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TYPE AND COTYPE INEQUALITIES FOR NON COMMUTATIVE L^p-SPACES

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INTRODUCTION

We say that a Banach space E is of *type r*, $1 \leqslant r \leqslant 2$ (resp. of *cotype s*, $2 \leqslant s \leqslant \infty$) if there exists a constant $C = C_r(E) > 0$ (resp. $K = K_s(E) > 0$) such that:

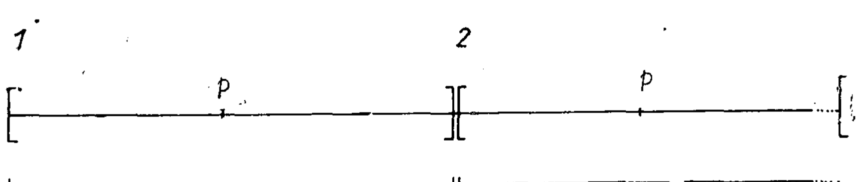
$$\mathbf{E}(\| \pm x_1 \pm x_2 \pm \ldots \pm x_n\|^2)^{1/2} \leqslant C\Big(\sum_{1\leqslant i\leqslant n} \|x_i\|^r\Big)^{1/r}$$

$$(\text{resp. } \Big(\sum_{1\leqslant i\leqslant n} \|x_i\|^s\Big)^{1/s} \leqslant K\mathbf{E}(\|\pm x_1 \pm x_2 \pm \ldots \pm x_n\|^2)^{1/2})$$

for all finite sequence $(x_1, x_2, \ldots, x_n)$ of elements in E, where $\mathbf{E}(\cdot)$ denotes the mean-value (2^n signs). These two properties are very useful in the study of the geometry of Banach spaces (cf. [15]). Recently, W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann have introduced (cf. [3]) the notion of uniform-*PL*-convexity. Recall that a Banach space $(E, \|\cdot\|)$ is said to be *p-uniformly-PL-convex* if there exist a real $q > 0$, and a constant $C > 0$ such that

$$\left((1/2\pi)\int_0^{2\pi} \|x + e^{i\theta}y\|^q \mathrm{d}\theta\right)^{1/q} \geqslant (\|x\|^p + C\|y\|^p)^{1/p}$$

for any x, y in E.



type p, contype 2, and 2-uniformly PL-convex

type 2, contype p, and p-uniformly PL-convex

It is well-known that the L^p-spaces of a measured space are of type $s = \min\{p, 2\}$ and of cotype $r = \max\{p, 2\}$. By [3], these classical L^p-spaces are also 2-uniformly-PL-convex for $1 \leq p \leq 2$ and p-uniformly-PL-convex for $2 \leq p < \infty$.

These results have been extended (cf. [14], [21], [3] and [22]) to the Banach spaces C_p of all compact operators T on a Hilbert space such that

$$\|T\|_p = (\mathrm{Trace}((T^*T)^{p/2}))^{1/p} < \infty.$$

Moreover, it is proved in [21] that the predual of any von Neumann algebra is of cotype 2, a fact which follows from the 2-uniform-PL-convexity of the dual of every C^*-algebra (a proof of this result, due to U. Haagerup, may be found in [3]).

In this paper, we shall prove that the "*non commutative*" L^p-spaces $L^p(M)$ associated with a *general* von Neumann algebra M have the same properties of type, cotype and uniform-PL-convexity as the usual L^p-spaces. "Non commutative" L^p-spaces arise naturally in the study of the leaf-space of a foliation from the measure theory point of view (cf. [2]). Our first result concerns the behavior of Rademacher averages of elements in $L^p(M)$ (*Khintchine inequalities*):

$$\Big(\sum_{1\leq i\leq n} \|T_i\|_p^p\Big)^{1/p} \leq \left(\int_0^1 \Big\|\sum_{1\leq i\leq n} r_i(t)T_i\Big\|_p^p dt\right)^{1/p} \leq$$

$$\leq \mathrm{A}_p\Big(\sum_{1\leq i\leq n} \|T_i\|_p^2\Big)^{1/2} \quad \text{if } 2 \leq p < \infty\,;$$

$$A_p\Big(\sum_{1\leq i\leq n} \|T_i\|_p^2\Big)^{1/2} \leq \left(\int_0^1 \Big\|\sum_{1\leq i\leq n} r_i(t)T_i\Big\|_p^p dt\right)^{1/p} \leq$$

$$\leq \Big(\sum_{1\leq i\leq n} \|T_i\|_p^p\Big)^{1/p} \quad \text{if } 1 \leq p \leq 2$$

where $(T_1, T_2, \ldots, T_n)$ is any finite sequence of elements in $L^p(M)$, A_p and A_p are constants depending only on p, and where r_i denotes the i-th Rademacher function, i.e.

$$r_i(t) = \mathrm{sgn}(\sin(2^i\pi t)) \quad \text{for } 0 \leq t \leq 1.$$

These inequalities, proved by combining complex interpolation methods together with rearrangement inequalities obtained in [8], imply that the "non commutative" L^p-spaces $L^p(M)$ associated with a von Neumann algebra M are of type $s = \min\{2, p\}$ and of cotype $r = \max\{2, p\}$. Our second result asserts that these "non commutative" L^p-spaces share also the same properties of uniform-PL-convexity as the classical L^p-spaces. These results allow us to prove that any operator from a C^*-algebra A into a "non commutative" L^p-space $L^p(M)$ factors through a Hilbert space, and is 2-C^*-summing in the sense of [16]. This solves a question raised by A. Connes in his IHES seminar. We would like to thank A. Connes for his question which is in fact the starting point of this paper. Note that our result is a well-known theorem of A. Grothendieck when both A and M are commutative (for more details on this subject, we refer to [15]).

The paper is organized as follows: after some preliminaries (Section 1) on the "non commutative" L^p-spaces $L^p(M)$ and on complex interpolation theory, we prove (Section 2) the rearrangement inequalities that we need to establish (Section 3) the Khintchine inequalities. The next section (Section 4) proves the uniform-PL-convexity of the "non commutative" L^p-spaces "from scratch". A last section (Section 5) gives applications to p-absolutely summing operators.

1. NOTATIONS AND PRELIMINARIES

1.1. NON COMMUTATIVE L^p-SPACES. Let M be a von Neumann algebra (we assume that M is σ-finite), and denote by N the crossed product of M by the modular automorphism group of a fixed weight on M. By [19], N admits a distinguished faithful semi-finite trace τ and a dual action θ_s ($s \in \mathbf{R}$) satisfying

$$\tau \circ \theta_s = e^{-s}\tau \quad \text{for any } s \in \mathbf{R}.$$

Following U. Haagerup ([10]), we put for $1 \leqslant p \leqslant \infty$:

$$L^p(M) = \{T \mid T \text{ is a } \tau\text{-measurable operator affiliated with } N \text{ such that } \theta_s(T) = \exp(-s/p)T \text{ for any } s \in \mathbf{R}\}.$$

Since all τ-measurable θ-invariant operators are bounded, we get $L^\infty(M) = M$ and the operator norm on M gives rise to a Banach space norm $\|\cdot\|_\infty$ on $L^\infty(M)$. For any φ in M_*^+, let H_φ be the Radon-Nikodym derivative $d\hat{\varphi}/d\tau$ of the dual weight $\hat{\varphi}$ on N relative to the trace τ. Each element T in $L^p(M)$ ($1 \leqslant p < \infty$) is a closed operator affiliated with N whose polar decomposition $T = U|T|$ satisfies: $U \in M$ and $|T| = (H_\varphi)^{1/p}$ for some unique φ in M_*^+. By [10], $L^1(M)$ is order isomorphic to the predual M_* of M via the map $T = UH_\varphi \mapsto U\varphi$. The "trace" tr is the positive linear form on $L^1(M)$ defined by $\mathrm{tr}(UH_\varphi) = \varphi(U)$. For T in $L^p(M)$, $1 \leqslant p < \infty$, we put:

$$\|T\|_p = \mathrm{tr}(|T|^p)^{1/p} < \infty.$$

We thus get Banach spaces $(L^p(M), \|\cdot\|_p)$ having all the expected properties such as duality. When M is semi-finite, these "non commutative" L^p-spaces are isometrically isomorphic to the well-known L^p-spaces associated with a trace (cf. [5] and [17]). In particular, they are generalizations of the Schatten classes C_p. Moreover, $L^\infty(M) = M$ and $L^1(M)$ is isometrically isomorphic to M_*. For more details on this theory, we refer the reader to [10] and [20].

1.2. GENERALIZED s-NUMBERS. Let N be a semi-finite von Neumann algebra with a (normal) faithful semi-finite trace τ. For any τ-measurable operator T in N, we call t-th *singular number of* T $(0 < t < \infty)$ the positive number

$$\mu_t(T) = \inf_{\substack{E = \text{projection in } N \text{ with} \\ \tau(1-E) \leqslant t}} \{ \sup_{\substack{x \in E(H) \\ \|x\| = 1}} \|Tx\| \}.$$

If f is an increasing continuous function on $[0, \infty]$ with $f(0) = 0$, we have (cf. [8]):

$$\tau(f(|T|)) = \int_0^\infty f(\mu_t(T))\mathrm{d}t$$

and many inequalities involving τ may be derived — using classical analysis — from inequalities involving μ_t. When N is the crossed product of a von Neumann algebra M by the modular automorphism group of a fixed weight on M, and τ the distinguished trace on N (cf. Section 1.1), we have

$$\mu_t(T) = t^{-1/p} \|T\|_p \quad (t > 0,\ 0 < p < \infty)$$

for any T in $L^p(M)$ (see [8] for instance). This remark is actually a powerfull trick to get L^p-norm inequalities for arbitrary von Neumann algebra just by integrating (semi-finite) s-numbers inequalities.

For any τ-measurable operator T in N, we put:

$$\Lambda_t(T) = \exp\left\{\int_0^t \log \mu_s(T)\mathrm{d}s\right\} \quad (t > 0).$$

It is clear that $\Lambda_t(T)$ is well defined (i.e., $\infty - \infty$ does not occur) when T satisfies the "Lorentz space" condition

$$(*) \qquad \mu_t(T) \leqslant Ct^{-\beta} \quad (C, \beta > 0) \text{ for any } t > 0$$

and, in particular, if T belongs to some $L^p(M)$, $p > 0$. We have:

LEMMA 1. (cf. [8], Theorem 4.2) *Let T, S be τ-measurable operators affiliated with N, satisfying the Lorentz space condition $(*)$. Then, we have:*

$$\Lambda_t(TS) \leqslant \Lambda_t(T)\Lambda_t(S) \quad \textit{for any } t > 0.$$

For more information on the theory of singular numbers, we refer to [6] and [8].

1.3. COMPLEX INTERPOLATION THEORY. For any pair (E_0, E_1) of compatible Banach spaces (i.e., there exists a third space such that both E_0 and E_1 can be considered as subspaces), we denote by $([E_0, E_1]_\theta, \|\cdot\|_\theta)$ the complex interpolation space of exponent θ, $0 < \theta < 1$ (cf. [1]), which is a subspace of $E_0 + E_1$ (algebraic sum). The main interest of this notion is the following result:

THEOREM 1. *Let (E_0, E_1) and (F_0, F_1) be two pairs of compatible Banach spaces. Let L be a linear map from $E_0 + E_1$ into $F_0 + F_1$. We assume that L maps E_i into F_i with*

$$\|L(x)\|_{F_i} \leqslant C_i \|x\|_{E_i} \quad (x \in E_i),\ i = 0, 1.$$

Then L maps $[E_0, E_1]_\theta = E_\theta$ into $[F_0, F_1]_\theta = F_\theta$ for each θ, $0 < \theta < 1$, and we have:

$$\|L(x)\|_{F_\theta} \leqslant C_0^{1-\theta} C_1^\theta \|x\|_{E_\theta} \quad (x \in E_\theta).$$

For a proof of this result, see for instance [1], Theorem 4.1.2.

Let ω be a fixed faithful normal state on M and imbed $M = E_0$ into $M_* = E_1$ via the map

$$T \in M \mapsto T\omega \in M_*.$$

Denote by $L^p(M, \omega)$ the interpolation space $([M, M_*]_{\theta=1/p}, \|\cdot\|_{\theta=1/p})$ (also $L^1(M, \omega) = M_*$ and $L^\infty(M, \omega) = M$). Let n be a fixed integer. For $1 \leqslant p, q < \infty$, denote by $L^p_{q,n}(M, \omega)$ the space

$$L^p(M, \omega) \times \ldots \times L^p(M, \omega) \quad (n \text{ terms})$$

equipped with the norm

$$\|(T_1, \ldots, T_n)\|_{p,q} = \Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^q\Big)^{1/q}.$$

By classical complex interpolation theory, we know that

$$[L^p_{q,n}(M, \omega), L^{p'}_{q',n}(M, \omega)] = L^{p''}_{q''}(M, \omega) \quad \text{(with equal norms)}$$

if

$$1/p'' = (1 - \theta)/p + \theta/p'$$

and

$$1/q'' = (1 - \theta)/q + \theta/q' \quad (0 < \theta < 1;\ 1 \leqslant p, q, p', q' < \infty).$$

We have:

THEOREM 2. ([12], Theorem 9.1). *$L^p(M, \omega)$ is isometrically isomorphic to $L^p(M)$.*

2. SOME REARRANGEMENT INEQUALITIES

The aim of this section is to establish the rearrangement inequalities that we shall need to prove the Khintchine inequalities.

2.1. LEMMA 2. *Let N be a von Neumann algebra with a faithful semi-finite trace τ. Let $T_1, \ldots, T_n$ be τ-measurable operators affiliated with N. Then, we have*

$$\int_0^t f(\mu_s(T_1T_2 \ldots T_n))\mathrm{d}s \leqslant \int_0^t f(\mu_s(T_1)\mu_s(T_2) \ldots \mu_s(T_n))\mathrm{d}s \quad (t > 0)$$

for any increasing function $f\colon \mathbf{R}_+ \to \mathbf{R}$ such that $t \mapsto f(\mathrm{e}^t)$ is convex.

Proof. Assume first that $T_1, \ldots, T_n$ are elements in N. By Lemma 1, we get

$$\Lambda_t(T_1 \ldots T_n) \leqslant \Lambda_t(T_1) \ldots \Lambda_t(T_n)$$

i.e.

$$\int_0^t \log(\mu_s(T_1 \ldots T_n))\,\mathrm{d}s \leqslant \int_0^t \log(\mu_s(T_1) \ldots \mu_s(T_n))\mathrm{d}s \quad (t > 0).$$

Using [6] (Corollary 3.2), we get the result when $T_1, \ldots, T_n$ are elements in N. If the T_i's are only τ-measurable operators, consider the polar decomposition

$$T_i = U_i \int_0^\infty \lambda \mathrm{d}E_\lambda^i$$

of T_i and put

$$T_{i,k} = U_i \int_0^k \lambda \mathrm{d}E_\lambda^i \quad (k \in \mathbf{N}).$$

The $T_{i,k}$ are elements in N satisfying

$$\mu_t(T_{i,k}) = \mu_t(|T_{i,k}|) \leqslant \mu_t(|T_i|) = \mu_t(T_i).$$

We thus have:

$$\int_0^t f(\mu_s(T_{1,k} \dots T_{n,k}))ds \leqslant \int_0^t f(\mu_s(T_{1,k}) \dots \mu_s(T_{n,k}))ds \leqslant$$

$$\leqslant \int_0^t f(\mu_s(T_1) \dots \mu_s(T_n))ds \quad \text{(because } f \text{ is increasing).}$$

But $T_{1,k}T_{2,k} \dots T_{n,k}$ converges to $T_1T_2 \dots T_n$ in the measure topology when $k \to \infty$ and hence

$$\mu_s(T_1 \dots T_n) \leqslant \liminf_{k\to\infty} \mu_s(T_{1,k} \dots T_{n,k}) \quad (s > 0)$$

by Fatou's lemma for singular numbers (cf. [8], Lemma 3.4). Then, the conclusion follows from Fatou's lemma for positive measurable functions. ▨

Let F be a finite subset of a von Neumann algebra N. For any integer k, we put:

$$\Pi_k(F, F^*) = \{S = S_1S_2 \dots S_{2k-1}S_{2k} \mid S_i \in F \text{ if } i \text{ is odd, } S_i \in F^* \text{ if } i \text{ is even and } \operatorname{card}\{i \mid S_i \in \{T, T^*\}\} \text{ is even for any } T \text{ in } F\}.$$

2.2. LEMMA 3. *Let N be a von Neumann algebra with a faithful semi-finite trace τ. Let F be a finite set of τ-measurable operators in N, and k be some integer.*

Then we have for any real α $(0 < \alpha < 1)$ and any $t > 0$:

$$\left(\sum_{S\in\Pi(F,F^*)} \int_0^t \mu_s(S)^\alpha ds\right)^{1/2k} \leqslant$$

$$\leqslant C_{2k}^{1/2k}\left[\sum_{T\in F}\left(\int_0^t \mu_s(T)^{2\alpha k}\right)^{1/k}\right]^{1/2}$$

where $C_{2k} = (2k)!/2^k k!$

Proof. Let $S = \prod_{1\leqslant i\leqslant 2k} S_i$ be some element in $\Pi(F, F^*) = \Pi_k(F, F^*)$. By Lemma 2 we have:

$$\int_0^t \mu_s(S)^\alpha ds \leqslant \int_0^t [\prod_{1\leqslant i\leqslant 2k} \mu_s(S_i)^\alpha] ds$$

and hence:

$$\sum_{S\in\Pi(F,F^*)}\int_0^t \mu_s(S)^\alpha\,\mathrm{d}s \leqslant \int_0^t \Big[\sum_{S\in\Pi(F,F^*)} \Big(\prod_{1\leqslant i\leqslant 2k}\mu_s(S_i)^\alpha\Big)\Big]\,\mathrm{d}s.$$

Assuming that $F=\{T_1,\ldots,T_n\}$, we see that $\mu_s(S_i)=\mu_s(S_i^*)$ is equal to $\mu_s(T_j)$ for some j, and hence:

$$\sum_{S\in\Pi(F,F^*)}\int_0^t \mu_s(S)^\alpha\,\mathrm{d}s \leqslant$$

$$\leqslant \int_0^t \Big\{\sum_{\substack{\beta\in\mathbf{N}^n\\ |\beta|=k}} [(2k)!/(2\beta)!]\prod_{1\leqslant i\leqslant n}\mu_s(T_i)^{2\alpha\beta_i}\Big\}\,\mathrm{d}s.$$

But $[(2k)!/(2\beta)!]\leqslant C_{2k}[k!/\beta!]$ where $C_{2k}=[(2k)!/2^k k!]$, so that we get:

$$\sum_{S\in\Pi(F,F^*)}\int_0^t \mu_s(S)^\alpha\,\mathrm{d}s \leqslant$$

$$\leqslant \int_0^t C_{2k}\Big[\sum_{\substack{\beta\in\mathbf{N}^n\\ |\beta|=k}} (k!/\beta!)\prod_{1\leqslant i\leqslant n}\mu_s(T_i)^{2\alpha\beta_i}\Big]\,\mathrm{d}s =$$

$$= C_{2k}\int_0^t \Big(\sum_{T\in F}\mu_s(T)^{2\alpha}\Big)^k\,\mathrm{d}s$$

and the result follows from Minkowski's inequality. ∎

Now, let M be a von Neumann algebra and denote by N the associated semi-finite von Neumann algebra (cf. Section 1.1). We have:

2.3. LEMMA 4. *Let k be a positive integer and F be a finite subset of elements in $L^{2k}(M)$. Then, we have:*

$$\Big[\sum_{S\in\Pi(F,F^*)}|\mathrm{tr}(S)|\Big]^{1/2k}\leqslant C_{2k}^{1/2k}\Big(\sum_{T\in F}\|T\|_{2k}^2\Big)^{1/2}$$

where C_{2k} is the constant in Lemma 3.

Proof. Let S be an element in $\Pi(F, F^*) = \Pi_k(F, F^*)$. Then S is in $L^1(M)$ and hence $\mu_s(S)^\alpha = s^{-\alpha}\|S\|_1^\alpha$ for any α, $0 < \alpha < 1$. We thus get:

$$\int_0^1 \mu_s(S)^\alpha \, ds = (1/(1-\alpha))\|S\|_1^\alpha .$$

On the other hand, any T in F being in $L^{2k}(M)$, we have:

$$\int_0^1 \mu_s(T)^{2\alpha k} \, ds = (1/(1-\alpha))\|T\|_{2k}^{2\alpha k} .$$

By Lemma 3, we get:

$$\Big(\sum_{S \in \Pi(F, F^*)} \|S\|_1^\alpha\Big)^{1/2k} \leqslant C_{2k}^{1/2k}\Big(\sum_{T \in F} \|T\|_{2k}^{2\alpha}\Big)^{1/2} .$$

Letting $\alpha \to 1$, we get:

$$\Big(\sum_{S \in \Pi(F, F^*)} \|S\|_1\Big)^{1/2k} \leqslant C_{2k}^{1/2k}\Big(\sum_{T \in F} \|T\|_{2k}^{2}\Big)^{1/2} .$$

But $|\mathrm{tr}(S)| \leqslant \|S\|_1$, and the lemma is proved. ▨

3. THE KHINTCHINE INEQUALITIES

3.1. THEOREM 3. *Let M be a von Neumann algebra. For any p, $1 \leqslant p \leqslant 2$ (resp. $2 \leqslant p < \infty$) there exists a constant $A_p > 0$ (resp. $B_p > 0$) such that we have for any finite sequence $(T_1, \ldots, T_n)$ of elements in $L^p(M)$:*

$$\Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^p\Big)^{1/p} \leqslant \left(\int_0^1 \Big\|\sum_{1 \leqslant i \leqslant n} r_i(t) T_i\Big\|_p^p \, dt\right)^{1/p} \leqslant$$

$$\leqslant B_p\Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^2\Big)^{1/2} \quad \text{if } 2 \leqslant p < \infty;$$

$$A_p\Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^2\Big)^{1/2} \leqslant \left(\int_0^1 \Big\|\sum_{1 \leqslant i \leqslant n} r_i(t) T_i\Big\|_p^p \, dt\right)^{1/p} \leqslant$$

$$\leqslant \Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^p\Big)^{1/p} \quad \text{if } 1 \leqslant p \leqslant 2$$

where r_i denotes the i-th Rademacher function.

The proof is based on the following lemmas:

3.2. LEMMA 5. *The Khintchine inequalities*

$$\left(\int_0^1 \Big\| \sum_{1\leqslant i\leqslant n} r_i(t)T_i \Big\|_p^p \,\mathrm{d}t\right)^{1/p} \leqslant B_p\Big(\sum_{1\leqslant i\leqslant n} \|T_i\|_p^2\Big)^{1/2}$$

hold for all even integer $p = 2k \geqslant 2$, *with a constant*

$$B_p = [(2k)!/2^k k!]^{1/2k}.$$

Proof. Let $p = 2k$ be an even integer. Let $T_1, \ldots, T_n$ be elements in $L^p(M)$ and put $F = \{T_1, \ldots, T_n\}$. We have:

$$\Big\| \sum_{1\leqslant i\leqslant n} r_i(t)T_i \Big\|_p^p = \mathrm{tr}\Big(\Big[\Big(\sum_{1\leqslant i\leqslant n} r_i(t)T_i^*\Big)\Big(\sum_{1\leqslant i\leqslant n} r_i(t)T_i\Big)\Big]^k\Big) =$$

$$= \mathrm{tr}\Big(\sum_{S\in\Omega(F,F^*)} S \prod_{1\leqslant i\leqslant n} (r_i(t))^{v_S(i)}\Big),$$

where $\Omega(F, F^*)$ is the set of all operators S of the form $S = \prod_{1\leqslant i\leqslant 2k} S_i$ such that $S_i \in F$ (resp. $S_i \in F^*$) if i is odd (resp. if i is even), and where $v_S(i)$ is the number of indices j such that $S_j \in \{T_i, T_i^*\}$. Using the identity:

$$\int_0^1 \prod_{1\leqslant i\leqslant n} (r_i(t))^{v_i}\,\mathrm{d}t = \begin{cases} 1 & \text{if all } v_i \text{ are even} \\ 0 & \text{if not} \end{cases}$$

we get:

$$\int_0^1 \Big\| \sum_{1\leqslant i\leqslant n} r_i(t)T_i \Big\|_p^p \,\mathrm{d}t = \sum_{S\in\Pi(F,F^*)} \mathrm{tr}(S).$$

We thus have:

$$\left(\int_0^1 \Big\| \sum_{1\leqslant i\leqslant n} r_i(t)T_i \Big\|_p^p \,\mathrm{d}t\right)^{1/p} \leqslant \Big(\sum_{S\in\Pi(F,F^*)} |\mathrm{tr}(S)|\Big)^{1/p} \leqslant$$

$$\leqslant C_p^{1/p}\Big(\sum_{1\leqslant i\leqslant n} \|T_i\|_p^2\Big)^{1/2} \qquad \text{(by Lemma 4).}$$

But $C_p^{1/p} = B_p$. ∎

3.3. REMARK. For $p = 2k$, our constant is the best possible.

3.4. LEMMA 6. *The Khintchine inequalities*

$$\left(\int_0^1 \|\sum_{1\leqslant i\leqslant n} r_i(t)T_i\|_p^p \,\mathrm{d}t\right)^{1/p} \leqslant B_p\big(\sum_{1\leqslant i\leqslant n}\|T_i\|_p^2\big)^{1/2}$$

hold for all p with $2 \leqslant p < \infty$.

Proof. By the preceding lemma, we may assume that p is not an even integer. Using Theorem 2, we may freely replace $L^p(M)$ by $L^p(M, \omega)$, where ω is some fixed normal faithful state on M. Let $k \in \mathbf{N}$ such that $2k < p < 2k + 2$. By Lemma 5, we know that the inequality is true for $p_0 = 2k$ and $p_1 = 2k + 2$. To end the proof, we shall interpolate between p_0 and p_1.

Let G be the group (of order 2^n) of all maps $\sigma: \{1, \ldots, n\} \to \{-1, +1\}$. For any sequence $(x_1, \ldots, x_n)$ of elements in a vector space, let us put:

$$\hat{x}(\sigma) = \sum_{1\leqslant i\leqslant n} \sigma(i)x_i.$$

For any sequence $(T_1, \ldots, T_n)$ of elements in $L^p(M)$, we have:

$$\left(\int_0^1 \|\sum_{1\leqslant i\leqslant n} r_i(t)T_i\|_p^p \,\mathrm{d}t\right)^{1/p} = 2^{-n/p}\big(\sum_{\sigma\in G}\|\hat{T}(\sigma)\|_p^p\big)^{1/p}.$$

For $i = 0, 1$, let $E_1 = L_2^{p_i}(M, \omega)$ be the space

$$L^{p_i}(M, \omega) \times \ldots \times L^{p_i}(M, \omega) \qquad (n \text{ terms})$$

with the norm $\|(T_1, \ldots, T_n)\|_{E_i} = \big(\sum_{1\leqslant j\leqslant n}\|T_j\|_{p_i}^2\big)^{1/2}$.

Let $F_i = L_{p_i}^{p_i}(M, \omega)$ be the space

$$L^{p_i}(M, \omega) \times \ldots \times L^{p_i}(M, \omega) \qquad (2^n \text{ terms})$$

with the norm $\|(S_1, \ldots, S_{2^n})\|_{F_i} = \big(\sum_{1\leqslant j\leqslant 2^n}\|S_j\|_{p_i}^{p_i}\big)^{1/p_i}$.

Put $\theta = (k + 1)/(p - 2k)p$, so that we have:

$$(*) \qquad 1/p = (1 - \theta)/p_0 + \theta/p_1.$$

Let L be the linear map from $E_0 + E_1$ into $F_0 + F_1$ defined by:

$$L((T_1, \ldots, T_n)) = (\hat{T}(\sigma))_{\sigma\in G}.$$

This map is bounded from E_i into F_i and satisfies:

$$\|L(x)\|_{F_i} \leqslant 2^{n/p_i} B_{p_i} \|x\|_{E_i} \quad \text{for } i = 0, 1 \qquad \text{(by Lemma 5)}.$$

By Theorem 1, L maps $[E_0, E_1]_\theta = E_\theta$ into $[F_0, F_1]_\theta = F_\theta$, is bounded on E_θ, and satisfies:

$$\|L(x)\|_{F_\theta} \leqslant 2^{n/p} B_{p_0}^{1-\theta} B_{p_1}^{\theta} \|x\|_{E_\theta} \qquad \text{(because of } (*)).$$

But E_θ (resp. F_θ) is the space $L_2^p(M, \omega)$ (resp. $L_p^p(M, \omega)$) as noticed in Section 1.3 and the proof of the lemma is complete. ▨

3.5. *End of the proof of Theorem 3.* Let us first assume that $2 \leqslant p < \infty$. By [8], we have for any T, S in $L^p(M)$:

$$2(\|T\|_p^p + \|S\|_p^p) \leqslant \|T + S\|_p^p + \|T - S\|_p^p \quad \text{if } 2 \leqslant p < \infty.$$

The inequality

$$\Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^p\Big)^{1/p} \leqslant \left(\int_0^1 \Big\|\sum_{1 \leqslant i \leqslant n} r_i(t) T_i\Big\|_p^p \, dt\right)^{1/p}$$

follows immediately by induction. By Lemma 6, we get the Khintchine inequalities when $2 \leqslant p < \infty$.

Assume now that $1 \leqslant p \leqslant 2$. As we have for any T, S in $L^p(M)$ (cf. [12], Proposition 5.5):

$$\|T + S\|_p^p + \|T - S\|_p^p \leqslant 2(\|T\|_p^p + \|S\|_p^p) \quad \text{if } 1 \leqslant p \leqslant 2,$$

we get easily:

$$\left(\int_0^1 \Big\|\sum_{1 \leqslant i \leqslant n} r_i(t) T_i\Big\|_p^p \, dt\right)^{1/p} \leqslant \Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^p\Big)^{1/p}.$$

It remains to prove that there exists for any p, $1 \leqslant p \leqslant 2$, a constant $A_p > 0$ such that:

$$A_p \Big(\sum_{1 \leqslant i \leqslant n} \|T_i\|_p^2\Big)^{1/2} \leqslant \left(\int_0^1 \Big\|\sum_{1 \leqslant i \leqslant n} r_i(t) T_i\Big\|_p^p \, dt\right)^{1/p}.$$

In other words, we have to prove that $L^p(M)$ has cotype 2 for $1 \leqslant p \leqslant 2$.

By [28] (Proposition 3.2), this is true for $p = 1$, because $L^1(M)$ is the predual of M. Assume $p \neq 1$. Then, the dual $L^{p'}(M)$ of $L^p(M)$ $(1/p + 1/p' = 1)$ is of type 2 by Lemma 6, so that $L^p(M)$ is of cotype 2 by a well-known duality argument (cf. [16]). ▧

Thus, we see that the non commutative L^p-spaces $L^p(M)$ associated with a von Neumann algebra M are of type $s = \min\{p, 2\}$ and of cotype $r = \max\{p, 2\}$.

4. UNIFORM-PL-CONVEXITY OF NON COMMUTATIVE L^p-SPACES

4.1. Let $(E, \|\cdot\|)$ be a Banach space. For $p, \varepsilon > 0$, we put:

$$H_p^E(\varepsilon) =$$

$$= \inf\left\{\left((1/2\pi)\int_0^{2\pi}\|x + e^{i\theta}y\|^p \,d\theta\right)^{1/p} - 1 \;\middle|\; x, y \in E;\ \|x\| = 1 \text{ and } \|y\| = \varepsilon\right\}.$$

Let q be a real number $2 \leqslant q < \infty$. Following [3] we say that $(E, \|\cdot\|)$ is *q-uniformly-PL-convex* if one of the following equivalent conditions is fulfilled:

i) for some $p > 0$, there exists a constant $A > 0$ such that $H_p^E(\varepsilon) \geqslant A\varepsilon^q$ for ε small enough,

ii) there exists a constant $B > 0$ such that $H_1^E(\varepsilon) \geqslant B\varepsilon^q$ for ε small enough,

iii) for some $p > 0$, there exists a constant $C > 0$ such that

$$\left((1/2\pi)\int_0^{2\pi}\|x + e^{i\theta}y\|^p \,d\theta\right)^{1/p} \geqslant (\|x\|^q + C\|y\|^q)^{1/q}.$$

By [3], we know that the usual L^p-spaces associated with a measure space are 2-uniformly-PL-convex for $1 \leqslant p \leqslant 2$ and p-uniformly-PL-convex for $2 \leqslant p < \infty$. In this section, we shall extend these results to non commutative L^p-spaces $(1 \leqslant p < \infty)$ associated with a von Neumann algebra M. When M is the algebra of all bounded operators on a Hilbert space, our result is just Theorem 1 of [22] for $p \geqslant 2$, but is more precise for $1 \leqslant p \leqslant 2$. When $p = 1$, our result follows from the fact — due to U. Haagerup — that the dual of any C^*-algebra is 2-uniformly-PL-convex (see [3], Theorem 4.3, for a proof). Finally, it is shown in [3] that a q-uniformly-PL-convex space is of cotype q, so that the results of this section give us again all the information about the cotype of the non commutative L^p-spaces.

4.2. LEMMA 8. *Let M be a von Neumann algebra. Let T, S be self-adjoint elements in $L^p(M)$ $(0 \leqslant p \leqslant 2)$. Then, we have:*

$$\|T + iS\|_p^p \leqslant \|T\|_p^p + \|S\|_p^p.$$

Proof. For $0 < p \leqslant 1$, the result follows (without any extra assumption on T and S) from [8], Theorem 4.7 (see also [7], Proposition 3.1). Let us assume that $1 \leqslant p \leqslant 2$. By [12], Proposition 5.5 we have:

$$\|T + iS\|_p^p + \|T - iS\|_p^p \leqslant 2(\|T\|_p^p + \|iS\|_p^p).$$

Since S and T are self-adjoint, we get:

$$\|T + iS\|_p = \|(T + iS)^*\|_p = \|T - iS\|_p.$$

On the other hand, we have $\|iS\|_p = \|S\|_p$, so that we get:

$$2\|T + iS\|_p^p \leqslant 2(\|T\|_p^p + \|S\|_p^p).$$

The lemma is proved. ∎

The author would like to thank the referee for pointing out a mistake in a first proof of this result.

The following two lemmas generalize two results of [21] (Lemma 1 and Proposition 1).

4.3. LEMMA 9. *Let N be a von Neumann algebra with a normal semi-finite trace τ. Let T, S be τ-measurable operators. Assume that T is positive and S is self-adjoint. Then, we have:*

$$\mu_t(T + iS) \geqslant \mu_t(T) \quad \textit{for any } t > 0.$$

Proof. Let E be a projection in N with $\tau(1 - E) \leqslant t$. For any vector ξ in $E(H)$ with $\|\xi\| = 1$, we have:

$$(T\xi \mid \xi) \leqslant |((T + iS)\xi \mid \xi)| \leqslant \|(T + iS)E\|$$

and hence:

$$\sup_{\substack{\xi \in E(H) \\ \|\xi\| = 1}} (T\xi \mid \xi) \leqslant \|(T + iS)E\|.$$

Taking the inf over all projections E in M with $\tau(1 - E) \leqslant t$, we get the result by [8] (remark after Definition 2.1). ∎

4.4. LEMMA 10. *Let N be a von Neumann algebra with a normal semi-finite trace τ. Let T, S be τ-measurable operators with $\lim_{t\to\infty} \mu_t(T) = \lim_{t\to\infty} \mu_t(S) = 0$. Assume that T is positive and S is self-adjoint. For any p with $0 \leqslant p \leqslant 2$, we* *have:*

$$\int_0^t \mu_s(T + iS)^p \, ds \leqslant \int_0^t \mu_s(T)^p \, ds + 2\int_0^t \mu_s(S)^p \, ds \quad (t > 0).$$

Proof. We may assume without loss of generality that $\int_0^t \mu_s(T)^p \, ds$ and $\int_0^t \mu_s(S)^p \, ds$ are both finite. We may also assume that T and S are τ-compact elements in N in the sense of [6] (cf. 1.8, p. 314). Indeed, let $T = \int_0^\infty \lambda \, dE_\lambda$ and $S = U\int_0^\infty \lambda \, dF_\lambda$ be the spectral decompositions of T, S, and put for $n = 1, 2, \ldots$:

$$T_n = \int_0^n \lambda \, dE_\lambda, \quad S_n = U\int_0^n \lambda \, dF_\lambda.$$

Then, T_n and S_n are τ-compact elements in N with T_n positive and S_n self-adjoint. Assuming that the lemma is true for T_n, S_n and using the inequalities

$$\mu_s(T_n) \leqslant \mu_s(T), \quad \mu_s(S_n) \leqslant \mu_s(S) \quad (s > 0),$$

we get:

$$\int_0^t \mu_s(T_n + iS_n)^p \, ds \leqslant \int_0^t \mu_s(T)^p \, ds + \int_0^t \mu_s(S)^p \, ds$$

for any $t > 0$. But $T_n \to T$ and $S_n \to S$ in measure so that $T_n + iS_n \to T + iS$ in measure. By [8] (Lemma 3.4) we get

$$\mu_s(T + iS) \leqslant \liminf_{n\to\infty} \mu_s(T_n + iS_n)$$

and hence

$$\int_0^t \mu_s(T + iS)^p \, ds \leqslant \int_0^t \mu_s(T)^p \, ds + \int_0^t \mu_s(S)^p \, ds \quad (t > 0)$$

for $0 < p \leqslant 2$ by Fatou's lemma. We thus may assume that T, S are τ-compact elements in N such that $\int_0^t \mu_s(T)^p \, ds$ and $\int_0^t \mu_s(S)^p \, ds$ are both finite. Let us make two more reductions. Firstly, we may easily assume that $\mu_s(T + iS) > 0$ for $s < t$. Secondly, replacing N by $N \otimes L^\infty([0, 1], dx)$, the trace τ by its tensor product with the trace $f \mapsto \int_0^1 f(s)ds$ on $L^\infty([0, 1], dx)$, and using the equality

$$\mu_s(T \otimes 1) = \mu_s(T) \quad (s > 0),$$

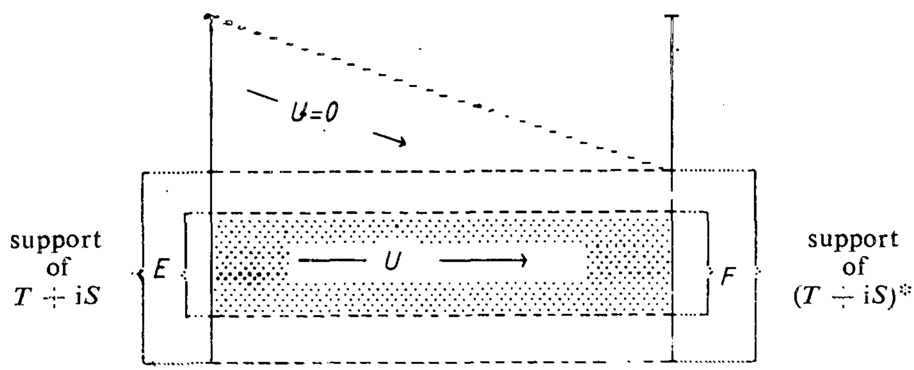
we may and do assume that N has no minimal projection. Then, there exists by [6] (Lemma 1.13, p. 317) a projection E in N which commutes with $|T + iS|$ and satisfies the following conditions:

i) $E \leqslant \operatorname{supp}(T + iS)$,

ii) $\tau(E) = t$, and

iii) $\mu_s(T + iS) = \mu_s(E|T + iS|E) = \mu_s(E|T + iS|^2E)^{1/2}$ for $0 \leqslant s < t$.

Let $T + iS = U|T + iS|$ be the polar decomposition of $T + iS$ and set $F = UEU^*$, $G = E \vee F$.



We have:

$$\tau(G) = \tau(E \vee F) \leqslant \tau(E) + \tau(F) = 2\tau(E) = 2t.$$

On the other hand, we have:

$$|G(T + iS)G|^2 = G(T + iS)^*G(T + iS)G =$$

$$= G|T + iS|U^*GU|T + iS|G \geqslant G|T + iS|U^*FU|T + iS|G =$$

$$= G|T + iS|E|T + iS|G = GE|T + iS|^2EG =$$

(because E commutes with $|T + iS|$)

$$= E|T + iS|^2E.$$

We then deduce:

$$\mu_s(G(T+\mathrm{i}S)G) = \mu_s(|G(T+\mathrm{i}S)G|^2)^{1/2} \geqslant$$

$$\geqslant \mu_s(E|T+\mathrm{i}S|^2E)^{1/2} = \mu_s(T+\mathrm{i}S) \quad \text{for } 0 \leqslant s \leqslant t.$$

But we clearly have:

$$\mu_s(G(T+\mathrm{i}S)G) \leqslant \mu_s(T+\mathrm{i}S)$$

so that finally:

$$(*) \qquad \int_0^t \mu_s(T+\mathrm{i}S)^p \,\mathrm{d}s = \int_0^t \mu_s(G(T+\mathrm{i}S)G)^p \,\mathrm{d}s.$$

By [8] (Lemma 2.6), we have $\mu_s(G(T+\mathrm{i}S)) = 0$ for $s \geqslant \tau(G)$, and hence:

$$\int_0^{2t} \mu_s(G(T+\mathrm{i}S)G)^p \,\mathrm{d}s = \int_0^{\infty} \mu_s(G(T+\mathrm{i}S)G)^p \,\mathrm{d}s =$$

$$= \|G(T+\mathrm{i}S)G\|_p^p \leqslant \qquad \text{([8], Corollary 2.8)}$$

$$\leqslant \|GTG\|_p^p + \|GSG\|_p^p = \qquad \text{(Lemma 8)}$$

$$= \int_0^{2t} \mu_s(GTG)^p \,\mathrm{d}s + \int_0^{2t} \mu_s(GSG)^p \,\mathrm{d}s \leqslant$$

$$\leqslant \int_0^{t} \mu_s(T)^p \,\mathrm{d}S + \int_t^{2t} \mu_s(GTG)^p \mathrm{d}s + 2\int_0^{t} \mu_s(S)^p \,\mathrm{d}s.$$

We thus have:

$$\int_0^{t} \mu_s(T+\mathrm{i}S)^p \,\mathrm{d}S + \int_t^{2t} \mu_s(GTG)^p \,\mathrm{d}s =$$

$$= \int_0^{t} \mu_s(G(T+\mathrm{i}S)G)^p \,\mathrm{d}s + \int_t^{2t} \mu_s(GTG)^p \,\mathrm{d}s \leqslant \qquad \text{(use}(*)\text{)}$$

$$\leqslant \int_0^{2t} \mu_s(G(T+\mathrm{i}S)G)^p \,\mathrm{d}s \leqslant \qquad \text{(use Lemma 9)}$$

$$\leqslant \int_0^{t} \mu_s(T)^p \,\mathrm{d}s + \int_t^{2t} \mu_s(GTG)^p \,\mathrm{d}s + 2\int_0^{t} \mu_s(S)^p \,\mathrm{d}s.$$

Substracting $\int_t^{2t} \mu_s(GTG)^p \, ds$ from both sides of this last inequality, we get our lemma. ▨

4.5. LEMMA 11. *Let M be a von-Neumann algebra, and T, S be elements in $L^p(M)$. We assume that T is positive and S is self-adjoint. Then, we have:*

$$\| T + iS\|_p^{\min(p,2)} \leqslant \begin{cases} \|T\|_p^p + \|S\|_p^p & \text{if } 0 < p \leqslant 2 \\ \|T\|_p^2 + 2\|S\|_p^2 & \text{if } 2 \leqslant p < \infty. \end{cases}$$

Proof. The case $0 < p \leqslant 2$ follows from Lemma 8. When $2 < p < \infty$, we have by Lemma 10:

$$(*) \qquad \int_0^1 \mu_s(T + iS)^2 \, ds \leqslant \int_0^1 \mu_s(T)^2 \, ds + 2\int_0^1 \mu_s(S)^2 \, ds.$$

But $\mu_s(T) \leqslant s^{-1/p}\|T\|_p$ for any T in $L^p(M)$, and hence:

$$\int_0^1 \mu_s(T)^2 \, ds = \|T\|_p^2 \int_0^1 s^{-2/p} \, ds = (p/(p-2))\|T\|_p^2.$$

Using this last equality, we deduce from $(*)$:

$$\|T + iS\|_p^2 \leqslant \|T\|_p^2 + 2\|S\|_p^2.$$ ▨

The following lemma generalizes a result (Theorem 1) of [9].

4.6. LEMMA 12. *Let M be a von Neumann algebra and T, S be elements in $L^p(M)$ with $\|T\|_p = 1$. Assume that the projections $1 - \operatorname{supp}(T)$ and $1 - \operatorname{supp}(T^*)$ are comparable in the Murray-von Neumann sense. Then, we have:*

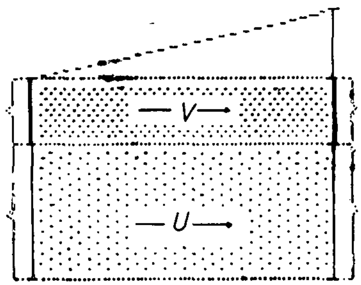
$$(1/2\pi) \int_0^{2\pi} \|T + e^{i\theta}S\|_p \, d\theta \geqslant \begin{cases} (1 + (1/8)\|S\|_p^2)^{1/2} & \text{if } 1 < p \leqslant 2 \\ (1 + (1/2^p)\|S\|_p^p)^{1/p} & \text{if } 2 \leqslant p < \infty. \end{cases}$$

Proof. Note first that these inequalities are true for T and any S if and only if they are true for T^* and any S. So, replacing T by T^* if necessary, we may assume that:

$$1 - \operatorname{supp}(T) \prec 1 - \operatorname{supp}(T^*)$$

in the Murray-von Neumann sense. Let V be a partial isometry in M having $1 - \operatorname{supp}(T)$ as support, and whose range projection is majorized (in the usual sense) by

$1 - \operatorname{supp}(T^*)$. Let $T = U|T|$ be the polar decomposition of T, and set $W = U + V$.



We have:

$$W^* W = (U^* + V^*)(U + V) = U^*U + V^*V = 1$$

and

$$|T^*| = UT^* = U(\operatorname{supp}(T))T^* = W(\operatorname{supp}(T))T^* = WT^*.$$

Set $X = |T^*|^{p-1}$, and denote by $p' = p/(p-1)$ the conjugate exponent of p. Then X is a *positive* element in $L^{p'}(M)$ satisfying:

$$\|X\|_{p'}^{p'} = \operatorname{tr}(X^{p'}) = \|T^*\|_p^p = 1 \tag{1}$$

and

$$\operatorname{tr}(XWT^*) = \operatorname{tr}(|T^*|^{p-1}|T^*|) = \|T^*\|_p^p = 1. \tag{2}$$

Let $Z = \|S\|_p^{1-p}|S|^{p-1}C^*$, where $S = C|S|$ is the polar descomposition of S. We thus define an element in $L^{p'}(M)$ such that:

$$\|Z\|_{p'} = 1 \quad \text{and} \quad \operatorname{tr}(ZS) = \|S\|_p.$$

Set $Y = ZW^*$. We thus define an element in $L^{p'}(M)$ satisfying:

$$\|Y\|_{p'} \leqslant 1 \quad \text{and} \quad \operatorname{tr}(YWS) = \|S\|_p \qquad (\text{because } W^*W = 1). \tag{3}$$

Set

$$\alpha = \begin{cases} (1/8)\|S\|_p & \text{if } 1 < p \leqslant 2 \\ (1/2^p)\|S\|_p^{p-1} & \text{if } 2 \leqslant p < \infty. \end{cases}$$

We clearly have:

$$1 + \alpha\|S\|_p = \begin{cases} [1 + 2(2\alpha)^2]^{1/2}[1 + (1/8)\|S\|_p^2]^{1/2} & \text{if } 1 < p \leqslant 2 \\ [1 + (2\alpha)^{p'}]^{1/p'}[1 + (1/2^p)\|S\|_p^p]^{1/p} & \text{if } 2 \leqslant p < \infty. \end{cases} \tag{4}$$

For any $\theta \in [0, 2\pi]$, set:

$$Y_\theta = \mathrm{i}e^{\mathrm{i}\theta} Y^* - \mathrm{i}e^{-\mathrm{i}\theta} Y.$$

Using Lemma 11 together with (1) and (3), we get:

$$\|X + \mathrm{i}\alpha Y_\theta\|_{p'}^{p'} \leqslant 1 + (2\alpha)^{p'} \quad \text{if } 2 \leqslant p < \infty \tag{5}$$

$$\|X + \mathrm{i}\alpha Y_\theta\|_{p'}^{2} \leqslant 1 + 2(2\alpha)^{2} \quad \text{if } 1 < p \leqslant 2. \tag{5 bis}$$

From (2) and (3), we deduce:

$$(1 + \alpha\|S\|_p) = \mathrm{tr}(XWT^* + \alpha YWS) =$$

$$= (1/2\pi)\int_0^{2\pi} \mathrm{tr}[(X + \mathrm{i}\alpha Y_\theta)(WT^* + e^{\mathrm{i}\theta}WS)]\,d\theta \leqslant$$

$$\leqslant (1/2\pi)\int_0^{2\pi} \|(X + \mathrm{i}\alpha Y_\theta\|_{p'} \|W(T^* + e^{\mathrm{i}\theta}S)\|_p\,d\theta$$

and hence, by (5) and (5bis):

$$(1 + \alpha\|S\|_p) \leqslant$$

$$\leqslant \begin{cases} (1/2\pi)(1 + (2\alpha)^{p'})^{1/p'} \displaystyle\int_0^{2\pi} \|T^* + e^{\mathrm{i}\theta}S\|_p\,d\theta, & \text{if } 2 \leqslant p < \infty \\ (1/2\pi)(1 + 2(2\alpha)^2)^{1/2} \displaystyle\int_0^{2\pi} \|T^* + e^{\mathrm{i}\theta}S\|_p\,d\theta & \text{if } 1 < p \leqslant 2. \end{cases}$$

Using (4), we get finally:

$$(1/2\pi)\int_0^{2\pi} \|T^* + e^{\mathrm{i}\theta}S\|_p\,d\theta \geqslant \begin{cases} (1 + (1/2^p)\|S\|_p^p)^{1/p} & \text{if } 2 \leqslant p < \infty \\ (1 + (1/8)\|S\|_p^2)^{1/2} & \text{if } 1 < p \leqslant 2. \end{cases}$$

The proof of the lemma is complete. ▨

If M is a factor, the hypothesis of Lemma 12 is always fulfilled by the comparability theorem for projections (cf. [4], Corollary 1, p. 218). It then follows that $L^p(M)$ is q-uniformly-PL-convex, where $q = \max\{2, p\}$. Using the reduction theory for von Neumann algebras together with results of [11], [13] and [18], it is not difficult to extend this last result to von Neumann algebras with separable predual. Actually, we first proved the result by this way, and the more elegant Proof presented below is largely due to U. Haagerup. We would like to thank Professor Haagerup for the welcome idea of using the 2×2 matrix trick in order to handle the non-factor case.

4.7. THEOREM 4. *Let M be a von Neumann algebra acting on a separable Hilbert space. Let p be a positive real number with $1 \leqslant p < \infty$. Then $L^p(M)$ is q-uniformly-PL-convex, where $q = \max\{2, p\}$.*

Proof. The case $p = 1$ follows from [3], so that we may assume that $p > 1$. Set $\tilde{M} = M \otimes M_2(\mathbf{C})$. Let φ be a fixed normal faithful weight on M and put:

$$\tilde{\varphi} = \varphi \otimes \mathrm{Tr},$$

where Tr is the normalized trace on $M_2(\mathbf{C})$, i.e., $\mathrm{Tr}(1) = 1$. We clearly have

$$\sigma_t^{\tilde{\varphi}} = \sigma_t^{\varphi} \otimes \mathrm{Id},$$

and $\tilde{N} = \tilde{M} \rtimes_{\sigma^{\tilde{\varphi}}} \mathbf{R}$ identifies naturally with $N \otimes M_2(\mathbf{C})$, where N denotes the crossed product of M by the modular automorphism group φ. The dual action $\tilde{\theta}$ and the distinguished semi-finite trace $\tilde{\tau}$ are then given by:

$$\tilde{\theta}_\lambda = \theta_\lambda \otimes \mathrm{Id} \quad (\lambda \in \mathbf{R})$$

and

$$\tilde{\tau} = \tau \otimes \mathrm{Tr}.$$

Let T, S be elements in $L^p(M)$ and set:

$$\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix}.$$

We thus define elements $\tilde{T}$, $\tilde{S}$ in $L^p(M)$. Moreover, we have:

$$\|\tilde{T}\|_p = \tilde{\tau}(E_1(|\tilde{T}|^p))^{1/p} =$$

$$= [(1/2)\{\tau(E_1(|T|^p)) + \tau(E_1(|T^*|^p))\}]^{1/p} = \|T\|_p,$$

and, similarly, $\|\tilde{S}\|_p = \|S\|_p$.

But now, the two projections $1-\text{supp}(\tilde{T})$ and $1-\text{supp}(\tilde{T}^*)$ are clearly equivalent, so that there exists by Lemma 12 a constant $C_p > 0$ such that we have:

$$(*) \qquad (1/2\pi)\int_0^{2\pi} \|\tilde{T} + e^{i\theta}\tilde{S}\|_p \, d\theta \geqslant (1 + C_p\|S\|_p^r)^{1/r},$$

where $r = \max\{2, p\}$. For $0 < q < \infty$, set:

$$H_q(A, B) = \left[(1/2\pi)\int_0^{2\pi} \|A + e^{i\theta}B\|_p^q \, d\theta\right]^{1/q}.$$

By [3] (Theorem 2.4), we have:

$$H_{2q}(A, e^{-1/2}B) \leqslant H_q(A, B) \quad \text{for any } A, B \text{ in } L^p(M).$$

Choosing n with $1/2^n \leqslant p$ and using the fact that $q \mapsto H_q(A, B)$ is increasing, we get:

$$H_1(\tilde{T}, \tilde{S}) \leqslant H_{1/2^n}(\tilde{T}, e^{1/2^n}\tilde{S}) \leqslant H_p(\tilde{T}, e^{1/2^n}\tilde{S}) =$$

$$= \left[(1/2\pi)\int_0^{2\pi} (1/2)\{\|T + e^{i\theta}e^{1/2^n}S\|_p^p + \|T^* + e^{i\theta}e^{1/2^n}S^*\|_p^p\} \, d\theta\right]^{1/p} =$$

$$= \left[(1/2\pi)\int_0^{2\pi} \|T + e^{i\theta}e^{1/2^n}S\|_p^p \, d\theta\right]^{1/p}.$$

From $(*)$, we deduce:

$$(1 + C_p\|S\|_p^r)^{1/r} \leqslant \left[(1/2\pi)\int_0^{2\pi} \|T + e^{i\theta}e^{1/2^n}S\|_p^p \, d\theta\right]^{1/p},$$

i.e.,

$$(1 + B_p\|S\|_p^r)^{1/r} \leqslant \left[(1/2\pi)\int_0^{2\pi} \|T + e^{i\theta}S\|_p^p \, d\theta\right]^{1/p},$$

where $B_p = \exp(-\, r/2^n)C_p$ and $r = \max\{p, 2\}$.

This proves that $L^p(M)$ is r-uniformly-PL-convex for $1 \leqslant p < \infty$. ▨

5. APPLICATIONS

A well-known theorem of A. Grothendieck asserts that every bounded operator T from $C(K)$ into $L^p(K, \lambda)$ (K is a compact set and λ a probability measure on K) factors through a Hilbert space if $1 \leqslant p \leqslant 2$. In his IHES seminar, A. Connes asked whether this result remains true or not when $C(K)$ is replaced by a C^*-algebra A, λ by a state φ on A and $L^p(K, \lambda)$ by the "non commutative" L^p-space associated with the von Neumann algebra generated by the image of A in the GNS representation of φ. We have in fact:

5.1. THEOREM 5. *Let M be a von Neumann algebra acting on a separable Hilbert space. Then, every bounded operator T from a C^*-algebra A into $L^p(M)$, $1 \leqslant p \leqslant 2$, factors through a Hilbert space and is q-C^*-summing for $q \geqslant 2$ in the following sense: there exists a constant $C > 0$ such that we have, for any finite sequence $(x_1, \ldots, x_n)$ of elements in A:*

$$\Big(\sum_{1 \leqslant i \leqslant n} \|T(x_i)\|_p^q\Big)^{1/q} \leqslant C \Big\|\Big(\sum_{1 \leqslant i \leqslant n} |x_i|^q\Big)^{1/q}\Big\|_A . ^{*)}$$

Proof. By Theorem 3, the dual of $L^p(M)$ is of type 2. But the dual of A is of cotype 2 (cf. [15], Proposition 9.3) so that T^* factors through a Hilbert space by Kwapien's result (see for instance [15], Corollary 3.6). Then T factors through a Hilbert space and is 2-C^*-summing by [15] (Remark 9.7 after Theorem 9.6 and Proposition 9.8). But a 2-C^*-summing operator with values in a Banach space is in fact q-C^*-summing by [16] (Proposition 1.3). ▨

This result also follows from the 2-uniformly-PL-convexity of $L^p(M)$ (Theorem 4) by using Theorem 9.11 of [15]. The less sophisticated proof given above was communicated to the author by G. Pisier, who also noticed that Theorem 5 for $p = 1$ is an immediate corollary of a result due to U. Haagerup about the factorization of operators from a C^*-algebra into a dual of some C^*-algebra. We would like to thank G. Pisier for these valuable remarks.

The following applications are well-known from the specialists of the geometry of Banach spaces. We shall state them briefly, hoping that they may be useful for C^*-algebraists too.

5.2. Recall that a sequence (x_n) of elements of a Banach space is said to be *unconditionally summable* if the series $\sum_{n \geqslant 1} \varepsilon_n x_n$ converge for any choice signs $\varepsilon_n = \pm 1$.

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*) Here, the modulus $|x|$ of an element x in A is defined by $|x| = [(1/2)(xx^* + x^*x)]^{1/2}$.

Let (T_n) be a unconditionally summable sequence of elements in $L^p(M)$, where M is a von Neumann algebra. Then we have:

$$\sum_{n\geq 1} \|T_n\|_p^{\max\{p,2\}} < \infty.$$

This follows immediately from Theorem 3.

5.3. Let M be a von Neumann algebra. If $1 \leqslant p \leqslant 2 \leqslant q < \infty$, then any bounded operator $T\colon L^q(M) \to L^p(M)$ factors through a Hilbert space. This follows immediately from Theorem 3 and Corollary 3.6 of [15].

5.4. Let M be a von Neumann algebra. Then every bounded operator from c_0 into $L^p(M)$ $(1 \leqslant p \leqslant 2)$ is q-summing for $2 \leqslant q < \infty$. For $q = 2$, this follows from a result of B. Maurey together with Theorem 3. But $L^p(M)$ is of cotype 2 and any 2-summing operator from c_0 to $L^p(M)$ is also q-summing for $2 < q < \infty$.

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