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ON QUASI-EQUIVALENCE OF QUASIFREE STATES OF GAUGE INVARIANT CAR ALGEBRAS

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1. INTRODUCTION AND MAIN RESULTS

In this article we give a criterion of quasi-equivalence for two gauge invariant quasifree states of a GICAR (gauge invariant CAR) algebra. A GICAR is the fixed point subalgebra by an action of any compact group via Bogoliubov automorphisms of a CAR (canonical anti-commutation relations) algebra. Quasifree states of a GICAR are the restriction of quasifree states of the CAR algebra to this subalgebra.

The representations associated to quasifree states have been studied by several authors in connection with application to the Ising model (Araki-Evans [2]) and to the representation theory of $U(\infty)$ (Strătilă-Voiculescu [5]). In a different setting, similar problems have been discussed by Baker and Powers [3], [4]. Their results are restricted to actions of specific groups $G = \mathbf{Z}_2$, U(1) or SU(2), and their methods are applicable only to factor states.

In this paper, we present a criterion of quasi-equivalence for arbitrary compact gauge group and for more or less general quasifree states with a minor technical condition. We believe that our proof is also simpler than earlier proof on this subject.

To describe our results, we now introduce some notations.

Let $\mathscr{K}$ be a separable Hilbert space with an antiunitary involution J:

$$(Jf, Jg) = (g, f) \quad (f, g \in \mathscr{K}), \ J^2 = \mathbf{1}. \tag{1.1}$$

Let $\mathfrak{A}(\mathscr{K}, J)$ be a selfdual Clifford C^*-algebra (see [1]) generated by $\{B(h)\ ;\ h \in \mathscr{K}\}$, where $B(h)$ is linear in h and satisfies the selfdual CAR:

$$B(h)^* = B(Jh), \tag{1.2a}$$

$$\{B(h),\ B(h')^*\} \equiv B(h)B(h')^* + B(h')^*B(h) = (h,\ h')\mathbf{1}. \tag{1.2b}$$

Suppose a compact group G and its unitary representation $u(g)$ on $\mathscr{K}$ are given. If $u(g)$ fulfils the condition

$$u(g)J = Ju(g), \tag{1.3}$$

the action α of G on $\mathfrak{A}(\mathscr{K}, J)$ is defined via the equation

$$\alpha_g(B(h)) = B(u(g)h) \quad (h \in \mathscr{K},\ g \in G). \tag{1.4}$$

The fixed point algebra by this action is denoted by $\mathfrak{A}(\mathscr{K}, J)^G$:

$$\mathfrak{A}(\mathscr{K}, J)^G := \{Q \in \mathfrak{A}(\mathscr{K}, J)\,;\, \alpha_g(Q) = Q \text{ for any } g \in G\}. \tag{1.5}$$

We sometimes write $\mathfrak{A}$ or $\mathfrak{A}^G$ instead of $\mathfrak{A}(\mathscr{K}, J)$ or $\mathfrak{A}(\mathscr{K}, J)^G$ if there is no danger of confusion.

For a state w of $\mathfrak{A}$, w^G denotes its restriction to $\mathfrak{A}^G$,

$$w^G = w \mid \mathfrak{A}^G. \tag{1.6}$$

Next we define quasifree states.

DEFINITION 1. $\mathscr{S}$ denotes the set of all selfadjoint operators S on $\mathscr{K}$ satisfying

$$\begin{aligned} &\text{(i) } 0 \leqslant S \leqslant \mathbf{1}, \text{ and} \\ &\text{(ii) } JSJ = \mathbf{1} - S. \end{aligned} \tag{1.7}$$

$$\mathscr{S}_G := \{S \in \mathscr{S}\,;\, [u(g), S] = 0 \text{ for all } g \in G\}. \tag{1.8}$$

DEFINITION 2. For each $S \in \mathscr{S}$, φ_S is a state of $\mathfrak{A}(\mathscr{K}, J)$ defined by

$$\varphi_S(B(h_1)B(h_2) \ldots B(h_{2k+1})) = 0 \quad (h_j \in \mathscr{K}), \tag{1.9a}$$

$$\varphi_S(B(h_1) \ldots B(h_{2k})) = \sum_p \operatorname{sgn} P \prod_{j=1}^{k} (Jh_{p(2j-1)}, Sh_{p(2j)}), \tag{1.9b}$$

where the sum runs over all permutations p satisfying

$$\begin{aligned} &p(2j-1) < p(2j) \quad \text{for all } j \text{ and} \\ &p(1) < p(3) < p(5) < \ldots < p(2k-1). \end{aligned} \tag{1.10}$$

If $E \in \mathscr{S}$ is a projection we call it a *basis projection*.

Our results are the following. (Quasi-equivalence and equivalence of states refer to those of associated cyclic representations.)

THEOREM A. (i) *Let* $E_1, E_2 \in \mathscr{S}_G$ *be basis projections.* $\varphi^G_{E_1}$ *and* $\varphi^G_{E_2}$ *are equivalent if and only if the following conditions are both satisfied.*

(1.11)

(a) $E_1 - E_2$ *is in the Hilbert-Schmidt class.*

(b) $\det_{E_1 \wedge JE_2J}(g) = 1$ *for all* $g \in G$.

By the first condition (a), $E_1 \wedge JE_2J$ *has a finite dimensional range which is* $u(g)$ *invariant, and* $\det_{E_1 \wedge JE_2J}(g)$ *is the determinant of* $u(g)$ *restricted to this range* $(E_1 \wedge JE_2J)\mathscr{K}$.

(ii) *Let* $S_1, S_2 \in \mathscr{S}_G$ *and* $0 < S_i < \mathbf{1}$ *(*i.e. S_i *has no point spectrum at* 0 *and* 1*). Then* $\varphi^G_{S_1}$ *and* $\varphi^G_{S_2}$ *are quasi-equivalent if and only if* $\sqrt{S_1} - \sqrt{S_2}$ *is in the Hilbert-Schmidt class.*

We shall prove the first statement in Section 2 and the second in Section 3.

2. PROOF FOR FOCK STATES

Before going to the proof we list some known facts about the cyclic representation of $\mathfrak{A}(\mathscr{K}, J)$ associated to a quasifree state φ_S.

(a) φ_S is pure if and only if S is a basis projection.

(b) φ_{S_1} and φ_{S_2} are quasi-equivalent if and only if $\sqrt{S_1} - \sqrt{S_2}$ is in the Hilbert-Schmidt class.

For a basis projection P in $\mathscr{S}_G$ the state φ_P is G-invariant, so φ^G_P is pure due to (a).

The following lemma will be used for our proof.

LEMMA 1. *Let* $\mathfrak{A}$ *be a* C^**-algebra and* $\mathfrak{B}$ *be its subalgebra with a conditional expectation* χ *from* $\mathfrak{A}$ *onto* $\mathfrak{B}$. *If two states* φ_1, φ_2 *of* $\mathfrak{B}$ *are quasi-equivalent,* $\varphi_1 \circ \chi$ *and* $\varphi_2 \circ \chi$ *are quasi-equivalent as states of* $\mathfrak{A}$. (See Strătilă-Voiculescu [5].)

Lemma 1 is applied to $\mathfrak{A} = \mathfrak{A}(\mathscr{K}, J)$, $\mathfrak{B} = \mathfrak{A}^G$ and $\chi = \int \mathrm{d}g\, \alpha_g(\cdot)$ where $\mathrm{d}g$ denotes the normalized Haar measure on G. By this lemma and (b), we have to study only the case that $P_1 - P_2$ is in the Hilbert-Schmidt class:

LEMMA 2. *Let* $P_1, P_2 \in \mathscr{S}_G$ *be basis projections. If* $\varphi^G_{P_1}$ *and* $\varphi^G_{P_2}$ *are equivalent,* $P_1 - P_2$ *is in the Hilbert-Schmidt class. Otherwise, the two states are disjoint.*

Let $\{\Omega, \pi(\cdot), \mathscr{H}\}$ be the cyclic representation of $\mathfrak{A}(\mathscr{K}, J)$ associated to the Fock state φ_{P_1}.

As φ_{P_1} is G-invariant, we have a unitary representation $V(g)$ of G such that

(2.1a) $$V(g)\Omega = \Omega,$$

(2.1b) $$V(g)\pi(Q) = \pi(\alpha_g(Q))V(g), \quad g \in G,\ Q \in \mathfrak{A}(\mathscr{K}, J).$$

Let $\mathscr{E}$ be the conditional expectation of $\mathfrak{B}(\mathscr{H})$ onto $V(G)'$ defined by

(2.2) $$\mathscr{E}(A) = \int dg\, V(g)AV(g)^* \quad (A \in \mathfrak{B}(\mathscr{H})).$$

Then

(2.3) $$\pi(\mathfrak{A}(\mathscr{K}, J)^G)'' = \pi(\mathfrak{A}(\mathscr{K}, J))'' \cap V(G)' = V(G)'.$$

The second identity is due to the purity of φ_{P_1}. The first identity is seen as follows.

Let A be in $\pi(\mathfrak{A}(\mathscr{K}, J))'' \cap V(G)'$. Then there exists a sequence $\{A_\alpha\}$ in $\mathfrak{A}(\mathscr{K}, J)$ such that $A = \lim_\alpha \pi(A_\alpha)$. As $\mathscr{E}(A) = A$ and $\mathscr{E}(\pi(A_\alpha)) = \pi(\chi(A_\alpha))$,

(2.4) $$A = \lim_\alpha \pi(\chi(A_\alpha)).$$

Since $\chi(A_\alpha)$ is in $\mathfrak{A}(\mathscr{K}, J)^G$, we obtain $\pi(\mathfrak{A}(\mathscr{K}, J)^G)'' \supset \pi(\mathfrak{A}(\mathscr{K}, J))'' \cap V(G)'$. The converse inclusion is obvious.

COROLLARY 3. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ be $\pi(\mathfrak{A}^G)$ irreducible subspaces of $\mathscr{H}$ invariant under $V(G)$. Then the restriction of $\pi(\mathfrak{A}^G)$ on $\mathscr{H}_1$ and on $\mathscr{H}_2$ are equivalent if and only if the restrictions of $V(G)$ on $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent.*

This is an immediate consequence of (2.3).

Note that quasi-equivalence is replaced by equivalence by irreducibility in the case of $\pi(\mathfrak{A}^G)$ and by $\dim\mathscr{H}_1 = \dim\mathscr{H}_2$ as well as proportionality to identity in the case of $V(G)$.

Proof of Theorem A (i). By Lemma 2, we can assume that $p_1 - p_2$ is in the Hilbert-Schmidt class and φ_{P_1} and φ_{P_2} yield unitarily equivalent representations of $\mathfrak{A}(\mathscr{K}, J)$. For $\{\Omega, \pi(\cdot), \mathscr{H}\}$ defined as above, there exists a unit vector Ω' implementing the state φ_{P_2}, which is unique up to a scalar factor. To see the uniqueness suppose we have another vector Ω'' implementing the same state φ_{P_2}. By irreducibility of $\pi(\cdot)$, both Ω' and Ω'' are cyclic for $\pi(\mathfrak{A})$. The uniqueness of GNS cyclic representation implies the existence of a unitary W on $\mathscr{H}$ satisfying

(2.5a) $$W\pi(Q)W^* = \pi(Q) \quad (Q \in \mathfrak{A}(\mathscr{K}, J)),$$

(2.5b) $$W\Omega' = \Omega''.$$

By irreducibility of $\pi(\mathfrak{A})W = c\mathbf{1}$ for some scalar c, which shows the uniqueness.

Claim 1. $V(g)\Omega' = c(g)\Omega'$ where $c(g)$ is a character of G.

In fact, $V(g)\Omega'$ implements the state $\varphi_{P_2} \circ \alpha_g = \varphi_{P_2}$ of $\mathfrak{A}(\mathscr{K}, J)$, so $V(g)\Omega' = c(g)\Omega'$ for some character $c(g)$ of G due to the uniqueness just proved above.

The cyclic representation of $(\varphi^G_{P_1}, \mathfrak{A}(\mathscr{K}, J)^G)$ (resp. $(\varphi^G_{P_2}, \mathfrak{A}^G)$) is equivalent to the restriction of $\pi(\mathfrak{A}(\mathscr{K}, J)^G)$ to $\mathscr{H}_1 \equiv [\pi(\mathfrak{A}^G)\Omega]$ (resp. $\mathscr{H}_2 \equiv [\pi(\mathfrak{A}^G)\Omega']$) and is irreducible by (2.3). By Corollary 3, $\pi(\mathfrak{A}^G) \,|\, \mathscr{H}_1$ is equivalent to $\pi(\mathfrak{A}^G) \,|\, \mathscr{H}_2$ if and only if $V(G) \,|\, \mathscr{H}_1$ is equivalent to $V(G) \,|\, \mathscr{H}_2$. By (2.1), $V(g) = \mathbf{1}$ on $\mathscr{H}_1$. Claim 1 implies $V(g) = c(g)\mathbf{1}$ on $\mathscr{H}_2$. Thus Theorem A(i) is proved if the following holds.

Claim 2.

(2.6) $$c(g) = \det_{P_1 \wedge JP_2 J}(g).$$

Proof of Claim 2. Let $\sin\theta = |P_1 - P_2|$, $0 \leqslant \theta \leqslant \pi/2$. As we assume that $P_1 - P_2$ is in the Hilbert-Schmidt class, θ is compact. Moreover $[u(g), \sin\theta] = [u(g), \theta] = 0$ due to $[P_i, u(g)] = 0$ $(i = 1, 2)$. Let

(2.7a) $$E_{\pi/2} = P_1 \wedge (\mathbf{1} - P_2) + (\mathbf{1} - P_1) \wedge P_2$$

(2.7b) $$E_0 = P_1 \wedge P_2 + (\mathbf{1} - P_1) \wedge (\mathbf{1} - P_2),$$

and

(2.8) $$H(P_2/P_1) = \frac{i\theta}{\sin\theta\cos\theta}[P_1, P_2].$$

It is easy to check

(2.9) $$H(P_2/P_1)^2 = \theta^2(\mathbf{1} - E_0 - E_{\pi/2}).$$

By the inequality $(2/\pi)\theta \leqslant \sin\theta$ for $0 \leqslant \theta \leqslant \pi/2$,

(2.10) $$\operatorname{tr} H(P_2/P_1)^2 = \operatorname{tr}\theta^2(\mathbf{1} - E_0 - E_{\pi/2}) \leqslant \frac{\pi^2}{4}\operatorname{tr}\sin^2\theta = \frac{\pi^2}{4}\|P_1 - P_2\|^2_{\mathrm{HS}}.$$

Thus $H(P_2/P_1)$ and $e^{iH(P_2/P_1)} - \mathbf{1}$ are in the Hilbert-Schmidt class. Note that

(2.11) $$[e^{iH(P_2/P_1)}, u(g)] = 0.$$

By Lemma 9.3 of [1], the Bogoliubov automorphism associated to $e^{iH(P_2/P_1)}$ is implemented by a unitary W_1 in $\pi(\mathfrak{A}^G)''$,

$$W_1\pi(B(h))W_1^* = \pi(B(e^{iH(P_2/P_1)}h)), \quad (h \in \mathscr{K}). \tag{2.12}$$

Let $e_1, \ldots, e_n$ be an orthogonal basis of $P_1 \wedge (\mathbf{1} - P_2)$ and U be a unitary on $\mathscr{K}$ determined by

$$Ue_j = Je_j \quad UJe_j = e_j, \tag{2.13a}$$

$$Uf = f \quad \text{for } f \in (\mathbf{1} - E_{\pi/2})\mathscr{K}. \tag{2.13b}$$

Then $R = e^{iH(P_2/P_1)}U$ satisfies

$$RP_1R^* = P_2. \tag{2.14}$$

Let W_2 be a unitary on $\mathscr{K}$ defined by

$$W_2 = W_1\pi\left(\prod_{j=1}^{n}(B(e_j - Je_j))\right) = W_1W_0. \tag{2.15}$$

The following can be easily checked (see [1]):

$$W_2\pi(B(f))W_2^* = \pi(B((-1)^nRf)) \quad (f \in \mathscr{K}). \tag{2.16}$$

By (2.14) and (2.16), $W_2\Omega$ implements the state φ_{P_2} and we may take $\Omega' = W_2\Omega$.

We can now make the following computation:

$$c(g) = (W_2\Omega,\ V(g)W_2\Omega) = (W_1W_0\Omega,\ V(g)W_1W_0\Omega) = (W_0\Omega,\ V(g)W_0\Omega). \tag{2.17}$$

The last identity is due to $W_1 \in \pi(\mathfrak{A}^G)''$. As $e_j \in P_1\mathscr{K}$, $\pi(B(Je_j))\Omega = 0$. Therefore

$$W_0\Omega = \pi\left(\prod_{j=1}^{n} B(e_j)\right)\Omega. \tag{2.18}$$

Combined with (2.17), we obtain

$$\begin{aligned} c(g) &= \varphi_{P_1}(B(e_1)^*B(e_2)^* \ldots B(e_n)^*B(u(g)e_n) \ldots B(u(g)e_1)) = \\ &= \det(e_i,\ u(g)e_j)_{1\leq i,j\leq n} \end{aligned} \tag{2.19}$$

The last line follows from the quasifree property of φ_{P_1}. The equation (2.19) is nothing but $\det_{P_1\wedge JP_2J} u(g)$, thus we have verified Claim 2. Q.E.D.

The following are some examples of the determinant appearing in Theorem A(i).

EXAMPLE (a). Let $G = \mathbf{Z}_2 = \{1, -1\}$ and $u(-1)f = -f$ for $f \in \mathscr{K}$. Then $\det_{P_1 \wedge (1-P_2)}(-1) = (-1)^{\dim(P_1 \wedge (1-P_2))}$. This is the $\mathbf{Z}_2$ index in [2].

EXAMPLE (b). Let $G = \mathrm{U}(1)$ and its unitary representation be determined by

$$u(e^{i\theta}) = e^{i\theta}E + e^{-i\theta}(\mathbf{1} - E) \tag{2.20}$$

where E is a basis projection on $\mathscr{K}$. (Note that $[u(e^{i\theta}), P_1] = [u(e^{i\theta}), P_2] = 0$ imply $[E, P_1] = [E, P_2] = 0$.)

In this case, the determinant appearing in Theorem A is given by the formula:

$$\det_{P_1 \wedge (1-P_2)} u(e^{i\theta}) = e^{im\theta}, \tag{2.21a}$$

$$m = \dim EP_1 \wedge (\mathbf{1} - P_2) - \dim (\mathbf{1} - E)P_1 \wedge (\mathbf{1} - P_2). \tag{2.21b}$$

By (1.7) for basis projections,

$$J(\mathbf{1} - E)(P_1 \wedge (\mathbf{1} - P_2))J = E(\mathbf{1} - P_1) \wedge P_2. \tag{2.22}$$

It is easy to show

$$\dim EP_1 \wedge (\mathbf{1} - P_2) = \dim_{EP_1\mathscr{K}} \ker P_2P_1, \tag{2.23}$$

where $\dim_{EP_1\mathscr{K}} \ker P_2P_1$ denotes the dimension of the kernel for the operator P_2P_1 restricted to $EP_1\mathscr{K}$. By (2.22) and (2.23), (2.21 b) is expressed as

$$m = \dim_{EP_1\mathscr{K}} \ker P_2P_1 - \dim_{EP_2\mathscr{K}} \ker P_1P_2. \tag{2.24}$$

By Theorem A(i), vanishing of (2.24) is the criterion for equivalence of $\varphi^G_{P_1}$ and $\varphi^G_{P_2}$ (when $P_1 - P_2$ is in the Hilbert-Schmidt class). This recovers a result of [5].

THEOREM B. *Let P be a basis projection in $\mathscr{S}_G$ and $\{\Omega, \pi(\cdot), \mathscr{H}\}$ be the Fock representation of $\mathfrak{A}(\mathscr{K}, J)$ associated to P. Consider a unitary w on $\mathscr{K}$ with properties $Jw = wJ$ and $wu(g) = u(g)w$ for all g in G. The Bogoliubov automorphism associated to w is unitarily implemented by a unitary in $\pi(\mathfrak{A}^G)''$ if and only if conditions*

(i) *$wPw^* - P$ is in the Hilbert-Schmidt class*

(ii) *$\det_{wPw^* \wedge (1-P)} u(g) = 1$ for all g in G*

are both satisfied.

Proof. If the automorphism is implemented by a unitary in $\pi(\mathfrak{A}(\mathscr{K}, J)^G)''$, $\varphi^G_{wPw^*}$ is realized by a vector in the space of $\mathfrak{A}(\mathscr{K}, J)^G$ for φ^G_P and hence Theorem A(i) implies the conditions (i) and (ii).

Next we prove the converse. Let $\{\Omega, \pi(\cdot), \mathscr{H}\}$ be the triple associated to φ_P and $V(g)$ defined as in (2.1). There exists a implementer W of the Bogoliubov automorphism associated to w due to (i). As we assume $wu(g) = u(g)w$, $WV(g) = c(g)'V(g)W$ by the irreducibility of $\pi(\mathfrak{A}(\mathscr{K}, J))$ for some character $c(g)'$. The vector state of $W\Omega$ is φ_{wPw^*} and

$$c(g)'^{-1}WV(g)\Omega = c(g)'^{-1}W\Omega = V(g)W\Omega. \tag{2.25}$$

The right hand side of (2.25) can be computed in the same way as in the proof of Theorem A(i). Thus

$$c(g)'^{-1} = \det_{wPw^* \wedge (1-P)} u(g) = 1, \tag{2.26}$$

which implies $W \in \pi(\mathfrak{A}(\mathscr{K}, J)^G)''$. (Note that $\det_{P \wedge (1-wPw^*)} u(g) = \det_{wPw^* \wedge (1-P)} u(g)$ due to $[J, u(g)] = 0$.) Q.E.D.

3. PROOF FOR CASE (ii)

In this section we give a simple proof of Theorem A(ii). We use a Fock space realization of the cyclic representation for φ_S.

Let $S \in \mathscr{S}_G$ with $0 < S < 1$. Consider the Hilbert space $\hat{\mathscr{K}} = \mathscr{K} \oplus \mathscr{K}$ and the antiunitary involution $\hat{J} = J \oplus -J$. We introduce a unitary representation of G on $\hat{\mathscr{K}}$ defined by

$$\hat{u}(g) = u(g) \oplus u(g). \tag{3.1}$$

Define a basis projection R_S on $\hat{\mathscr{K}}$ by

$$R_S = \begin{bmatrix} S & \sqrt{(1-S)S} \\ \sqrt{(1-S)S} & 1-S \end{bmatrix}. \tag{3.2}$$

(The matrix notation refers to the first and second summand of $\hat{\mathscr{K}}$.) Consider the Fock state φ_{R_S} of $\mathfrak{A}(\hat{\mathscr{K}}, \hat{J})$ and its cyclic representation $\{\Omega, \pi(\cdot), \mathscr{H}\}$.

Now we regard $\mathfrak{A}(\mathscr{K}, J)$ as a subalgebra of $\mathfrak{A}(\hat{\mathscr{K}}, \hat{J})$ by

$$B(h) \in \mathfrak{A}(\mathscr{K}, J) \to B(h \oplus 0) \in \mathfrak{A}(\hat{\mathscr{K}}, \hat{J}). \tag{3.3}$$

Then

$$(3.4)\qquad (\Omega,\ \pi(Q)\Omega) = \varphi_S(Q) \quad \text{for } Q \in \mathfrak{A}(\mathscr{K},\ J).$$

Let T be a unitary on $\mathscr{H}$ defined by

$$(3.5a)\qquad T\Omega = \Omega,$$

$$(3.5b)\qquad T\pi(B(h \oplus f)) = -\pi(B(h \oplus f))T.$$

We introduce another representation π_{-1} of $\mathfrak{A}(\mathscr{K},\ J)$ by

$$(3.6)\qquad \pi_{-1}(B(h)) = \pi(B(0 \oplus h))T.$$

It is easy to see

$$(3.7)\qquad (\Omega,\ \pi_{-1}(Q)\Omega) = \varphi_{1-S}(Q),$$

and

$$(3.8)\qquad \pi(\mathfrak{A}(\mathscr{K},\ J))'' \subset \pi_{-1}(\mathfrak{A}(\mathscr{K},\ J))'.$$

Ω is known to be cyclic and separating for both $\pi(\mathfrak{A}(\mathscr{K},\ J))$ and $\pi_{-1}(\mathfrak{A}(\mathscr{K},\ J))$ if $0 < S < \mathbf{1}$. (See [1], Corollary 4.10.)

Proof of Theorem A(ii). By Lemma 1, if $\varphi^G_{S_1}$ and $\varphi^G_{S_2}$ are quasi-equivalent, $\sqrt{S_1} - \sqrt{S_2}$ is in the Hilbert-Schmidt class, so we prove the converse. If $\sqrt{S_1} - \sqrt{S_2}$ is in the Hilbert-Schmidt class, so is $R_{S_1} - R_{S_2}$ (see [1], Lemma 5.1). Then $\varphi_{R_{S_1}}$ and $\varphi_{R_{S_2}}$ yield unitarily equivalent representations of $\mathfrak{A}(\hat{\mathscr{K}},\ \hat{J})$. So we have a Hilbert space $\mathscr{H}$ and a representation π of $\mathfrak{A}(\hat{\mathscr{K}},\ \hat{J})$, in which $\varphi_{R_{S_1}}$ and $\varphi_{R_{S_2}}$ are vector states of $\mathscr{H}$ implemented by Ω_1 and Ω_2. (As we assume $0 < S_i < \mathbf{1}$ $(i = 1, 2)$ Ω_i is cyclic and separating for $\pi(\mathfrak{A}(\mathscr{K},\ J))$ as mentioned before.)

The cyclic representation of $\mathfrak{A}(\mathscr{K}, J)^G$ associated to $\varphi^G_{S_i}$ $(i = 1, 2)$ is $\pi(\mathfrak{A}(\mathscr{K},\ J)^G)$ restricted to the $\mathfrak{A}(\mathscr{K},\ J)^G$ invariant subspace $\mathscr{H}_i$:

$$(3.9)\qquad \mathscr{H}_i = [\pi(\mathfrak{A}(\mathscr{K},\ J))^G\Omega_i].$$

Let P_i be the projection to $\mathscr{H}_i$. Then $\{\pi(\cdot)\,|\,\mathscr{H}_1, \mathfrak{A}(\mathscr{K},\ J)^G\}$ and $\{\pi(\cdot)\,|\,\mathscr{H}_2, \mathfrak{A}(\mathscr{K},\ J)^G\}$ are quasi-equivalent if and only if $c(P_1) = c(P_2)$, where $c(P)$ denotes the central support of the projection P in the von Neumann algebra $\pi(\mathfrak{A}(\mathscr{K},\ J)^G)'$.

Actually $c(P_1) = c(P_2) = \mathbf{1}$, because $c(P_i)\mathscr{H}$ is invariant under $\pi(\mathfrak{A}(\mathscr{K},\ J)^G)'$ which contains $\pi_{-1}(\mathfrak{A}(\mathscr{K},\ J))$. But our assumption $0 < S_i < \mathbf{1}$ implies that Ω_i is cyclic for $\pi_{-1}(\mathfrak{A}(\mathscr{K},\ J))$, so $c(P_i)\mathscr{H} = \mathscr{H}$. Thus $\varphi^G_{S_1}$ and $\varphi^G_{S_2}$ are quasi-equivalent.

Q.E.D.

REMARK. We proved a criterion for quasi-equivalence of two quasifree states $\varphi_{S_1}^G$, $\varphi_{S_2}^G$ under the technical condition that S_1 and S_2 have no point spectrum at 0. This technical condition may be eliminated in examples of § 2. See [6], where the factoriality of states is also discussed.

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