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SHIFTS ON THE HYPERFINITE II_1-FACTOR

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INTRODUCTION

In [3], Powers discussed the identity preserving $*$-endomorphisms of $B(H)$ and of the hyperfinite II_1-factor R. He called such an endomorphism σ of R a *shift* of R if σ has the property that the intersection of the ranges of σ^n for $n = 1, 2, \ldots$ consists only of scalar multiples of the identity and defined the *index* of a shift σ of R by the Jones index $[R : \sigma(R)]$. He studied a class of shifts of R with the index 2, which are called binary shifts. He showed the existence of uncountably many pairwise non-conjugate binary shifts of R and that there are at least a countable infinity of binary shifts which are pairwise not outer conjugate.

In this paper, we shall study shifts of the hyperfinite II_1-factor R.

By Jones' result, the set of indices of all shifts of R is contained in the set

$$I(R) = \{4\cos^2(\pi/n)\ ;\ n = 3, 4, \ldots\} \cup [4, \infty).$$

In Section 2 it is shown that, for a given $\lambda \in I(R)$, there are at least a countable infinity of shifts of R with the index λ which are pairwise not outer conjugate. These shifts are obtained from sequences of projections in R satisfying some conditions.

In Section 3, we shall discuss shifts of R with an integer index n. These shifts are given by sequences of unitaries in R with certain conditions, which are a modification of binary shifts of Powers. In the case where $n = 2$, our result coincides with his result. The key point which Powers used in order to define the conjugacy invariant is a result of Jones index theory that the relative commutant of a subfactor with index < 4 is trivial. First, we shall show that if σ is a binary shift of R then the trivialness of the relative commutant of $\sigma(R)$ in R follows from the property of the integer 2, i.e., 2 is a prime number. Next, a class of shifts of R with a given integer index n are studied. We shall call such shifts *n-unitary shifts* and we show that if σ is an n-unitary shift of R then there is an orthonormal basis of unitaries with respect to σ in R. As an application of it, we show that, for an integer n, there are uncountably many n-unitary shifts of R which are pairwise non-conjugate.

This work was motivated by Powers' talk at the closing workshop in MSRI in Berkeley in June 1985. The author would like to express her hearty thankss to MSRI, and to R. T. Powers for sending her his preprint and also to the members of the seminar in Osaka Kyoiku University for a number of discussions on this subject.

1. PRELIMINARIES

Throughout this paper, R will denote the hyperfinite II_1 factor and tr is the canonical trace of R.

In this section, we describe the terminology of Powers. A $*$-endomorphism σ of R is a *shift* of R if $\sigma(1) = 1$ and $\bigcap_{n=1}^{\infty} \sigma^n(R) = \mathbf{C}1$, where 1 is the identity of R and $\mathbf{C}$ denotes the complex numbers. If σ is a shift of R, then $\sigma(R)$ is a subfactor of R. Then the Jones index $[R:\sigma(R)]$ of $\sigma(R)$ in R is defined. The *index* of σ is defined as $[R:\sigma(R)]$. By Jones' index theory, if $[R:\sigma(R)] < \infty$, then $\sigma(R)' \cap R$ is finite dimensional. If the index of a shift σ is finite, we call the dimension of $\sigma(R)' \cap R$ the *multiplicity* of σ. Two shifts σ_1 and σ_2 of R are *conjugate* if there is a $*$-automorphism θ of R such that $\theta\sigma_1 = \sigma_2\theta$ and *outer conjugate* if there are a $*$-automorphism θ of R and a unitary $u \in R$ such that $\theta\sigma_1 \mathrm{Ad}\, u = \sigma_2\theta$, where $\mathrm{Ad}\, u(x) = uxu^*$ for all $x \in R$.

2. NON OUTER CONJUGATE SHIFTS

If two shifts σ_1 and σ_2 of R are outer conjugate, then the index of σ_1 equals the one of σ_2. In this section, we shall show for a given index λ that there are at least a countably infinity of shifts of R with the index λ which are pairwise not outer conjugate.

Let us consider a $\lambda \in \{4\cos^2(\pi/n)\ ;\ n = 3, 4, \ldots\} \cup [4, \infty)$.

DEFINITION 2.1. A shift σ of R is called a *λ-projection shift* if there are a projection $p \in R$ and a positive integer k satisfying the following:

(i) R is generated by $\{p, \sigma(p), \sigma^2(p), \ldots\}$;

(ii) $\sigma^i(p)\sigma^j(p) = \sigma^j(p)\sigma^i(p)$, if $|i-j| \neq k$

$$\sigma^i(p)\sigma^j(p)\sigma^i(p) = (1/\lambda)\sigma^i(p) \quad \text{if } |i-j| = k$$

for all $i, j = 0, 1, 2, \ldots$;

(iii) $\mathrm{tr}(w\sigma^i(p)) = (1/\lambda)\mathrm{tr}(w)$ if w is an associative word on $\{1, p, \sigma(p), \ldots, \ldots, \sigma^{i-1}(p)\}$.

EXAMPLE 2.2. For a given $\lambda \in \{4\cos^2(\pi/n) \,;\, n = 3, 4, \ldots\} \cup [4, \infty)$ and a positive integer k, we shall show the existence of a λ-projection shift σ of R. In [1], Jones gives a family $\{e_i \,;\, i = 1, 2, \ldots\}$ of projections satisfying the following:

(a)
$$e_i e_j = e_j e_i \quad \text{for} \quad |i-j| \geqslant 2,$$

$$e_i e_{i\pm 1} e_i = (1/\lambda) e_i \quad \text{for all } i = 1, 2, \ldots;$$

(b) the von Neumann algebra generated by $\{e_i \,;\, i = 1, 2, \ldots\}$ is a hyperfinite II_1 factor P;

(c) $\mathrm{tr}(we_i) = (1/\lambda)\mathrm{tr}(w)$ if w is a word on $\{1, e_1, \ldots, e_{i-1}\}$.

For $h = 1, 2, \ldots, k$, we put $P_h = P$ and $e(h; i) = e_i$. Let R be the tensor product of $P_1, P_2, \ldots,$ and P_k. The trace of R is the tensor product of the traces of the P_h's. Then we have a $*$-endomorphism σ of R such that $\sigma^m(p) = e(j; i)$ for the positive integer $m = (i-1)k + (j-1)$. It is clear that σ satisfies the condition in Definition 2.1 for the projection $p = e(1;1)$ and the number k.

LEMMA 2.3. *Suppose σ is a λ-projection shift of R. Let a projection $p \in R$ and a positive integer k satisfy the conditions* (i), (ii) *and* (iii). *Then the index of σ is λ and k depends only on the shift σ. In the case where $\lambda > 4$,*

$$k + 1 = \min\{j;\, \sigma^j(R)' \cap R \text{ is non commutative}\}.$$

In the case where $\lambda \leqslant 4$,

$$k + 1 = \min\{j\,;\, \sigma^j(R)' \cap R \text{ is non trivial}\}.$$

Proof. For $j = 1, 2, \ldots, k$, and $i = 1, 2, \ldots$, we put $e(j; i) = \sigma^{(i-1)k+j-1}(p)$. Then the sequence $\{e(j; i) \,;\, i = 1, 2, \ldots\}$ satisfies the properties (a), (b) and (c) for $j = 1, 2, \ldots, k$. The factor R is generated by the family $\{P_j \,;\, j = 1, 2, \ldots k\}$ of factors which are pairwise commutative, where P_j is the hyperfinite II_1 factor generated by $\{e(j; i) \,;\, i = 1, 2, \ldots\}$. Let Q_j be the von Neumann algebra generated by $\{e(j; i) \,;\, i = 2, 3, \ldots\}$. Then, by [1], Q_j is a subfactor of P_j such that $[P_j : Q_j] = \lambda$. The subfactor $\sigma(R)$ is generated by the family $\{Q_1, P_2, \ldots, P_k\}$. Hence

$$\text{the index of } \sigma = [R : \sigma(R)] = [P_1 : Q_1] = \lambda.$$

For $j = 1, 2, \ldots, k$, $\sigma^j(R)' \cap R$ is generated by the family $\{Q_1' \cap P_1, \ldots, Q_{j-1}' \cap P_{j-1}, \mathbf{C}1, \ldots, \mathbf{C}1\}$ which are pairwise commutative.

Assume that $\lambda \leqslant 4$. Then Jones showed that $Q_j' \cap P_j = \mathbf{C}1$ for $j = 1, 2, \ldots, k$. Hence $\sigma^j(R)' \cap R = \mathbf{C}1$ for $j = 1, 2, \ldots, k$. On the other hand $\sigma^{k+1}(R)' \cap R$ contains the projection p. Hence $k + 1$ is the minimal positive integer j such that $\sigma^j(R)' \cap R$ is non trivial.

Assume that $\lambda > 4$. Then for each $j = 1, 2, \ldots, k$, there is a projection $f_j \in Q_j' \cap P_j$ such that $Q_j = \{x + \theta_j(x) ; x \in (f_j P_j f_j)\}$ for some $*$-isomorphism θ_j of $f_j P_j f_j$ onto $(1 - f_j)P_j(1 - f_j)$ by [2; Corollary 5.3]. Hence $Q_j' \cap P_j$ is the abelian algebra $\mathbf{C}f_j + \mathbf{C}(1 - f_j)$, so that $\sigma^j(R)' \cap R$ is an abelian von Neumann algebra for all $j = 1, 2, \ldots, k$. On the other hand, the relative commutant of the subfactor generated by $\{e(j; i) ; i = 3, 4, \ldots\}$ in P_j contains f_j and $e(j; 1)$. Since f_j is contained in $Q_j' \cap P_j$ and f_j is non trivial, it follows that f_j does not commute with $e(j; 1)$ for $j = 1, 2, \ldots, k$. Hence $\sigma^{k+1}(R)' \cap R$ is not commutative.

DEFINITION 2.4. Suppose σ is a λ-projection shift of R. Let a projection p and a positive integer k satisfy the conditions in Definition 2.1. Then p is called a *generator* of σ and k is called the *anticommutator number* of σ.

LEMMA 2.5. *Suppose σ_1 and σ_2 are λ-projection shifts of R. If σ_1 is outer conjugate to σ_2, then the anticommutator number of σ_1 equals the one of σ_2.*

Proof. If σ_1 is outer conjugate to σ_2, then there are an $*$-automorphism θ of R and a unitary u in R such that $\sigma_1 \mathrm{Ad}\, u = \theta^{-1}\sigma_2\theta$. Take an integer j and put $w = \sigma_1(u) \ldots \sigma_1^j(u)$. Then

$$\sigma_1^j(R)' \cap R = \mathrm{Ad}\, w^*(\theta^{-1}(\sigma_2^j(R)' \cap R)).$$

Hence by Lemma 2.3 the anticommutator numbers of σ_1 and σ_2 coincide.

THEOREM 2.6. *For each $\lambda \in \{4\cos^2(\pi/n) ; n = 3, 4, \ldots\} \cup [4, \infty)$, there are at least a countable infinity of outer conjugacy classes among the λ-projection shifts of R.*

Proof. Let take a $\lambda \in [4\cos^2(\pi/n) ; n = 3, 4, \ldots\} \cap [4, \infty)$. Let k be a positive integer. Then by Example 2.2, there is a λ-projection shift of R, the anticommutator number of which is k. Hence by Lemma 2.5, there are at least a countable infinity of outer conjugacy classes among the λ-projection shifts of R.

3. UNCOUNTABLY MANY NON-CONJUGATE SHIFTS

In this section we shall discuss on shifts of R, the indices of which are integers.

Let n be a positive integer. We treat a pair of sets Q and S of integers satisfying the following condition $(*)$ for some integer m:

$$(*) \quad \begin{cases} Q = (i(1), i(2), \ldots, i(m)), & 0 \leqslant i(1) < i(2) < \ldots < i(m) \\ S = (j(1), j(2), \ldots, j(m)), & j(i) = 1, 2, \ldots, n-1, \text{ for } i = 1, 2, \ldots, m. \end{cases}$$

DEFINITION 3.1. A shift σ of R is called an *n-unitary shift of R* if there is a unitary $u \in R$ satisfying the following:

(i) $u^n = 1$;

(ii) R is generated by $\{u, \sigma(u), \sigma^2(u), \dots\}$;

(iii) $\sigma^k(u)u = u\sigma^k(u)$ or $\sigma^k(u)u = \gamma u\sigma^k(u)$ for all $k = 1, 2, \dots,$ where $\gamma = \exp(2\pi i/n)$;

(iv) for each (Q, S) satisfying $(*)$, there are an integer k ($\geqslant 0$) and a non trivial $\lambda \in \mathbf{T} = \{\mu \in \mathbf{C}; |\mu| = 1\}$ such that

$$\sigma^k(u)u(Q, S) = \lambda u(Q, S)\sigma^k(u),$$

where $u(Q, S)$ is defined by

$$u(Q, S) = (\sigma^{i(1)}(u))^{j(1)}(\sigma^{i(2)}(u))^{j(2)} \dots (\sigma^{i(m)}(u))^{j(m)}.$$

The unitary u is called a *generator* of σ.

REMARK 3.2. A σ-generator u of an n-unitary shift σ has the period n. In fact, by (ii) and (iii), there is a positive integer k such that $\sigma^k(u)u = \gamma u\sigma^k(u)$. Assume $u^m = 1$ for some $m = 1, 2, \dots, n-1$. Then $u = \sigma^k(u^m)u = \gamma^m u\sigma^k(u^m) = \gamma^m u$. This contradicts that $\gamma^m \neq 1$ for such an m.

If (Q_1, S_1) and (Q_2, S_2) satisfy $(*)$, then $u(Q_1, S_1)u(Q_2, S_2) = \mu u(Q_2, S_2)u(Q_1, S_1)$ for some $\mu \in \mathbf{T}$ and $u(Q_1, S_1)u(Q_2, S_2)$ is $\mu u(Q, S)$ for some $\mu \in \mathbf{T}$ and a pair (Q, S) with $(*)$, otherwise $u(Q_1, S_1)u(Q_2, S_2)$ is a scalar multiple of 1.

PROPOSITION 3.3. *If n is a prime number, then the condition* (iv) *follows from the conditions* (i), (ii) *and* (iii).

Proof. Suppose (Q, S) satisfies $(*)$. Assume that no such k and λ as in (iv) exist. Then $u(Q, S)$ belongs to $R' \cap R$. Hence $u(Q, S) = \mu 1$ for some $\mu \in \mathbf{T}$. Therefore there are integers p and q which satisfy the condition

$$\sigma^p(u^q) = \nu\, \sigma^{i(1)}(u^{j(1)}) \dots \sigma^{i(s)}(u^{j(s)}) \quad \text{for some } \nu \in \mathbf{T},$$

where $1 \leqslant q \leqslant n-1$, $0 \leqslant i(1) < i(2) < \dots < i(s) < p$, $1 \leqslant j(\cdot) \leqslant n-1$. Let us consider the decomposition $n = qa + b$, where $1 \leqslant a \leqslant n$ and $0 \leqslant b < q$. In the case where $b \neq 0$, there is an h such that $u^{j(h)} \neq 1$ by Remark 3.2. Then $\sigma^p(u^b)$ can be expressed in a similar form to the above one. Continuing like this, we may assume that $b = 0$, so that q must be 1 because n is prime. Hence $\sigma_p(u)$ is contained in the von Neumann algebra B generated by $\{\sigma^i(u); 0 \leqslant i \leqslant p-1\}$. It implies that $\sigma^j(u)$ is contained in B for all $j = 0, 1, 2, \dots$. This is a contradiction because the hyperfinite II_1 factor R is not contained in a finite dimensional algebra B.

THEOREM 3.4. *Let σ be an n-unitary shift of R and $u \in R$ a σ-generator. Then each $x \in R$ has a unique expansion:*

$$x = x_0 1 + \sum x(Q, S)u(Q, S), \quad (x_0, x(Q, S) \in \mathbf{C})$$

in the pointwise $\|\cdot\|_2$-convergence topology, where $\sum$ is taken over the set of all (Q, S)'s satisfying the condition $(*)$, and $\|x\|_2 = (\mathrm{tr}(x^* x))^{1/2}$ for all $x \in R$.

Proof. If (Q_1, S_1) and (Q_2, S_2) satisfy $(*)$ and $(Q_1, S_1) \neq (Q_2, S_2)$, then there is a (Q, S) with $(*)$ so that $u(Q_1, S_1)^* u(Q_2, S_2) = \mu u(Q, S)$ for some $\mu \in \mathbf{T}$. On the other hand, by (iv), there are a $\lambda \in \mathbf{T}$ ($\lambda \neq 1$) and an integer k such that $u(Q, S) = \lambda \sigma^k(u)^* u(Q, S)\sigma^k(u)$. Hence $\mathrm{tr}(u(Q_1, S_1)^* u(Q_2, S_2)) = \mu \mathrm{tr}(u(Q, S)) = 0$. Thus all $u(Q, S)$'s and 1 form an orthonormal unitary basis in R, because the linear span of the $u(Q, S)$'s and 1 is $\|\cdot\|_2$-dense in R.

COROLLARY 3.5. *Let σ be an n-unitary shift of R. Then the index of σ is n.*

Proof. It is clear that $\sigma(R)$ is a subfactor of R. By Theorem 3.4, each $x \in R$ is expressed in the form

$$x = a_0 + ua_1 + \dots + u^{n-1}a_{n-1},$$

where

$$a_0 = x_0 1 + \sum x(Q, S)u(Q, S)$$

for $Q = (i(1), \dots, i(m))$ with $i(1) \neq 0$ and

$$a_i = \sum x(Q, S)u(Q, S)$$

for Q and S with $i(1) = 0$ and $j(1) = i$. Then $\{a_0, \dots, a_{n-1}\}$ is contained in $\sigma(R)$ and $E(u^i a_i) = 0$ for all $i = 1, 2, \dots, n-1$, where E is the conditional expectation of R onto $\sigma(R)$. Hence $[R : \sigma(R)] = n$.

Next, we shall consider the following condition (iv)′ for n-unitary shifts of R:

(iv)′ For each (Q, S) satisfying $(*)$, there are a positive integer k and a non trivial $\lambda \in \mathbf{T}$ such that

$$\sigma^k(u)u(Q, S) = \lambda u(Q, S)\sigma^k(u).$$

PROPOSITION 3.6. *If n is a prime number, then the condition* (iv)′ *follows from the condition* (i), (ii) *and* (iii).

Proof. Suppose (Q, S) satisfies $(*)$.

Case (1). $u(Q, S) = u^j$ for some $j \neq n$. By Proposition 3.3, the condition (iv) is satisfied. Then the integer k in (iv) for u^j must be non-zero. Hence (iv)′ is satisfied.

Case (2). $u(Q, S) \in \sigma(R)$. If there are no such k and λ as in the statement, then the unitary $u(Q, S)$ must be in the center of the factor $\sigma(R)$. This contradicts that $u(Q, S) = \mathrm{tr}(u(Q, S))1 = 0$ by Proposition 3.3.

Case (3). $u(Q, S) = u^j w$ for some integer $j = 1, 2, \ldots, n-1$ and $w = u(Q', S') \in \sigma(R)$ for some (Q', S') with $(*)$. Assume that there are no such k and λ in (iv)'. Then $u^j w \in \sigma(R)' \cap R$. Hence, for each integer i, we have that

$$[\sigma^i(u), u] = 0 \quad \text{if and only if} \quad [\sigma^i(u), w] = 0$$

and

$$\sigma^i(u)u = \gamma u \sigma^i(u), \quad \text{if and only if} \quad \sigma^i(u)w = \gamma^i w \sigma^i(u).$$

Since $w = u(Q', S')$ is contained in $\sigma(R)$, it follows that there are non-zero integer h and non trivial $\mu \in \mathbf{T}$ such that $\sigma^h(u)w = \mu w \sigma^h(u)$. Then $\mu = \gamma^j$ and

$$\sigma^h(u)u^j w = \gamma^j u^j \sigma^h(u)w = \gamma^{2j} u^j w \sigma^h(u).$$

Since $u^j w \in \sigma(R)' \cap R$ and $\sigma^h(u) \in \sigma(R)$, we have that $\gamma^{2j} = 1$. Hence $2j = n$ because j is in the set $\{1, 2, \ldots, n-1\}$. This contradicts that n is a prime number. Therefore, the statement (iv)' is satisfied.

Geoffrey Price kindly pointed out to me some typographical errors in the proof of Proposition 3.6 in a preliminary version of this paper and that he has another proof of this proposition.

THEOREM 3.7. *Suppose σ is an n-unitary shift of R with a generator $u \in R$ satisfying* (iv)'. *Then the relative commutant of $\sigma(R)$ in R is trivial.*

Proof. Let take an $x \in \sigma(R)' \cap R$. Let $\{x(Q, S)\}$ be the coefficients of x in the expansion provided by Theorem 3.4. For each non negative integer k and each (Q, S) with the property $(*)$, there is a $\lambda(k, (Q, S)) \in \mathbf{T}$ such that

$$\sigma^k(u)u(Q, S)\sigma^k(u)^* = \lambda(k, (Q, S))u(Q, S).$$

Since $\sigma^k(u)x = x\sigma^k(u)$ for all positive integer k, we have that $x(Q, S) = x(Q, S)\lambda(k, (Q, S))$ for all k. By (iv)', for each (Q, S) with the property $(*)$, there is a positive integer k with $\lambda(k, (Q, S)) \neq 1$, which implies that $x(Q, S) = 0$. Hence $x = x_0 1$. Thus $\sigma(R)' \cap R = \mathbf{C}1$.

Let σ be a shift of R. A unitary $w \in R$ is called a *normalizer* of σ if $w\sigma^k(R)w^* = \sigma^k(R)$ for all $k = 1, 2, \ldots$. The set $N(\sigma)$ of all normalizer of σ is also called a normalizer of σ ([3]).

COROLLARY 3.8. *Suppose σ is an n-unitary shift of R with a generator u satisfying* (iv)'. *If w is a normalizer of σ, then there are a $\lambda \in \mathbf{T}$ and (Q, S) with $(*)$ such that $w = \lambda u(Q, S)$.*

Proof. For an $x \in R$, consider the expansion $x = \sum_{i=0}^{n-1} x_i u^i$ where $x_i \in \sigma(R)$. Put

$$\theta(x) = \sum_{i=0}^{n-1} \gamma^i x_i u^i.$$

Then θ is an automorphism of R and $\sigma(R) = \{x \in R\ ;\ \theta(x) = x\}$. Since $w\sigma(R)w^* = \sigma(R)$, there is a $*$-automorphism α of R such that $w\sigma(x)w^* = \sigma(\alpha(x))$. Then

$$\theta(w)\sigma(x)\theta(w)^* = \theta(\sigma(\alpha(x))) = w\sigma(x)w^* \quad \text{for all } x \in R.$$

Hence $w^*\theta(w) \in \sigma(R)' \cap R$. Since $\sigma(R)' \cap R = \mathbf{C}1$ by Theorem 3.7 and $\theta^n(x) = x$ for all $x \in R$, there is an integer $k(1)$ $(1 \leqslant k(1) \leqslant n)$ such that $\theta(w) = \gamma^{k(1)}w$. Hence $\theta(u^{-k(1)}w) = u^{-k(1)}w$, so that $u^{-k(1)}w \in \sigma(R)$. Then there is a unique w_1 in $N(\sigma)$ because $w\sigma^2(R)w^* = \sigma^2(R)$. By the same argument, w_1 can be expressed in the form $w_1 = u^{k(2)}\sigma(w_2)$ with $w_2 \in N(\sigma)$. Put

$$p = \sup\{s\ ;\ w = u^{k(1)}\sigma(u^{k(2)}) \dots \sigma^s(u^{k(s+1)})\sigma^{s+1}(w_s),\ k(s+1) \neq n\}.$$

Assume that p is infinite. Take an (Q, S) with $(*)$. Then there is an $s > i(m)$ such that

$$w = u^{k(1)}\sigma(u^{k(2)}) \dots \sigma^s(u^{k(s+1)})\sigma^{s+1}(w_s), \quad 1 \leqslant k(s+1) \underset{\neq}{<} n.$$

Since there is a unitary v in the von Neumann algebra generated by $\{u, \sigma(u), \dots, \sigma^{s-1}(u)\}$ which satisfies that

$$u(Q, S)^*w = \lambda v\sigma^s(u^{k(s+1)})\sigma^{s+1}(w_s) \quad \text{for some } \lambda \in \mathbf{T},$$

it follows that $\mathrm{tr}(u(Q, S)^*w) = 0$, because either $v = u(Q_1, S_1)$ for some (Q_1, S_1) with $(*)$ or v is a scalar multiple of the identity. By Theorem 3.4, this is a contradiction. Hence p is finite and we have that there is a (Q, S) with $(*)$ such that $w = u(Q, S)\sigma^{k+1}(w_k)$ for all $k > i(m)$, where $Q = (i(1), \dots, i(m))$. Hence $u(Q, S)^*w$ is contained in $\sigma^k(R)$ for all k such that $k > i(m)$. Since σ is a shift, $u(Q, S)^*w = \lambda 1$ for some $\lambda \in \mathbf{T}$. Thus $w = \lambda u(Q, S)$.

Let σ be an n-unitary shift of R with a generator u. Put $S(\sigma; u) = \{k\ ;\ \sigma^k(u)u = \gamma u\sigma^k(u)\}$.

Now, we consider the following condition $(**)$ for an infinite set $\{k_i\ ;\ i = 1, 2, \dots\}$ of positive integers:

$$(**) \qquad k_{i+1} > k_i \quad \text{and} \quad k_{i+2} - k_{i+1} > k_{i+1} - k_i \quad \text{for all } i.$$

COROLLARY 3.9. *Suppose u and v are generators of an n-unitary shift σ, and that each $S(\sigma; u)$ or $S(\sigma; v)$ satisfies $(**)$ when it is infinite. Then $u = \gamma^k v^m$ for some k and m.*

Proof. If $S(\sigma; u)$ is either finite or satisfies $(**)$, then the condition (iv)′ is satisfied by Lemma 3.16. Hence by Corollary 3.8, $u = \lambda\sigma^{i(1)}(v^{j(1)}) \dots \sigma^{i(m)}(v^{j(m)})$ and $v = \mu\sigma^{p(1)}(u^{q(1)}) \dots \sigma^{p(s)}(u^{q(s)})$, where all $j(k)$ and $q(h)$ are in $\{1, 2, \dots, n-1\}$, $i(1) < i(2) < \dots < i(m)$, $p(1) < p(2) < \dots < p(s)$ and $\lambda, \mu \in \mathbf{T}$. Substituting the expression for u in the expression for v, we have $u = \nu\sigma^{i(1)+p(1)}(u^{j(1)q(1)}) \dots \dots \sigma^{i(m)+p(s)}(u^{j(m)q(s)})$ for some scalar ν. Since the expression is unique, $i(1) = p(1) = 0$ and $j(1)q(1) = na + 1$ for some integer a. Suppose $m \neq 1$. Then $j(m)q(s) = nb$ for some integer b. If $S(\sigma; v)$ is finite, put $i = i(m) + \max S(\sigma; v)$. Then $\sigma^i(u)u = \gamma^{j(m)j(1)}u\sigma^i(u)$. Since $\gamma^{j(m)j(1)q(s)} = 1$ and $\gamma^{j(m)j(1)q(1)} = \gamma^{j(m)} \neq 1$, it contradicts the condition (iii) in Definition 3.1. If $S(\sigma; v) = \{k_i ; i = 1, 2, \dots\}$ is infinite, there exists j such that $k_{j+1} - k_j > 2i(m)$. Put $i = k_{j+1} - i(m)$. Then $\sigma^i(u)$ and u can not satisfy (iii). Hence $m = 1$. Similarly $s = 1$.

In the case of a prime number n, the condition for $S(\sigma; u)$ and $S(\sigma; v)$ in Corollary 3.9 is not necessary.

DEFINITION 3.10. Let σ be an n-unitary shift of R with a generator u such that $S(\sigma; u)$ satisfies $(**)$ when it is infinite. Put

$$S(\sigma) = \{k ; \sigma^k(u)u = \gamma u\sigma^k(u)\}.$$

We shall call $S(\sigma)$ the *γ-set* or the *anticommutator set of σ*.

REMARK 3.11. The γ-set of σ does not depend on σ-generators by Corollary 3.9.

THEOREM 3.12. *Suppose σ_1 and σ_2 are n-unitary shifts with generators satisfying the condition in Definition* 3.10. *Then σ_1 and σ_2 are conjugate if and only if $S(\sigma_1) = S(\sigma_2)$.*

Proof. Let u be a generator of σ_1 satisfying the condition. For an automorphism θ of R with $\sigma_1 = \theta^{-1}\sigma_2\theta$, the unitary $\theta(u)$ is a generator of σ_2. Hence $S(\sigma_1) = S(\sigma_2)$.

Conversely let u_i be a σ_i-generator with the property (iv)′ for $i = 1,2$. Then there is an automorphism θ of R such that $\theta(\sigma_1^k(u_1)) = \sigma_2^k(u_2)$, for all $k = 0, 1, \dots$. Hence $\theta\sigma_1 = \sigma_2\theta$ because the trace tr on R is determined by unitaries $u_i(Q, S)$'s.

Now we shall give examples of n-unitary shifts of R with generators satisfying the condition in Definition 3.10.

Let Σ be a subset of positive integers. For all integer k, we put

$$f(\Sigma; k) = \begin{cases} 1, & \text{when } |k| \in \Sigma \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 3.13. *Let Σ be a subset of positive integers. For a positive integer n, there is a sequence $\{u_i ; i = 0, 1, \ldots\}$ of unitaries satisfying the relations:*

$$u_i^n = 1 \tag{1}$$

$$u_i u_j = \gamma^{f(\Sigma; i-j)} u_j u_i, \quad \textit{where } \gamma = \exp(2\pi i/n). \tag{2}$$

Proof. Let A be a type I_n factor. Then there is a pair $\{v, w\}$ of unitaries in A so that $\{v^i w^j ; i, j = 0, 1, \ldots, n\}$ form an orthonormal basis in A with respect to the canonical trace. Considering the tensor products of A, we have $\{u_0, \ldots, \ldots, u_m\}$ with (1) and (2), where $m = \min \Sigma - 1$. Let B be the von Neumann algebra generated by $\{u_0, \ldots, u_m\}$. Then there is a $*$-automorphism g_{m+1} of B such that $g_{m+1}(u_j) = u_j$ for $j = 1, \ldots, m$ and $g_{m+1}(u_0) = \gamma u_0$. Considering the crossed product C of B by g_{m+1}, we have $\{u_i ; i = 0, 1, \ldots, m+1\}$ with (1) and (2). Since an element in C has a unique expression as a linear combination of the group generated by $\{u_i ; i = 0, \ldots, m+1\}$, there is an automorphism g_{m+2} of C so that $g_{m+2}(u_i) = \gamma^{f(\Sigma; m+2-i)} u_i$. Thus we have the desired sequence of unitaries.

DEFINITION 3.14. The sequence $\{u_i ; i = 0, 1, \ldots\}$ in Lemma 3.13 is called an *n-unitary sequence over Σ.*

We denote by the same notation $u(Q, S)$ as in Definition 3.1 the unitary defined as follows:

$$u(Q, S) = (u_{i(1)})^{j(1)} (u_{i(2)})^{j(2)} \ldots (u_{i(m)})^{j(m)},$$

where (Q, S) satisfies $(*)$. We put

$$f(\Sigma; (Q, S), k) = \sum_{h=1}^{m} f(\Sigma; k - i_h) j_h.$$

THEOREM 3.15. *Suppose Σ is a subset of positive integers and $\{u_i ; i = 0, 1, \ldots\}$ is an n-unitary sequence over Σ. Then the following statements are equivalent:*

(1) *The C^*-algebra A generated by $\{u_i ; i = 0, 1, \ldots\}$ is simple.*

(2) *The center of A is the multiples of the identity.*

(3) *A has a unique trace τ such that $\tau(1) = 1$.*

(4) *If (Q, S) satisfies $(*)$, then there are an integer k $(k \geq 0)$ and a non trivial $\lambda \in \mathbf{T}$ such that $u_k u(Q, S) = \lambda u(Q, S) u_k$.*

Proof. Suppose the statement (4) is false. Then there is a (Q, S) with $(*)$ such that $u(Q, S) u_k = u_k u(Q, S)$ for all $k = 0, 1, \ldots$. Since $u(Q, S)$ is not a scalar multiple of the identity, the statements (1) through (3) are false.

Suppose (4) is true. Then the set of all $u(Q, S)$ and 1 is linearly independent. For all (Q, S) with $(*)$, put $\tau(u(Q, S)) = 0$. Put $\tau(1) = 1$. Since A is the C^*-algebra completion of the linear span of all $u(Q, S)$ and $1, \tau$ is a trace on A. Take a trace φ on A. By (4) $\varphi(u(Q, S)) = 0$ for (Q, S) with $(*)$. Hence the tracial state of A is unique. Thus (3) and (4) are equivalent. In order to prove that (2) is true, take a z in the center of A and $\varepsilon > 0$. Then there are complex numbers $\{\lambda_0, \ldots, \lambda_m\}$ and $\{(Q_i, S_i) ; i = 1, 2, \ldots, m\}$, each of which satisfying $(*)$ and

$$\left\| z - \lambda_0 1 - \sum_{i=1}^{m} \lambda_i u(Q_i, S_i) \right\| < \varepsilon.$$

By (4), there are $k(i)$ and $p(i)$ such that

$$u_{k(i)} u(Q_i, S_i) = \gamma^{p(i)} u(Q_i, S_i) u_{k(i)}, \quad \text{for} \quad i = 1, 2, \ldots, m.$$

For all $x \in R$, put

$$E_i(x) = \sum_{j=1}^{n} (1/n)(u_{k(i)})^j x (u_{k(i)})^{-j}.$$

Then $\|E_i(x)\| \leqslant \|x\|$ for $i = 1, 2, \ldots, m$ and $E_i(z - \lambda_0 1) = z - \lambda_0 1$. Since $p(i)$ is in $\{1, 2, \ldots, n-1\}$ for all $i = 1, 2, \ldots, m$, we have that $\sum_{j=1}^{n} \gamma^{p(i)j} = 0$. Hence we infer

$$E_1\left(E_2\left(\ldots\left(E_m\left(z - \lambda_0 1 - \sum_{i=1}^{m} \lambda_i u(Q_i, S_i)\right) \ldots\right)\right)\right) = z - \lambda_0 1,$$

so that $\|z - \lambda_0 1\| < \varepsilon$. Since ε is arbitrary, we have $z = \lambda_0 1$. Thus (2), (3) and (4) are equivalent.

Suppose A is not simple and (4) is true. Let y be in a two sided ideal J such that $\tau(y^* y) = 1$. Then we have $\{\lambda_i ; i = 0, 1, \ldots, m\}$ and $\{(Q_i, S_i) ; i = 1, 2, \ldots, m\}$ with $(*)$ such that

$$\left\| y^* y - \lambda_0 1 - \sum_{i=1}^{m} \lambda_i u(Q_j, S_i) \right\| < 1/4.$$

Let E_i be the same map as above for $i = 1, 2, \ldots, m$. Put $y' = E_1(E_2(\ldots \ldots (E_m(y^* y)) \ldots)$, then $y' - \lambda_0 1 \in J$ and $\|y' - \lambda_0 1\| < 1/2$. Since $\tau(y') = \tau(y^* y) = 1$, we have that $\|y' - 1\| < 1$, which implies that y' is invertible so that $J = A$. Thus all statements are equivalent.

LEMMA 3.16. *Let Σ be a set of positive integers which is finite or satisfies $(**)$ and let $\{u_i ; i = 0, 1, \ldots\}$ be an n-unitary sequence over Σ. Then for a (Q, S)*

with (*), *there are a positive integer* k *and a non trivial* $\lambda \in \mathbf{T}$ *such that*

$$u_k u(Q, S) = \lambda u(Q, S) u_k .$$

Proof. Let $Q = (i(1), \ldots, i(m))$ and $S = (j(1), \ldots, j(m))$ satisfy (*). Then

$$u_k u(Q, S) = \mu_k u(Q, S) u_k \quad \text{and} \quad \mu_k = \gamma^{f(\Sigma;(Q,S),k)}$$

for $k = 0, 1, \ldots$.

Suppose Σ is a finite set. Put $k = i(m) + \max \Sigma$. Then $f(\Sigma;(Q, S), k) = j(m)$. Since $j(m) \in \{1, \ldots, n-1\}$, $\mu_k \neq 1$. Thus the lemma holds in the case where Σ is finite.

Suppose Σ satisfies (**). Then there is an integer p such that

$$k_{p+1} - k_p > \max\{i(m) - i(j) \, ; j = 1, 2, \ldots, m-1\} .$$

Put $k = i(m) + k_p$. Then $k - i(m) = k_p$ is in Σ and $k - i(j)$ is not in Σ for all $j = 1, \ldots, m-1$. Hence $f(\Sigma;(Q, S), k) = j(m)$ and $\mu_k = \gamma^{j(m)} \neq 1$. Thus Lemma is true when Σ satisfies (**).

THEOREM 3.17. *Let* Σ *be a set of positive integers which is either finite or satisfies* (**). *Then there is an n-unitary shift* σ *of* R *such that* $\sigma(R)' \cap R = \mathbf{C}1$ *and* $S(\sigma) = \Sigma$.

Proof. Let $\{u_i \, ; i = 0, 1, 2, \ldots\}$ be an n-unitary sequence over Σ. Let A be a C^*-algebra generated by $\{u_i \, ; i = 0, 1, \ldots\}$. Then A has a unique tracial state τ by Lemma 3.16 and Theorem 3.15. Let π be the cyclic $*$-representation of A induced by τ. Let R be the von Neumann algebra generated by $\pi(A)$. Since the trace is unique, the algebra R is a finite factor. On the other hand by the requirements for the n-unitary sequence, the C^*-algebra A is the union of an ascending sequence of finite dimensional algebras. Hence R is a hyperfinite II_1 factor. Put $u = \pi(u_0)$ and $\sigma^i(\pi(u)) = \pi(u_i)$ for $i = 1, 2, \ldots$. Then by Theorem 3.15, Lemma 3.16 and Theorem 3.4, the mapping σ is extended to all elements of R. It is clear that σ is a shift of R. By the conditions for Σ, Lemma 3.16 and Theorem 3.7, we have that $\sigma(R)' \cap R = \mathbf{C}1$. By Lemma 3.13, we have that $S(\sigma) = \Sigma$.

COROLLARY 3.18. *Let* n *be a positive integer. Then there are uncountably many n-unitary shifts of* R *with multiplicity* 1 *which are pairwise non-conjugate and there are at least a countable infinity of n-unitary shifts which are pairwise not outer conjugate.*

Proof. Since there are uncountably many sets of positive integers satisfying the condition (**), by Theorem 3.17, we have uncountably many n-unitary shifts of R with multiplicity 1.

Let σ be an n-unitary shift of R such that $S(\sigma) = \{k\}$ for some positive integer k. By a similar way as in Section 2, it is shown that

$$k + 1 = \min\{\text{integer } j \,;\, \sigma^j(R)' \cap R \neq \mathbf{C}1\}.$$

Hence we have a family $\{\sigma_k \,;\, k = 1, 2, \ldots\}$ of n-unitary shifts of R which are pairwise not outer conjugate.

REMARK 3.19. In the above discussion in Section 3, we fixed the primitive n-th root γ of 1 as $\gamma = \exp(2\pi i/n)$. If we take another primitive n-th root of 1, the same results as above are true and it is easy to show that two n-unitary shifts, which are associate with different primitive n-th roots of 1, are not conjugate.

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