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THE HOMOTOPY GROUPS OF THE UNITARY GROUPS OF NON-COMMUTATIVE TORI

MARC A. RIEFFEL

In studying the homotopy groups of the ordinary unitary groups U_n, an important tool [26] is provided by the fibration of U_n over the sphere S^{2n-1} with fiber U_{n-1} obtained by letting U_n act on the unit sphere in the complex vector space $\mathbf{C}^n$ and viewing U_{n-1} as the stability subgroup of the standard basis vector e_n. Recently a corresponding fibration has been used by Corach and Larotonda [6] to study the homotopy groups of the group of invertible elements of a Banach algebra, and by Schröder [22] to study the homotopy groups of the group of unitary elements of a von Neumann algebra. In the present paper we use this fibration to calculate the homotopy groups of the group of unitary elements of a non-commutative torus for which there is at least some irrationality in the structure constants.

We recall [8, 9, 21] that a non-commutative m-torus A_θ (where $m \geqslant 2$) is determined by a real-valued skew bilinear form θ on $\mathbf{Z}^m$, and is the universal C^*-algebra generated by unitary elements u_x for $x \in \mathbf{Z}^m$ subject to the relation

$$u_y u_x = \exp(\pi i \theta(x, y)) u_{x+y}$$

for all $x, y \in \mathbf{Z}^m$. We say that θ is not rational if there exists at least one pair, x and y, in $\mathbf{Z}^m$, such that $\theta(x, y)$ is not a rational number. In [21] much information was amassed about the non-stable K-theory of non-commutative tori A_θ for which θ is not rational. We here combine some of that information with the fibration alluded to above, to calculate the homotopy groups of the groups $U_n(A_\theta)$ of unitary matrices over A_θ. Our main result is:

THEOREM. *Assume that the skew bilinear form θ on $\mathbf{Z}^m$ is not rational. Then*

$$\pi_k(U_n(A_\theta)) \cong \begin{Bmatrix} \mathrm{K}_1(A_\theta) & \text{for } k \text{ even} \\ \mathrm{K}_0(A_\theta) & \text{for } k \text{ odd} \end{Bmatrix} \cong \mathbf{Z}^{2^{m-1}}$$

for all $k \geqslant 0$ and $n \geqslant 1$.

The results of Corach and Larotonda [6] are based on the Bass stable rank, denoted $\mathrm{sr}(A)$, and give the above result only for $n \geqslant \mathrm{sr}(A_\theta) + k + 1$. (It can be shown that for θ not rational, $\mathrm{sr}(A_\theta) \leqslant 2$, with equality if A_θ is not simple, while $\mathrm{sr}(A_\theta) = 1$ for some simple A_θ [18,1] though it is not known how widely this happens.) The results of Schröder [22] depend on special properties of von Neumann algebras which do not hold for the A_θ's. In Section 4 we will discuss a somewhat wider context in which the techniques of Schröder apply.

In the course of our discussion we obtain a variety of new results about the non-stable K-theory of C^*-algebras. For example, Theorems 2.9 and 2.10 give information about when $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is isomorphic to $\mathrm{K}_1(A)$, while Theorem 4.7 gives information about how the connected stable rank behaves under the formation of matrix algebras.

1. THE FIBRATION

Let A be a unital C^*-algebra, and let $\mathrm{GL}_n(A)$ and $U_n(A)$ denote the groups of invertible and unitary elements respectively in $M_n(A)$, the algebra of $n \times n$ matrices with entries in A. Now $U_n(A)$ is a deformation retract of $\mathrm{GL}_n(A)$, as will be discussed early in Section 5, and so for the purposes of computing homotopy groups we can use either one. We find it technically slightly more convenient to use $\mathrm{GL}_n(A)$ (as in [5,6,7]), and so we will use it throughout the next sections. We return to $U_n(A)$ in Section 5.

In this section we will establish some of our notation, and will review the fibration in [6], in a form suited to our needs.

Let e_n denote the last standard basis vector in the free right A-module A^n, and let $\mathrm{Lc}_n(A)$ denote the orbit of e_n under the evident left action of $\mathrm{GL}_n(A)$, where the elements of A^n are viewed as column vectors. Then $\mathrm{Lc}_n(A)$ will consist of exactly the last columns of the various matrices in $\mathrm{GL}_n(A)$, which explains our choice of the notation Lc_n. The stability subgroup of e_n must then be the subgroup of matrices whose last column is e_n, that is, matrices in $\mathrm{GL}_n(A)$ of the triangular form

$$\begin{pmatrix} x & 0 \\ c & 1 \end{pmatrix}$$

where $x \in \mathrm{GL}_{n-1}(A)$ and c is any row of elements of A of length $n-1$. We will denote this subgroup by $\mathrm{TL}_n(A)$. Thus we have the identification

$$\mathrm{GL}_n(A)/\mathrm{TL}_n(A) = \mathrm{Lc}_n(A)$$

for $n \geqslant 1$ (if we let $\mathrm{TL}_1(A)$ denote the group with one element).

We will equip A^n and $\mathrm{GL}_n(A)$, and any subsets of them, with the topology coming from the norm on A. Then the action of $\mathrm{GL}_n(A)$ on A^n is jointly continuous. In [20] we considered the subset $\mathrm{Lg}_n(A)$ of A^n consisting of elements (a_i) of A^n for which there exists a (b_i) in A^n such that $\sum b_i a_i = 1$. It is easily seen that the action of $\mathrm{GL}_n(A)$ on A^n carries $\mathrm{Lg}_n(A)$ into itself (and by 1.4.8 of [14] the orbits in $\mathrm{Lg}_n(A)$ correspond to certain isomorphism classes of stably free modules). Since e_n is in $\mathrm{Lg}_n(A)$, it is clear that $\mathrm{Lc}_n(A)$ is a subset of $\mathrm{Lg}_n(A)$. Now in the proof of Theorem 8.3 of [20] it is shown that for any $\xi \in \mathrm{Lg}_n(A)$ the map $x \mapsto x\xi$ from $\mathrm{GL}_n(A)$ to $\mathrm{Lg}_n(A)$ is an open mapping. Consequently, $\mathrm{Lc}_n(A)$ will be an open (and closed) subset of $\mathrm{Lg}_n(A)$, and the mapping γ from $\mathrm{GL}_n(A)$ onto $\mathrm{Lc}_n(A)$ defined by $\gamma(x) = xe_n$ will be an open mapping. Furthermore, as indicated in [22], Theorem 7.2 of [15] applies to show that γ is a Serre fibration, so that by, for example, Theorem 10 of Section 2 of Chapter 7 of [25], one has the homotopy exact sequence

$$\to \pi_{k+1}(\mathrm{Lc}_n(A)) \to \pi_k(\mathrm{TL}_n(A)) \to \pi_k(\mathrm{GL}_n(A)) \to \pi_k(\mathrm{Lc}_n(A)) \to$$

where the base points in the groups are taken to be their identity elements, while the base point in $\mathrm{Lc}_n(A)$ is taken to be e_n. As is made clear in [25], this long exact sequence ends with

$$\pi_0(\mathrm{TL}_n(A)) \to \pi_0(\mathrm{GL}_n(A)) \to \pi_0(\mathrm{Lc}_n(A))$$

viewed as pointed sets. (See also 17.11 of [26].)

Now if $\mathrm{GL}_{n-1}(A)$ is viewed as embedded in $\mathrm{GL}_n(A)$ by the embedding map φ defined by

$$\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

then it is evident that one obtains a deformation retraction of $\mathrm{TL}_n(A)$ onto $\mathrm{GL}_{n-1}(A)$ by carrying the off-diagonal entry of any element of $\mathrm{TL}_n(A)$ linearly to zero. Thus the inclusion of $\mathrm{GL}_{n-1}(A)$ into $\mathrm{TL}_n(A)$ gives an isomorphism of homotopy groups, and so the above homotopy exact sequence can be rewritten as

$$\pi_{k+1}(\mathrm{Lc}_n(A)) \to \pi_k(\mathrm{GL}_{n-1}(A)) \to \pi_k(\mathrm{GL}_n(A)) \to \pi_k(\mathrm{Lc}_n(A)) \to,$$

ending, as before, with $\pi_0(\mathrm{Lc}_n(A))$. Thus if we can obtain information about $\pi_k(\mathrm{Lc}_n(A))$, this will help us to obtain information about $\pi_k(\mathrm{GL}_n(A))$.

2. THE SPACE OF LAST COLUMNS

It is an immediate consequence of Theorem 8.3 of [21] that if $A = A_\theta$ is a non-commutative torus with θ not rational, then for every $n \geqslant 2$ the map from $\mathrm{GL}_{n-1}(A)$ to $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is surjective, where for any C^*-algebra A the connected component of the identity element in $\mathrm{GL}_n(A)$ is denoted by $\mathrm{GL}_n^0(A)$.

2.1. PROPOSITION. *Let A be any unital C^*-algebra. For a given n, the map from* $\mathrm{GL}_{n-1}(A)$ *to* $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ *is surjective if and only if* $\mathrm{Lc}_n(A)$ *is connected, or equivalently, if and only if every element of* $\mathrm{Lc}_n(A)$ *is the last column of an element of* $\mathrm{GL}_n^0(A)$.

Proof. Suppose that the map is surjective. Let $\xi \in \mathrm{Lc}_n(A)$, and choose $z \in \mathrm{GL}_n(A)$ whose last column is ξ. By the surjectivity, there is an $x \in \mathrm{GL}_{n-1}(A)$ and $y \in \mathrm{GL}_n^0(A)$ such that $z = y\varphi(x)$. But $\varphi(x)e_n = e_n$, and so $ye_n = ze_n = \xi$. But y is connected by a path to the identity element, and so ξ is connected by a path to e_n. Thus $\mathrm{Lc}_n(A)$ is path connected.

Suppose, conversely, that $\mathrm{Lc}_n(A)$ is connected. Now according to Theorem 8.3 of [20] the connected components of $\mathrm{Lg}_n(A)$ are exactly the orbits for the action of $\mathrm{GL}_n^0(A)$, and so this must be true for $\mathrm{Lc}_n(A)$ also. Since $\mathrm{Lc}_n(A)$ is assumed to be connected, it must be the orbit of e_n under $\mathrm{GL}_n^0(A)$. Let z be any element of $\mathrm{GL}_n(A)$, and let ξ be the last column of z. From what we have just found, there is a $y \in \mathrm{GL}_n^0(A)$ such that $y\xi = e_n$. Then $yze_n = e_n$, so that yz is of the form $\begin{pmatrix} x & 0 \\ c & 1 \end{pmatrix}$ for some $x \in \mathrm{GL}_{n-1}(A)$. Let $w = \begin{pmatrix} 1 & 0 \\ -cx^{-1} & 1 \end{pmatrix}$, which is an element of $\mathrm{GL}_n^0(A)$. Then $wyz = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \varphi(x)$. Thus the coset of z in $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is in the image of the map from $\mathrm{GL}_{n-1}(A)$, as desired. Q.E.D.

The difficulty in working with $\mathrm{Lc}_n(A)$ is that in general it is hard to tell which elements of $\mathrm{Lg}_n(A)$ are in $\mathrm{Lc}_n(A)$. The following aspect of this difficulty is pertinent. Let $\mathbf{T}$ denote the 1-torus (= circle), and for any non-negative integer k let $\mathbf{T}^k$ denote the k-torus (with $\mathbf{T}^0$ being just a point). For any C^*-algebra A let $\mathbf{T}^kA$ denote the C^*-algebra of continuous functions from $\mathbf{T}^k$ into A. If A_θ is a non-commutative torus, then so is $\mathbf{T}^kA_\theta$, and if the skew bilinear form θ is not rational, then neither is that for $\mathbf{T}^kA_\theta$. Thus from Theorem 8.3 of [21] together with Proposition 2.1 we actually find:

2.2. LEMMA. *Let A_θ be a non-commutative torus for which θ is not rational. Then* $\mathrm{Lc}_n(\mathbf{T}^kA_\theta)$ *is connected for all integers $n \geqslant 2$ and $k \geqslant 0$.*

However, to calculate homotopy groups, we will see later that what we need is information about the space $C(\mathbf{T}^k, \mathrm{Lc}_n(A))$ of continuous functions from $\mathbf{T}^k$ into $\mathrm{Lc}_n(A)$. And in general, although it is clear that $\mathrm{Lc}_n(\mathbf{T}^kA)$ can be viewed as a subspace of $C(\mathbf{T}^k, \mathrm{Lc}_n(A))$, these spaces need not coincide. On the other hand, since $\mathrm{Lg}_n(A)$ is not defined in terms of $\mathrm{GL}_n(A)$, it does not share this difficulty. Specifically:

2.3. LEMMA. (See Corollary 2.4 of [7].) *For any unital C^*-algebra A and any integers $n \geqslant 1$ and $k \geqslant 0$, we have*

$$\mathrm{Lg}_n(\mathbf{T}^kA) = C(\mathbf{T}^k, \mathrm{Lg}_n(A)).$$

Proof. That $\mathrm{Lg}_n(\mathbf{T}^k A)$ is contained in $C(\mathbf{T}^k, \mathrm{Lg}_n(A))$ is clear. The other direction becomes evident when one uses the alternate characterization of $\mathrm{Lg}_n(A)$ obtained in Proposition 1 c) of [22], which says that (a_i) in A^n is in $\mathrm{Lg}_n(A)$ exactly if $\sum a_i^* a_i$ is invertible in A. Q.E.D.

Thus if $\mathrm{Lc}_n(A)$ happens to coincide with $\mathrm{Lg}_n(A)$, then we do avoid the difficulty indicated above. But it is evident that $\mathrm{Lc}_n(A)$ coincides with $\mathrm{Lg}_n(A)$ exactly if $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ (that is, A is $(n, 1)$-Hermite in the terminology of [7]; see also [14]). Now recall Definition 10.1 of [20] which defines $\mathrm{gsr}(A)$ to be the smallest integer m such that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for all $n \geqslant m$. Thus $\mathrm{gsr}(A)$ is the smallest integer m such that $\mathrm{Lc}_n(A)$ coincides with $\mathrm{Lg}_n(A)$ for all $n \geqslant m$. But recall also Proposition 10.5 of [20] which says that $\mathrm{gsr}(A)$ is the smallest integer m such that whenever W is a projective A-module such that $W \oplus A \cong A^n$ for some $n \geqslant m$ then $W \cong A^{n-1}$. (Throughout this paper, by "projective module" we always mean "finitely generated projective module".) Recall further that a unital C^*-algebra is said to be finite if left-invertible elements are invertible (and stably finite if $M_n(A)$ is finite for all n). Equivalently, $\mathrm{GL}_1(A)$ acts transitively on $\mathrm{Lg}_1(A)$, or $W \oplus A \cong A$ implies that $W = \{0\}$. Thus we have:

2.4. PROPOSITION. *Let A be a unital C^*-algebra. The following conditions are equivalent for an integer p:*

1. $\mathrm{gsr}(A) \leqslant p$,
2. *whenever W is a projective A-module such that $W \oplus A \cong A^n$ for some $n \geqslant p$ then $W \cong A^{n-1}$ (and, if $p = 1$, then A is finite),*
3. $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ *for all $n \geqslant p$.*

Now according to Theorem 7.1 of [21] A_θ satisfies cancellation for projective modules whenever θ is not rational, and so, in particular, stably free modules are free. Also A_θ is finite since it has a faithful trace. Thus we obtain:

2.5. PROPOSITION. *If θ is not rational, then* $\mathrm{gsr}(\mathbf{T}^k A_\theta) = 1$ *for all integers $k \geqslant 0$.*

Recall now Definition 4.7 of [20], which defines $\mathrm{csr}(A)$ to be the smallest integer m such that $\mathrm{GL}_n^0(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for all $n \geqslant m$, or equivalently by Corollary 8.5 of [20], such that $\mathrm{Lg}_n(A)$ is connected for all $n \geqslant m$. (Thus $\mathrm{gsr}(A) \leqslant \mathrm{csr}(A)$.) Clearly we have:

2.6. PROPOSITION. *Let A be a unital C^*-algebra. Then* $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ *for all $n \geqslant \mathrm{csr}(A)$, and $\mathrm{Lc}_n(A)$ is connected for all $n \geqslant \mathrm{csr}(A)$. Thus the map from $\mathrm{GL}_{n-1}(A)$ to $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is surjective for all $n \geqslant \mathrm{csr}(A)$. If $\mathrm{gsr}(A) = 1$, then $\mathrm{csr}(A)$ is the smallest integer m such that the usual map from $\mathrm{GL}_{n-1}(A)$ to $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is surjective for all $n \geqslant m$.*

Since $\mathrm{Lc}_1(A) = \mathrm{GL}_1(A)$, and since from Theorem 8.3 of [21] it follows that $\mathrm{GL}_1(\mathbf{T}^k A_\theta)$ is not connected, we obtain from Lemma 2.2 and Propositions 2.5 and 2.6:

2.7. PROPOSITION. *If θ is not rational then* $\mathrm{csr}(\mathbf{T}^k A_\theta) = 2$ *for all integers $k \geqslant 0$.*

It is thus appropriate to combine Lemma 2.3 and Proposition 2.6 to obtain:

2.8. PROPOSITION. *Let A be a unital C^*-algebra, and let p be an integer such that* $\mathrm{csr}(\mathrm{T}^k A) \leqslant p$ *for all integers* $k \geqslant 0$. *Then* $C(\mathrm{T}^k, \mathrm{Lc}_n(A))$ *is connected for all integers* $n \geqslant p$ *and* $k \geqslant 0$.

Clearly this proposition applies to A_θ for θ not rational, with $p = 2$. We will use this result in the next section to calculate homotopy groups. But we will first draw a consequence concerning the injectivity of the map from $\mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A)$ to $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$, or equivalently, from $\pi_0(\mathrm{GL}_{n-1}(A))$ to $\pi_0(\mathrm{GL}_n(A))$. From the long exact sequence at the end of Section 1 it is clear that this map is injective exactly if the map from $\pi_1(\mathrm{GL}_n(A))$ to $\pi_1(\mathrm{Lc}_n(A))$ is surjective. This can be seen directly as follows. Injectivity means that if $x \in \mathrm{GL}_{n-1}(A)$ and if $\varphi(x) \in \mathrm{GL}_n^0(A)$ then $x \in \mathrm{GL}_{n-1}^0(A)$. Now if $\varphi(x) \in \mathrm{GL}_n^0(A)$ then there is a path, $\{u_t\}$, from $\varphi(x)$ to 1. Let ξ_t be the last column of u_t for each t, so that ξ_t is a path in $\mathrm{Lc}_n(A)$ going from e_n to e_n, that is, a loop in $\mathrm{Lc}_n(A)$ representing an element of $\pi_1(\mathrm{Lc}_n(A))$. One then proceeds to examine whether this loop is homotopic to the loop of last columns of a loop in $\mathrm{GL}_n(A)$ from 1 to 1.

One situation in which injectivity will hold is when $\mathrm{Lc}_n(A)$ is simply-connected, that is, $\pi_1(\mathrm{Lc}_n(A)) = 0$. More generally, suppose that $n \geqslant \mathrm{csr}(A)$, so that $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ and these are connected. Let $f \in C(T, \mathrm{Lg}_n(A))$. Since $\mathrm{Lg}_n(A)$ is connected, f is homotopic to a function from T to $\mathrm{Lg}_n(A)$ which carries the base point 1 of T to the base point e_n of $\mathrm{Lg}_n(A)$. If the map from $\pi_1(\mathrm{GL}_n(A))$ to $\pi_1(\mathrm{Lc}_n(A))$ is surjective, this function has a homotopy preimage. Consequently there is a $g \in C(T, \mathrm{GL}_n(A))$ such that the function $t \mapsto g(t)e_n$ is homotopic to f. But $C(\mathrm{T}, \mathrm{GL}_n(A)) = \mathrm{GL}_n(\mathrm{T}A)$ and $C(T, \mathrm{Lg}_n(A)) = \mathrm{Lg}_n(\mathrm{T}A)$, and the usual map from $\mathrm{GL}_n(\mathrm{T}A)$ to $\mathrm{Lg}_n(\mathrm{T}A)$ has range which is open and closed. If we view f as an element of $\mathrm{Lg}_n(\mathrm{T}A)$, we see that the above argument shows that the component of f in $\mathrm{Lg}_n(\mathrm{T}A)$ meets the range of the map from $\mathrm{GL}_n(\mathrm{T}A)$. It follows that the map from $\mathrm{GL}_n(\mathrm{T}A)$ to $\mathrm{Lg}_n(\mathrm{T}A)$ is surjective.

Conversely, suppose that the map from $\mathrm{GL}_n(\mathrm{T}A)$ to $\mathrm{Lg}_n(\mathrm{T}A)$ is surjective. Let f be a loop in $\mathrm{Lg}_n(A)$ preserving the base point, and so representing an element of $\pi_1(\mathrm{Lg}_n(A))$. Then f can be viewed as an element of $\mathrm{Lg}_n(\mathrm{T}A)$ such that $f(1) = e_n$. By the surjectivity, there is a $g \in \mathrm{GL}_n(\mathrm{T}A)$ such that $g(t)e_n = f(t)$ for all t. In particular, $g(1)e_n = e_n$, so that $g(1)^{-1}e_n = e_n$. Define h by $h(t) = g(t)g(1)^{-1}$. Then $h(t)e_n = f(t)$, and $h(1) = I$. We see in this way that the map from $\pi_1(\mathrm{GL}_n(A))$ to $\pi_1(\mathrm{Lg}_n(A))$ is surjective.

Recall now that $\mathrm{gsr}(\mathrm{T}A)$ is the smallest integer m such that for all $n \geqslant m$ the map from $\mathrm{GL}_n(\mathrm{T}A)$ to $\mathrm{Lg}_n(\mathrm{T}A)$ is surjective. Then if we combine the above observations with Proposition 2.6 we obtain:

2.9. THEOREM. *Let A be a unital C^*-algebra, and let* $r = \max(\mathrm{csr}(A), \mathrm{gsr}(\mathrm{T}A))$. *Then for all* $n \geqslant r$ *the map from* $\mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A)$ *to* $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ *is an isomorphism, and in particular,* $\mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \cong \mathrm{K}_1(A)$.

This theorem is a refinement of Theorem 10.12 of [20], and provides a strengthening of that theorem with a much simpler proof. To see this, we must use the notion of topological stable rank, tsr, which was introduced in [20] as a generalization to Banach algebras of the classical covering dimension of a compact space, and which in [12] was shown in the case of C^*-algebras to coincide with the purely algebraic Bass stable rank (denoted Bsr in [20]). From Corollary 4.10 of [20] we know that $\mathrm{csr}(A) \leqslant \mathrm{tsr}(A) + 1$. On the other hand, as seen earlier, $\mathrm{gsr}(\mathbf{T}A) \leqslant \mathrm{csr}(\mathbf{T}A)$, while by Corollary 8.6 of [20] we have $\mathrm{csr}(\mathbf{T}A) \leqslant \mathrm{tsr}(A) + 1$. Thus we obtain from Theorem 2.9:

2.10. THEOREM. *Let A be a unital C^*-algebra. For all $n \geqslant \mathrm{tsr}(A)$ the map from $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ to $\mathrm{GL}_{n+1}(A)/\mathrm{GL}_{n+1}^0(A)$ is an isomorphism, and in particular $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A) \cong \mathrm{K}_1(A)$.*

The possibility of this improvement over the purely algebraic results used for Theorem 10.12 of [20] is presumably due in part to the fact that we are working over the complex numbers.

3. THE HOMOTOPY GROUPS

We wish to show that, in fact, $\pi_k(\mathrm{Lc}_n(A_\theta)) = 0$ for all integers $n \geqslant 2$ and $k \geqslant 0$. Now π_k is just the set of connected components of $C(\mathrm{S}^k, \mathrm{Lc}_n(A_\theta))$. So we need to show that this space is connected. Note that since $\mathrm{Lc}_n(A)$ is an open subset of A^n, the components of $C(\mathbf{T}^k, \mathrm{Lc}_n(A))$ are the same as the path components. Thus if $C(\mathbf{T}^k, \mathrm{Lc}_n(A))$ is connected, then any map from $\mathbf{T}^k$ to $\mathrm{Lc}_n(A)$ is homotopic to a constant map.

3.1. PROPOSITION. *Let X be a path-connected space such that for every integer $k \geqslant 1$ all maps from $\mathbf{T}^k$ to X are homotopic to constant maps. Then $\pi_k(X) = 0$ for all $k \geqslant 0$.*

Proof. The proof is by induction on k. Since X is path-connected, clearly $\pi_0(X) = 0$. Since $\mathbf{T}^1 = \mathrm{S}^1$, it follows also that $\pi_1(X) = 0$. For higher k we use the Hurewicz isomorphism theorem. (See, for example, [25]. I am indebted to Ed Spanier for showing me that the Hurewicz theorem was just what I needed here.) Since X is path-connected, we can use any base point. Suppose that we know that $\pi_j(X) = 0$ for $0 \leqslant j \leqslant k - 1$. Then by the Hurewicz isomorphism theorem the homology groups $H_j(X)$ are 0 for $0 \leqslant j \leqslant k - 1$, and $\pi_k(X) \cong H_k(X)$, where the isomorphism is given by the Hurewicz map. We now show that $H_k(X) = 0$. Let $\eta \in H_k(X)$, and let $[\mathrm{S}^k]$ denote a fundamental class in $H_k(\mathrm{S}^k)$. Then because $\pi_k(X) \cong H_k(X)$ under the Hurewicz map, there must (by the definition of the Hurewicz map) be a map f from S^k to X such that $f_*([\mathrm{S}^k]) = \eta$. Let $[\mathbf{T}^k]$ denote a fundamental class in $H_k(\mathbf{T}^k)$, and let g denote the usual map from $\mathbf{T}^k$ to S^k obtained by viewing

$\mathbf{T}^k$ as the k-cube $\mathbf{I}^k$ with certain faces identified, and collapsing all the faces to a point. (This is implicit in the alternate description of homotopy groups given on page 372 of [25].) Thus $g_*([\mathbf{T}^k]) = \pm [\mathbf{S}^k]$. Then $(fg)_*([\mathbf{T}^k]) = \pm \eta$. But fg is a map from $\mathbf{T}^k$ to X, and by hypothesis any such map is homotopic to a constant map. Thus $\eta = 0$. It follows that $H_k(X) = 0$. Conseqently $\pi_k(X) = 0$. Q.E.D.

Combining this with Proposition 2.8, we obtain:

3.2. PROPOSITION. *Let A be a unital C^*-algebra, and let p be an integer such that* $\operatorname{csr}(\mathbf{T}^k A) \leqslant p$ *for all integers* $k \geqslant 0$. *Then* $\pi_k(\mathrm{Lc}_n(A)) = 0$ *for all integers* $n \geqslant p$ *and* $k \geqslant 0$.

Combining this with the homotopy long exact sequence of Section 1, we obtain one of our main results:

3.3. THEOREM. *Let A be a unital C^*-algebra, and let p be an integer such that* $\operatorname{csr}(\mathbf{T}^k A) \leqslant p$ *for all integers* $k \geqslant 0$. *Then for all integers* $n \geqslant p - 1$ *and* $k \geqslant 0$

$$\pi_k(\mathrm{GL}_n(A)) = \begin{cases} \mathrm{K}_1(A) & \text{for } k \text{ even} \\ \mathrm{K}_0(A) & \text{for } k \text{ odd}. \end{cases}$$

Proof. Let $\mathrm{GL}_\infty(A)$ denote as usual the inductive limit of the $\mathrm{GL}_n(A)$'s with the embedding maps φ. Then by definition $\mathrm{K}_1(A) = \pi_0(\mathrm{GL}_\infty(A))$. By Bott periodicity (see III, 1.11 and 7.7 of [13]), $\mathrm{K}_0(A) \cong \pi_1(\mathrm{GL}_\infty(A))$ and $\pi_k(\mathrm{GL}_\infty(A)) \cong \pi_{k-2}(\mathrm{GL}_\infty(A))$. From Proposition 3.2 and the long exact sequence, we see that under the embeddings φ used earlier,

$$\pi_k(\mathrm{GL}_{n-1}(A)) \cong \pi_k(\mathrm{GL}_n(A))$$

for all integers $n \geqslant p$ and $k \geqslant 0$.Because of the inductive limit topology on $\mathrm{GL}_\infty(A)$, any maps from any $\mathbf{S}^k$ into $\mathrm{GL}_\infty(A)$ and any homotopies between such maps come from maps and homotopies into some $\mathrm{GL}_n(A)$. Thus we see that for the canonical maps of the $\mathrm{GL}_n(A)$ into $\mathrm{GL}_\infty(A)$ we have

$$\pi_k(\mathrm{GL}_{n-1}(A)) \cong \pi_k(\mathrm{GL}_\infty(A))$$

for all integers $n \geqslant p$ and $k \geqslant 0$. Q.E.D.

From the work of Pimsner and Voiculescu [17] it is easily seen that if A_θ is a non-commutative m-torus, then

$$\mathrm{K}_0(A_\theta) \cong \mathbf{Z}^{2^{m-1}} \cong \mathrm{K}_1(A_\theta).$$

Since we have seen that when θ is not rational the hypotheses of Theorem 3.3 are satisfied with $p = 2$, we obtain:

3.4. THEOREM. *If A_θ is a non-commutative m-torus, and if the skew form θ is not rational, then*

$$\pi_k(\mathrm{GL}_n(A_\theta)) \cong \mathbf{Z}^{2^{m-1}}$$

for all integers $n \geqslant 1$ and $k \geqslant 0$.

Since, as we will see in Section 5, each $U_n(A)$ is a deformation retract of $\mathrm{GL}_n(A)$ the above result will also hold for $\pi_k(U_n(A_\theta))$, giving the theorem stated in the introduction.

4. DIVISIBLE ALGEBRAS

In this section we treat a situation in which one can use arguments which are much closer to the original arguments of Schröder for von Neumann algebras [22] than are our arguments in the previous sections. Some of our main examples here are motivated by, and in the spirit of, the results of Blackadar in [4] (and 4.7.2 of [3]).

In the last sentence of Schröder's paper [22] he makes important use of the fact that any type II_1 von Neumann algebra can be expressed as the tensor product of a full matrix algebra of any size with another type II_1 von Neumann algebra. This suggests the following definition:

4.1. DEFINITION. A C^*-algebra A is said to be *divisible* if for every integer m there is an integer $n \geqslant m$ such that A can be expressed in the form $M_n(B)$ for some C^*-algebra B.

It is clear that any UHF C^*-algebra is divisible.

We actually need a much stronger condition than divisibility, given by the following definitions.

4.2. DEFINITION. Let A be a divisible C^*-algebra. We say that A is csr-*boundedly* divisible (resp. tsr-*boundedly* divisible) if there is a constant, K, such that for every integer m there is an integer $n \geqslant m$ such that A can be expressed as $M_n(B)$ for a C^*-algebra B such that $\mathrm{csr}(B) \leqslant K$ (resp. $\mathrm{tsr}(B) \leqslant K$).

Since for any C^*-algebra C we have $\mathrm{csr}(C) \leqslant \mathrm{tsr}(C) + 1$ by Corollary 4.10 of [20], it follows that:

4.3. PROPOSITION. *If A is* tsr-*boundedly divisible, then A is* crs-*boundedly divisible.*

As a first example we have:

4.4. PROPOSITION. *If A is a divisible C^*-algebra and if* $\mathrm{tsr}(A) = 1$, *then A is* tsr-*boundedly divisible.*

Proof. This follows immediately from Theorem 3.3 of [20], which tells us that $\operatorname{tsr}(C) = 1$ iff $\operatorname{tsr}(M_n(C)) = 1$. Q.E.D.

The results of Schröder [22] fit into the present context because $\operatorname{tsr}(A) = 1$ means exactly that invertible elements are dense, so that what he in effect uses is:

4.5. COROLLARY. *Any type* II_1 *von Neumann algebra is* tsr-*boundedly divisible.*

In fact, this corollary is also true for AW^*-algebras of type II_1, with the divisibility coming from § 19 of [2], and $\operatorname{tsr}(A) = 1$ coming from [11].

4.6. PROPOSITION. *If A is* tsr-*boundedly divisible, then* $\operatorname{tsr}(A) \leqslant 2$.

Proof. Let K be the constant in Definition 4.2. We can, by Theorem 6.1 of [20], find an integer m such that, for any integer $n \geqslant m$ and any C^*-algebra C with $\operatorname{tsr}(C) \leqslant K$, we have $\operatorname{tsr}(M_n(C)) \leqslant 2$. Q.E.D.

Actually, csr behaves in part the same way as does tsr with respect to forming matrix algebras (Theorem 6.1 of [20]). Specifically:

4.7. THEOREM. *Let A be a C^*-algebra. Then for any positive integer m*

$$\operatorname{csr}(M_m(A)) \leqslant \{(\operatorname{csr}(A) - 1)/m\} + 1,$$

where here $\{\ \}$ denotes "least integer greater than".

Proof. We consider first the case in which A is unital. Let $r = \operatorname{csr}(A)$, and let k be any integer such that $(k-1)m + 1 \geqslant r$. We show that $\mathrm{Lg}_k(M_m(A))$ is connected. Let $S = (S_1, \ldots, S_k) \in \mathrm{Lg}_k(M_m(A))$, and view S as a column, and thus as an $mk \times m$ matrix with entries in A. We will show that S is path connected to the "first standard basis vector" in $(M_m(A))^k$. Now the fact that $S \in \mathrm{Lg}_k(M_m(A))$ means that there is an $m \times mk$ matrix T with entries in A such that $TS = I_m$, the identity matrix in $M_m(A)$. This means in particular that the first column of S is in $\mathrm{Lg}_{mk}(A)$. Since by the choice of k we have $mk \geqslant r$, it follows that there is an element of $\mathrm{GL}^0_{mk}(A) = \mathrm{GL}^0_k(M_m(A))$ which carries this first column to the first standard basis vector, e_1, in A^{mk}. By right-multiplying S by this element, we see that S is path connected in $\mathrm{Lg}_k(M_m(A))$ to an element whose first column is e_1. So we can assume now that S itself has this property. Since $TS = I_m$, it follows that the first column of T must be the first standard basis vector in A^m. Thus the second row of T begins with a 0. But the "inner-product" of the second row of T and the second column of S must be 1. It follows that the second column of S with first element removed, say ξ, must be in $\mathrm{Lg}_{mk-1}(A)$. By the choice of k we have $mk - 1 \geqslant r$, and so there is an element, x, of $\mathrm{GL}^0_{mk-1}(A)$ which takes ξ to the first standard basis vector. Then $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ is in $\mathrm{GL}^0_{mk}(A)$ and carries S to a matrix whose first column is e_1, and whose second column has arbitrary first entry, 1 as second entry, and 0's for all remaining entries. By an elementary row operation with a matrix in $GL^0_{mk}(A)$ we can arrange that the second column is, in fact, e_2. That is, S is path connected to an element of $Lg_{mk}(A)$ whose first two columns are e_1 and e_2.

Continuing in this way, we can path-connect S to an element of $Lg_{mk}(A)$ whose p^{th} column is e_p for each p. When we reach the last (that is, m^{th}) column, we find that its last $mk - m + 1$ entries form an element of $Lg_{mk-m+1}(A)$, and so to ensure that the last column can be adjusted to be e_m we must have $GL^0_{mk-m+1}(A)$ acting transitively. Thus our requirement that $m(k-1) + 1 \geqslant r$ gives the smallest value of k that will make this technique work. But as long as this requirement is fulfilled, we see that we have path-connected S to the matrix which is the "first standard basis vector" in $(M_m(A))^k$. Thus $Lg_k(M_m(A))$ is connected when $m(k-1) + 1 \geqslant r$, that is when $k \geqslant ((r-1)/m) + 1$, and so when $k \geqslant \{(r-1)/m\} + 1$. That is,

$$\operatorname{csr}(M_m(A)) \leqslant \{(\operatorname{csr}(A) - 1)/m\} + 1.$$

To handle the non-unital case it suffices to show that if $\tilde{A}$ denotes A with identity element adjoined, then $\operatorname{csr}(M_m(A)) = \operatorname{csr}(M_m(\tilde{A}))$. Since we will not need this case in this paper, we leave the details to the reader. Q.E.D.

I do not know what can be said about the reverse inequality for the above theorem. However, in Theorem 3.10 of [24] Sheu shows that, for K the algebra of compact operators, one has $\operatorname{csr}(A \otimes K) \leqslant 2$ for any C^*-algebra A.

Just as in Proposition 4.6 for tsr, we immediately obtain:

4.8. Corollary. *If A is* csr-*boundedly divisible, then* $\operatorname{csr}(A) \leqslant 2$.

Since $\operatorname{gsr}(A) \leqslant \operatorname{csr}(A)$, and since whenever $\operatorname{gsr}(A) \leqslant 2$ and A is finite we have $\operatorname{gsr}(A) = 1$, we obtain:

4.9. Corollary. *If A is* csr-*boundedly divisible, and if A is finite, then* $\operatorname{gsr}(A) = 1$.

4.10. Corollary. *If A is* csr-*boundedly divisible and finite, then stably-free A-modules are free.*

For the purpose of determining homotopy groups we need:

4.11. Proposition. *If A is* tsr-*boundedly divisible, and if X is a finite-dimensional compact space, then $C(X, A)$ is* tsr-*boundedly divisible.*

Proof. Since X is finite dimensional, it can be embedded as a closed subset of some torus, $\mathbf{T}^d$. Setting $\mathbf{T}^d B = C(\mathbf{T}^d, B)$ for any C^*-algebra B, as before, we see that $C(X, B)$ is a quotient of $\mathbf{T}^d B$, and consequently $\operatorname{tsr}(C(X, B)) \leqslant \operatorname{tsr}(\mathbf{T}^d B)$

by Theorem 4.3 of [20]. But $\mathbf{T}^d B \cong B \times_\alpha \mathbf{Z}^d$ for the trivial action of $\mathbf{Z}^d$ on B, and so $\operatorname{tsr}(\mathbf{T}^d B) \leqslant d + \operatorname{tsr}(B)$ by Theorem 7.1 of [20].

By the hypothesis on A there is a constant K such that for any m there is an $n \geqslant m$ such that $A \cong M_n(B)$ with $\operatorname{tsr}(B) \leqslant K$. Then $C(X, A) \cong M_n(C(X, B))$, and $\operatorname{tsr}(C(X, B)) \leqslant d + K$ by the above discussion, which shows that $C(X, A)$ is tsr-boundedly divisible. Q.E.D.

We remark that the above proposition is our version, for the present context, of the corollary in [22].

4.12. COROLLARY. *If A is* tsr-*boundedly divisible, then for any finite dimensional compact space X we have* $\operatorname{csr}(C(X, A)) \leqslant 2$.

4.13. THEOREM. *Let A be a unital C^*-algebra which is* tsr-*boundedly divisible. Then for all $n \geqslant 1$ and $k \geqslant 0$ we have*

$$\pi_k(\mathrm{GL}_n(A)) = \begin{cases} \mathrm{K}_1(A) & \text{for } k \text{ even} \\ \mathrm{K}_0(A) & \text{for } k \text{ odd}. \end{cases}$$

Proof. From Corollary 4.12 we see that $\operatorname{csr}(C(\mathrm{S}^k, A)) \leqslant 2$ for all $k \geqslant 0$, so that $\mathrm{Lg}_n(C(\mathrm{S}^k, A))$ is connected and is equal to $\mathrm{Lc}_n(C(\mathrm{S}^k, A))$ for all $n \geqslant 2$ and $k \geqslant 0$. Thus $\pi_k(\mathrm{Lc}_n(A)) = 0$ for all $n \geqslant 2$ and $k \geqslant 0$. (Alternatively, we could have used Proposition 3.2 here instead of Corollary 4.12.) The rest of the proof is the same as that for Theorem 3.3. Q.E.D.

We now give further examples to which the above theorem applies. These involve AF C^*-algebras [10] and are motivated in part by the results of Blackadar in [4].

4.14. PROPOSITION. *Let A be any unital C^*-algebra for which* $\operatorname{tsr}(A) < \infty$, *and let B be a unital divisible* AF *C^*-algebra. Then $A \otimes B$ is* tsr-*boundedly divisible.*

Proof. By hypothesis, for any m there is an $n \geqslant m$ such that $B \cong M_n(C)$. Now C is a corner in B, and so C itself must be an AF C^*-algebra. Also, $A \otimes B \cong M_n(A \otimes C)$. But by using Theorems 5.1 and 6.1 of [20], it is easily seen that $\operatorname{tsr}(A \otimes C) \leqslant \operatorname{tsr}(A)$. Q.E.D.

We conclude with a similar situation where, however, $\operatorname{tsr}(A)$ need not be finite. For this we need:

4.15. LEMMA. *Let B be a unital divisible* AF *C^*-algebra, and suppose that B has been expressed as $M_k(C)$ for some k. Then C is a unital divisible* AF *C^*-algebra.*

Proof. Since corners of AF C^*-algebras are again AF C^*-algebras, C is an AF C^*-algebra. Let m be given. Then since B is divisible, there is a $q \geqslant mk$ such that B can be written in the form $M_q(D)$ for a unital AF C^*-algebra D.

Then there is a finite dimensional subalgebra, C', of C, having the same identity element, such that $M_k(C')$ is close enough to the matrix units for the decomposition $B = M_q(D)$, that $M_k(C')$ itself contains matrix units for an M_q. But C', being finite dimensional, is the direct sum of full matrix algebras, C_i', and the size of each $M_k(C_i')$ must be divisible by both k and $q \geqslant mk$. Then the size of each $M_k(C_i')$ must be divisible by the least common multiple, say p, of k and q. Let $n = p/k$, so $n \geqslant m$. Then the size of each C_i' must be divisible by n. Thus C' will contain matrix units for M_n, and so $C \cong M_n(E)$ for some AF C^*-algebra E. Q.E.D.

4.16. PROPOSITION. *Let $A = C(X) \times_\alpha G$ where X is a compact space, G is a discrete solvable group, and α is an action of G on X. Let B be any unital divisible* AF *C^*-algebra. Then $A \otimes B$ is* tsr-*boundedly divisible.*

Proof. For any m we can find $n \geqslant m$ such that $B = M_n(C)$ where C is divisible AF by Lemma 4.15. Then $A \otimes B \cong M_n(A \otimes C)$. Thus it suffices to show, with C replaced by B, that:

4.4. LEMMA. *If A is as above and if B is a unital divisible* AF *C^*-algebra, then* $\mathrm{tsr}(A \otimes B) \leqslant 2$.

Proof. This proof is just an elaboration of the proof of Corollary A5 of [4]. Since any compact space can be embedded in a product of intervals, and thus is a projective limit of finite-dimensional compact spaces, $C(X)$ is the inductive limit of $C(X_i)$'s with $\mathrm{tsr}(C(X_i)) < \infty$. By Theorem 6.1 of [20] and the assumption that B is divisible, we can, for any X_i, find a large enough integer n such that $\mathrm{tsr}(M_n(C(X_i))) \leqslant 2$ and $B = M_n(C)$. Then $C(X_i) \otimes B \cong M_n(C(X_i)) \otimes C$. Since C is an AF C^*-algebra, it follows from Theorem 5.1 (and 6.1) of [20] that $\mathrm{tsr}(C(X_i) \otimes B) \leqslant 2$. Then, again by Theorem 5.1 of [20], it follows that $\mathrm{tsr}(C(X) \otimes B) \leqslant 2$.

Notice that $A \otimes B \cong (C(X) \otimes B) \times_\alpha G$, where we use α to denote also the evident action of G on $C(X) \otimes B$ which leaves elements of B fixed. Let H be any finitely generated subgroup of G. Since H is solvable, we can find a finite composition series $\{H_n\}$ for H such that each H_n/H_{n-1} is cyclic. Let s be the length of this series. Then it is easily seen that $(C(X) \otimes B) \times_\alpha H$ is obtained by s successive crossed products with cyclic groups. If the cyclic group is infinite, then in forming the crossed product the tsr is raised by no more than 1 according to Theorem 7.1 of [20]. But if the cyclic group is finite, then by 7.8.1 of [3], or [16], the crossed product is a quotient of a crossed product by an infinite cyclic group, so that by Theorem 4.1 of [20] the tsr is again raised by no more than 1. We conclude that $\mathrm{tsr}((C(X) \otimes B) \times_\alpha H) \leqslant s + 2$.

By Theorem 6.1 of [20] we can choose an integer m such that if D is any C^*-algebra with $\mathrm{tsr}(D) \leqslant s + 2$ then $\mathrm{tsr}(M_n(D)) \leqslant 2$ for all $n \geqslant m$. Since B is divisible, we can find an $n \geqslant m$ such that $B = M_n(C)$ where C is an AF C^*-algebra. Then $(C(X) \otimes B) \times_\alpha H \cong M_n((C(X) \otimes C) \times_\alpha H)$. But the results of the previous paragraph apply when B is any divisible AF C^*-algebra, and C is divisible by Lemma 4.2. Thus $\operatorname{tsr}((C(X) \otimes C) \times_\alpha H) \leqslant s + 2$. From our hypothesis on n it follows that

$$\operatorname{tsr}(M_n((C(X) \otimes C) \times_\alpha H)) \leqslant 2.$$

Consequently, $\operatorname{tsr}((C(X) \otimes B) \times_\alpha H) \leqslant 2$. Now $(C(X) \otimes B) \times_\alpha G$ is the inductive limit of the $(C(X) \otimes B) \times_\alpha H$ as H ranges over the finitely generated subgroups of G. From Theorem 5.1 of [20] it follows that $\operatorname{tsr}((C(X) \otimes B \times_\alpha G) \leqslant 2$ as desired. Q.E.D.

5. UNITARY GROUPS

In this section we gather together a few facts relating unitary groups to the setting of the previous sections. For a unital C^*-algebra A we let $U_n(A)$ denote the group of unitary elements in $M_n(A)$. Following Schröder [22], we let $S_n(A)$ denote the set of (a_i) in A^n such that $\sum a_i^* a_i = 1$ (the "unit A-sphere"). Clearly $U_n(A) \subset \subset \mathrm{GL}_n(A)$ and $S_n(A) \subset \mathrm{Lg}_n(A)$, and the action of $\mathrm{GL}_n(A)$ on $\mathrm{Lg}_n(A)$ used earlier restricts to an action of $U_n(A)$ which carries $S_n(A)$ into itself. The map $t \mapsto ((1 - t)1 + t(vv^*)^{-1/2})v$ for $t \in [0, 1]$ and $v \in \mathrm{GL}_n(A)$ clearly defines a deformation retract of $\mathrm{GL}_n(A)$ onto $U_n(A)$.

We begin by showing that, just as we saw earlier that the action of $\mathrm{GL}_n(A)$ on $\mathrm{Lg}_n(A)$ is open (the proof of Theorem 8.3 of [20]), so also the action of $U_n(A)$ on $S_n(A)$ is open. A proof of this is already implicit in the proof of Proposition 3 of [23], using the theory of Banach Lie groups. We give here a direct proof. The proof we give here will be notationally simpler if we formulate it in a setting where matrices are not explicit:

5.1. Proposition. *Let A be a unital C^*-algebra, and let Ξ be a projective right A-module equipped with a Hermitian metric $\langle\ ,\ \rangle_A$. Let $E = \mathrm{End}_A(\Xi)$, with its structure as a C^*-algebra coming from the Hermitian metric. Let $U(\Xi)$ denote the group of unitary elements of E, and let*

$$S(\Xi) = \{\xi \in \Xi : \langle \xi, \xi \rangle_A = 1\},$$

so that $U(\Xi)$ acts on $S(\Xi)$. Then for each $\xi \in S(\Xi)$ the mapping $u \mapsto u\xi$ from $U(\Xi)$ to $S(\Xi)$ is an open mapping.

Proof. By translation it suffices to show that for every ξ in $S(\Xi)$ and for every open neighborhood N of 1_E the set $N\xi$ is a neighborhood of ξ in $S(\Xi)$. Let $\varepsilon > 0$ be given, with $\varepsilon < 1/48$. We show that if N is the ball in $U(\Xi)$ of radius 41ε about 1_E, then $N\xi$ contains the ball in $S(\Xi)$ of radius ε about ξ. (The norm on Ξ is given by $\|\xi\| = (\|\langle\xi, \xi\rangle_A\|)^{1/2}$.) So let $\eta \in S(\Xi)$ with $\|\eta - \xi\| < \varepsilon$. Let $p = \langle\xi, \xi\rangle_E$, where by definition

$$\langle\xi, \xi\rangle_E \zeta = \xi\langle\xi, \zeta\rangle_A$$

for $\zeta \in \Xi$. It is easily verified that p is a projection in E, and in fact is the projection on the A-submodule ξA of Ξ, which is a free A-submodule. In the same way, $q = \langle\eta, \eta\rangle_E$ is the projection on ηA. Then simple calculations using the generalized Cauchy-Schwarz inequality of Proposition 2.9 of [19] show that

$$\|p - q\| \leqslant \|\langle\xi, \xi\rangle_E - \langle\xi, \eta\rangle_E\| + \|\langle\xi, \eta\rangle_E - \langle\eta, \eta\rangle_E\| \leqslant 2\varepsilon.$$

Let $w = qp + (1 - q)(1 - p)$. Then a standard straightforward calculation shows that

$$\|1 - w\| \leqslant 2\|p - q\| \leqslant 4\varepsilon.$$

Thus w is invertible if $\varepsilon < 1/4$, which we now assume. Clearly $qw = qp = wp$, so that w restricts to an isomorphism from ξA onto ηA. Furthermore, $w\xi = qp\xi = \langle\eta, \eta\rangle_E \xi$, so that

$$\|w\xi - \eta\| = \|\eta\langle\eta, \xi\rangle_A - \eta\langle\eta, \eta\rangle_A\| \leqslant \|\xi - \eta\| < \varepsilon.$$

Note that $ww^* = qpq + (1 - q)(1 - p)(1 - q)$, which maps ηA into itself and $(\eta A)^\perp$ into itself, so that $(ww^*)^{-1/2}$ does also. Now $\|ww^* - 1\| \leqslant \|ww^* - w\| + \|w - 1\| \leqslant (1 + 4\varepsilon)4\varepsilon + 4\varepsilon < 12\varepsilon$. Assume now that $\varepsilon < 1/48$, so that $12\varepsilon < 1/4$. A little calculation shows that for $|1 - t| < 1/4$, one has $|1 - t^{-1/2}| < |1 - t|$, for any real number t. It follows that $\|(ww^*)^{-1/2} - 1\| \leqslant 12\varepsilon$. Let $u = (ww^*)^{-1/2}w$, so that $u \in U(\Xi)$. Then

$$\|u - 1\| \leqslant \|u - w\| + \|w - 1\| \leqslant 12\varepsilon(1 + 4\varepsilon) + 4\varepsilon \leqslant 28\varepsilon.$$

Furthermore, u carries ξA isometrically onto ηA. Let $\zeta = u\xi$, so that $\zeta \in \eta A$ and $\langle\zeta, \zeta\rangle_A = 1$. Then

$$\|\zeta - \eta\| \leqslant \|u\xi - w\xi\| + \|w\xi - \eta\| \leqslant \|(ww^*)^{-1/2} - 1\|\,\|w\xi\| + \varepsilon \leqslant 13\varepsilon.$$

Now the mapping $a \mapsto \eta a$ is an isomorphism of A onto ηA as right A-modules, preserving the Hermitian metrics. Thus there is a $b \in A$ such that $\zeta = \eta b$, and $b^*b = 1$. Furthermore

$$\|b - 1\| = \|\zeta - \eta\| \leqslant 13\varepsilon < 1,$$

so that b is actually invertible, and so is a unitary in A. Let $v \in U(\Xi)$ be defined to be the identity operator on $(\eta A)^\perp$, and on ηA to correspond to *left* multiplication by b on A. (Note that $\eta A + (\eta A)^{\perp} = \Xi$ because the inner-product is assumed to give a Hermitian metric, so that Ξ is self-dual.) Thus $v\eta = \zeta$, so that $v^*u\xi = \eta$. Furthermore,

$$\|v^*u - 1\| \leqslant \|v - 1\| + \|u - 1\| \leqslant 13\varepsilon + 28\varepsilon = 41\varepsilon.$$

We thus see that if, for $\varepsilon < 1/48$, we let N be the ball in $U(\Xi)$ of radius 41ε about 1, then $N\xi$ contains the ball in $S(\Xi)$ of radius ε about ξ, as desired. Q.E.D.

It is clear that the stability subgroup of e_n is U_{n-1}, and so if $U_n(A)$ acts transitively on $S_n(A)$ then we obtain a Serre fibration of $U_n(A)$ over $S_n(A)$ with fiber $U_{n-1}(A)$ just as in [22].

We can now supplement Proposition 2.4 as follows:

5.2. Proposition. *Let A be a unital C^*-algebra. Then* gsr(A) *is the smallest integer m such that $U_n(A)$ acts transitively on $S_n(A)$ for all $n \geqslant m$.*

Proof. Suppose that $n \geqslant \text{gsr}(A)$, and that $\xi \in S_n(A)$. As in the above proof, ξA is a free submodule of A^n, so that $(\xi A)^{\perp} \oplus A \cong A^n$. By Proposition 10.5 of [20] it follows that $(\xi A)^{\perp} \cong A^{n-1}$. Any such isomorphism can be adjusted to be unitary, and so $(\xi A)^{\perp}$ will possess a basis $\eta_1, \ldots, \eta_{n-1}$ which is orthonormal with respect to the Hermitian metric. Thus $\eta_1, \ldots, \eta_{n-1}, \xi$ is an orthonormal basis for A^n, and so there is a $u \in U_n(A)$ such that $ue_n = \xi$. That is, $U_n(A)$ acts transitively on $S_n(A)$.

Conversely, suppose that $U_n(A)$ acts transitively on $S_n(A)$ for all $n \geqslant m$. Now, as discussed earlier, the orbits of $\text{GL}_n(A)$ in $\text{Lg}_n(A)$ are all open, and so closed, and so they are unions of path components. But it is easily seen that every element of $\text{Lg}_n(A)$ is path-connected to an element of $S_n(A)$. (In fact $S_n(A)$ is a deformation retract of $\text{Lg}_n(A)$ by Proposition 1 b) of [22].) From the assumption that $U_n(A)$ acts transitively on $S_n(A)$ it then follows that $\text{GL}_n(A)$ acts transitively on $\text{Lg}_n(A)$. Thus $m \geqslant \text{gsr}(A)$ as desired. Q.E.D.

Let $U_n^0(A)$ denote the connected component of the identity in $U_n(A)$. We can supplement Proposition 2.6 as follows:

5.3. Proposition. *Let A be a unital C^*-algebra. Then* csr(A) *is the smallest integer m such that $S_n(A)$ is connected for all $n \geqslant m$, or equivalently, such that $U_n^0(A)$ acts transitively on $S_n(A)$ for all $n \geqslant m$. For $n \geqslant$* csr(A) *the usual map from $U_{n-1}(A)$ to $U_n(A)/U_n^0(A)$ is surjective. If* gsr$(A) = 1$, *then* csr(A) *is the smallest integer m such that for all $n \geqslant m$ the usual map of $U_{n-1}(A)$ to $U_n(A)/U_n^0(A)$ is surjective.*

Proof. Since, as indicated above, $S_n(A)$ is a deformation retract of $\text{Lg}_n(A)$ we see that $S_n(A)$ is connected exactly if $\text{Lg}_n(A)$ is. Thus the first assertion is true

Since from Proposition 5.1 we know that the mapping from $U_n(A)$ to S_n given by $u \mapsto ue_n$ is open, the image of $U_n^0(A)$ under this mapping must be a component of $S_n(A)$, and thus is all of $S_n(A)$ if $S_n(A)$ is connected. The proofs of the remaining assertions are similar to the proofs of Propositions 2.1 and 2.6.

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REFERENCES

1. Anderson, J.; Paschke, W., The rotation algebra, preprint.
2. Berberian, S. K., *Baer *-rings*, Springer-Verlag, Berlin—Heidelberg, 1972.
3. Blackadar, B., Notes on the structure of projections in simple C^*-algebras, in *Semesterbericht Functionalanalysis*, Tübingen, Wintersemester 1982/83.
4. Blackadar, B., A stable cancellation theorem for simple C^*-algebras, *Proc. London Math. Soc.*, **47**(1983), 303—305.
5. Corach, G.; Larotonda, A. R., Stable range in Banach algebras, *J. Pure Appl. Algebra*, **32**(1984), 289—300.
6. Corach, G.; Larotonda, A. R., A stabilization theorem for Banach algebras, *J. Algebra*, **101**(1986), 433—449.
7. Corach, G.; Larotonda, A. R., Unimodular matrices in Banach algebra theory, *Proc. Amer. Math. Soc.*, **96**(1986), 473—477.
8. Cuntz, J.; Elliott, G. A.; Goodman, F. M.; Jørgensen, P. E. T., On the classification of non-commutative tori. II, *C. R. Math. Rep. Acad. Sci. Canada*, 7(1985), 189—194.
9. Disney, S.; Elliott, G. A.; Kumjian, A.; Raeburn, I., On the classification of non-commutative tori, *C.R. Math. Rep. Acad. Sci. Canada*, 7(1985), 137—141.
10. Effros, E. G., *Dimensions and C^*-algebras*, Conf. Board Math. Sci. 46, Amer. Math. Soc., Providence, 1981.
11. Handelman, D., Stable range in AW^*-algebras, *Proc. Amer. Math. Soc.*, **76**(1979), 241—249.
12. Herman, R. H.; Vaserstein, L. N., The stable range of C^*-algebras, *Invent. Math.*, **77**(1984), 553—555.
13. Karoubi, M., *K-theory: An introduction*, Springer-Verlag, Berlin—Heidelberg—New York, 1978.
14. Lam, T. Y., *Serre's conjecture*, Lect. Notes Math., **635**, Springer-Verlag, Berlin—Heidelberg—New York, 1978.
15. Michael, E., Convex structures and continuous selections, *Canad. J. Math.*, **11**(1959), 556—575.
16. Olesen, D.; Pedersen, G. K., Partially inner C^*-dynamical systems, *J. Funct. Anal.*, **66**(1986), 262—281.
17. Pimsner, M. V.; Voiculescu, D., Exact sequences for K-groups and Ext-groups of certain crossed product C^*-algebras, *J. Operator Theory*, **4**(1980), 93—118.
18. Riedel, N., On the topological stable rank of irrational rotation algebras, *J. Operator Theory*, **13**(1985), 143—150.
19. Rieffel, M. A., Induced representations of C^*-algebras, *Adv. in Math.*, **13**(1974), 176—257.

20. RIEFFEL, M. A., Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc.*, **46**(1983), 301–333.
21. RIEFFEL, M. A., Projective modules over higher dimensional non-commutative tori, preprint.
22. SCHRÖDER, H., On the homotopy type of the regular group of a W^*-algebra, *Math. Ann.*, **267**(1984), 271–277.
23. SCHRÖDER, H., On the homotopy type of the regular group of a real W^*-algebra, *Integral Equation Operator Theory*, **9**(1986), 694–705.
24. SHEU, A. J.-L., *The cancellation property for modules over the group C^*-algebras of certain nilpotent Lie groups*, Doctoral dissertation, University of California, Berkeley, 1985.
25. SPANIER, E. H., *Algebraic topology*, McGraw-Hill, New York, 1966.
26. STEENROD, N., *The topology of fibre bundles*, Princeton U. Press, Princeton, 1951.

MARC A. RIEFFEL
Department of Mathematics,
University of California,
Berkeley, California 94720,
U.S.A.

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