^{2\pi i/n}\Omega, \ldots, e^{2\pi i(n-1)/n}\Omega$ where $\Omega$ is the open disc with center at $c$ and radius $r$ satisfying $0 < r \leqslant \min\{c \sin(\pi/n), 1 - c\}$. If $\tau_0' = \Pi \circ \mu_0'$ is a trivial extension of $K(\mathscr{H})$ by $C(X)$ where

$$X = \{(e^{2\pi i t}, z e^{2\pi i t/n}) \mid (t, z) \in [0, 1] \times K\},$$

$\mu_0'(Z_1) = U_0$, $\mu_0'(Z_2) = E_0$, $U = U_0 \oplus 1_{\mathscr{H}}$ and $E = E_0 \oplus (rU_+ + c)$ then $(U, E)$ is a commuting pair of essentially normal operators that is not jointly quasitriangular but for which every polynomial in $U, E$ of degree at most $n-1$ in $E$ is quasitriangular.

## 3. THE INFINITE CASE

In this section we prove Theorem 1.1 when $\mathbf{Z}_k^+ = \mathbf{Z}^+ = \{1, 2, \ldots\}$.

Using standard K-theory (see [10], 10.4, page 171) one can deduce that the abelian group $K_1(C(X))$ is freely generated by $\{[Z_1]\} \cup \{[Z_2^n - (c_l)^n Z_1] \mid l \in \mathbf{Z}^+\}$ or by $\{[Z_1], [Z_2]\} \cup \{[Z_2^n - (c_l)^n Z_1] \mid l \in \mathbf{Z}^+ \setminus \{1\}\}$ depending on whether 0 is in $K$ or 0 is not in $K$. In particular, the torsion free Ext-group $\operatorname{Ext}(C(X))$ $\left(\cong \operatorname{Hom}(K_1(C(X)), \mathbf{Z}) \cong \cong \prod_1^\infty \mathbf{Z}\right)$ is not a free abelian group. Hence the infinite version of Theorem 2.2 can not be true. However, we prove the infinite case of Theorem 1.1 by directly constructing a jointly quasitriangular, commuting pair $(\dot{U}, \dot{E})$ of essentially normal operators whose induced extension $\dot{\tau}$ is equivalent to the extension $\tau$ induced by the given pair $(U, E)$.

*Proof of the infinite case of Theorem 1.1.* If $(U, E)$ is jointly quasitriangular then as in Corollary 2.3 the operators $U$ and $E^n - (c_l)^n U$, $l \in \mathbf{Z}^+$, are all Fredholm and quasitriangular and so have nonnegative index. Now suppose $0 \in K$ and suppose the Fredholm operators $U$ and $E^n - (c_l)^n U$, $l \in \mathbf{Z}^+$ have nonnegative indices and that these are respectively given by $a_z$ and $a_l$.

Let $\tau_0' = \Pi \circ \mu_0'$ be a trivial extension by $C(X)$ and let $Z_1, Z_2 \colon X \to \mathbf{C}$ be the coordinate maps. Define $\dot{U} = U_+^{(a_z)} \oplus I \oplus \mu_0'(Z_1)$ and $\dot{E} = 1 \oplus T \oplus \mu_0'(Z_2)$ where $T = \bigoplus_{l=1}^{\infty} \bigoplus_{a_l} T_l$, $I = \bigoplus_{l=1}^{\infty} \bigoplus_{a_l} 1_{\mathscr{H}}$ and $T_l$ is the operator associated with $\Omega_l$ which in addition to the properties mentioned prior to Theorem 2.2 satisfies the property $T_l T_l^* - T_l^* T_l$ is a positive trace class operator (so compact) and $\operatorname{tr}(T_l T_l^* - T_l^* T_l) = m(\Omega_l)/\pi$ where $m(\Omega_l)$ is the Lebesgue measure of $\Omega_l$ (see [2]). In particular, $\|T_l T_l^* - T_l^* T_l\| \leqslant m(\Omega_l)/\pi$. The essential normality of $\dot{E}$ is a consequence of the fact that $m(\Omega_l) \to 0$ as $l \to \infty$ and the quasitriangularity of $T$ is a consequence of an Apostol, Foiaş and Voiculescu result (see [1]) or more directly of a Douglas and Pearcy result (see [6]). It is now clear that the commuting pair $(\dot{U}, \dot{E})$ is jointly quasitriangular and is made up of essentially normal operators. Since the joint essential spectrum of the pair $(U_+^{(a_z)}, 1)$ is $S^1 \times \{1\}$ and that of the pair $(I, T)$ is contained in $\bigcup_{l=1}^{\infty} \{1\} \times \partial\Omega_l$, it follows that the joint essential spectrum $\sigma_e(\dot{U}, \dot{E})$ of $(\dot{U}, \dot{E})$ is $X$. Let $\tau$ and $\dot{\tau}$ be respectively the extensions induced by $(U, E)$ and $(\dot{U}, \dot{E})$. Then $\operatorname{ind}(\dot{U}) = a_z = \operatorname{ind}(U)$ and for $l$ in $\mathbf{Z}^+$,

$$\operatorname{ind}(\dot{E}^n - (c_l)^n \dot{U}) = \operatorname{ind}(1 - (c_l)^n U_+^{(a_z)}) +$$

$$+ \operatorname{ind}(T^n - (c_l)^n) + \operatorname{ind}((\mu_0'(Z_2))^n - (c_l)^n \mu_0'(Z_1)) =$$

$$= \sum_{j=1}^{n} \operatorname{ind}(T - c_l e^{2\pi i j/n}) = \operatorname{ind}(T - c_l) + \sum_{j=1}^{n-1} \operatorname{ind}(T - c_l e^{2\pi i j/n}).$$

For each $l$ in $\mathbf{Z}^+$ and $j$ in $\{1, \ldots, n\}$, the operator $T - c_l e^{2\pi i j/n} = \bigoplus_{m=1}^{\infty} (\bigoplus_{a_m} (T_m - c_l e^{2\pi i j/n}))$ is Fredholm and $\operatorname{ind}(\bigoplus_{a_m} (T_m - c_l e^{2\pi i j/n})) = a_m \delta_{ml} \delta_{jn}$ for all $m$ in $\mathbf{Z}^+$. Hence $\operatorname{ind}(\dot{E}^n - (c_l)^n \dot{U}) = a_l = \operatorname{ind}(E^n - (c_l)^n U)$ for all $l$ in $\mathbf{Z}^+$. Since the $K_1$-group of $C(X)$ is freely generated by $\{[Z_1]\} \cup \{[Z_2^n - (c_l)^n Z_1] \mid l \in \mathbf{Z}^+\}$ and

$$\gamma_\infty([\bar{\tau}])([Z_1]) = \gamma_\infty([\tau])([Z_1]) = a_z,$$

$$\gamma_\infty([\bar{\tau}])([Z_2^n - (c_l)^n Z_1]) = \gamma_\infty([\tau])([Z_2^n - (c_l)^n Z_1]) = a_l$$

for each $l$ in $\mathbf{Z}^+$, it follows that $\gamma_\infty([\bar{\tau}]) = \gamma_\infty([\tau])$. We can now complete the proof as in Corollary 2.3.

In the case when zero does not belong to $K$ observe that $e^{2\pi i j/n}\Omega_1 = \Omega_1$ for each $j$ in $\{1, \ldots, n\}$. Since by hypothesis the operator $E^n - (c_1)^n U = E^n$ is Fredholm and has nonnegative index, the operator $E$ is also Fredholm and has nonnegative index. Let this nonnegative index of $E$ be given by $a_1$. Also by hypothesis the Fredholm operators $U$ and $E^n - (c_l)^n U$, $l \in \mathbf{Z}^+ \setminus \{1\}$ have nonnegative indices. Let these be respectively given by $a_z$ and $a_l$, $l \in \mathbf{Z}^+ \setminus \{1\}$. Now we can complete the proof as in the case when zero was assumed to belong to $K$.

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*SUDHIR GOKHALE*
*Department of Mathematics,*
*Western Illinois University,*
*Macomb, IL 61455,*
*U.S.A.*

*NORBERTO SALINAS*
*Department of Mathematics,*
*University of Kansas,*
*Lawrence, KS 66045,*
*U.S.A.*


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