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ON QUASIALGEBRAIC OPERATORS IN BANACH SPACES

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The concepts of quasialgebraic elements and capacity in a Banach algebra are due to Halmos [3]. Denote by $\mathscr{P}$ the set of all monic polynomials, i.e. polynomials with leading coefficient equal to 1. By $\deg p$ we denote the degree of $p \in \mathscr{P}$. Let a be an element of a Banach algebra A. Then the capacity of a is defined as $\operatorname{cap} a = \inf_{p \in \mathscr{P}} \|p(a)\|^{1/\deg p}$. An element $a \in A$ is called quasialgebraic if $\operatorname{cap} a = 0$.

Let K be a compact subset of the complex plane $\mathbf{C}$. Then the classical capacity of K may be defined as $\operatorname{cap} K = \inf_{p \in \mathscr{P}} \sup_{z \in K} |p(z)|^{1/\deg p}$. By [3], $\operatorname{cap} a = \operatorname{cap} \sigma(a)$ for every $a \in A$.

Clearly, every algebraic element is quasialgebraic (a is called algebraic if $p(a) = 0$ for some $p \in \mathscr{P}$).

Let X be a Banach space and T a bounded operator on X. The well-known theorem of Kaplansky [4] states that if T is locally quasialgebraic, i.e. for every $x \in X$ there exists $p_x \in \mathscr{P}$ such that $p_x(T)x = 0$ then T is algebraic.

Following [3], we call an operator T locally quasialgebraic if $\inf_{p \in \mathscr{P}} \|p(T)x\|^{1/\deg p} = 0$ for every $x \in X$. The aim of the present paper is to prove the version of Kaplansky's theorem for quasialgebraic operators: every locally quasi-algebraic operator is quasialgebraic. This gives an affirmative answer to a problem of Halmos [3].

Some other possible ways to define locally quasialgebraic operators were studied in [7] and [8]. We discuss some details at the end of this paper.

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X will denote a complex Banach space, $B(X)$ the algebra of all bounded operators on X. For $T \in B(X)$ we denote $\sigma_e(T)$, $\hat{\sigma}(T)$ and $\hat{\sigma}_e(T)$ the essential spectrum, the polynomially convex hull of $\sigma(T)$ and the polynomially convex hull of $\sigma_e(T)$, respectively. By ∂ we denote the topological boundary.

LEMMA 1. *Let* $T \in B(X)$. *Then* $\operatorname{cap} \sigma(T) = \operatorname{cap} \partial \hat{\sigma}_e(T)$.

Proof. It is well-known [2] that $\sigma(T)\setminus\sigma_e(T)$ contains only countably many isolated points in the unbounded component of $\mathbf{C}\setminus\sigma_e(T)$. Then

$$\operatorname{cap}\sigma(T) = \operatorname{cap}\hat{\sigma}(T) = \operatorname{cap}\hat{\sigma}_e(T) = \operatorname{cap}\partial\hat{\sigma}_e(T)$$

(see [5], p. 53—57).

LEMMA 2. *Let $T \in B(X)$, $\lambda \in \partial\hat{\sigma}_e(T)$. Let $Y \subset X$ be a closed subspace of finite codimension. Then*

$$\inf_{\substack{x \in Y \\ \|x\|=1}} \|(T-\lambda)x\| = 0.$$

Proof. The assertion follows from [1].

PROPOSITION 3. *Let $T \in B(X)$, $\lambda \in \partial\hat{\sigma}_e(T)$. Let $E \subset X$ be a finite dimensional subspace of X and let $\varepsilon_1, \varepsilon_2 > 0$. Then there exists $y \in X$, $\|y\| = 1$ such that*

(i) $\|(T-\lambda)y\| \leqslant \varepsilon_1$

(ii) $\|x+\alpha y\| \geqslant \|x\|(1-\varepsilon_2)$ *for every* $x \in E, \alpha \in \mathbf{C}$

(iii) $\|x+\alpha y\| \geqslant |\alpha|(1-\varepsilon_2)/2$ *for every* $x \in E$, $\alpha \in \mathbf{C}$.

Proof. Assume $\varepsilon_2 < 1$. The set $\{x \in E, \|x\| = 1\}$ is a compact metric space so we can find a finite ε_2-net $M \subset E$, i.e. M is a finite set such that $x \in E$, $\|x\| = 1$ implies $\operatorname{dist}\{x, M\} \leqslant \varepsilon_2$. There exists a linear functional $f_m \in X'$, $\|f_m\| = 1, f_m(m) = 1$ for every $m \in M$. Put $Y = \bigcap_{m \in M} \operatorname{Ker} f_m$. By Lemma 2, there exists $y \in Y$, $\|y\| = 1$ and $\|(T-\lambda)y\| \leqslant \varepsilon_1$. Let $x \in E$, $x \neq 0$ and let $\alpha \in \mathbf{C}$. Then there exists $m \in M$, $\left\|\frac{x}{\|x\|} - m\right\| \leqslant \varepsilon_2$. We have

$$\|x+\alpha y\| \geqslant |f_m(x+\alpha y)| = |f_m(x)| =$$

$$= \|x\| \left| f_m\left(\frac{x}{\|x\|} - m\right) + f_m(m)\right| \geqslant \|x\|(1-\varepsilon_2);$$

hence (ii) holds. Further, $\dfrac{\|x+\alpha y\|}{1-\varepsilon_2} \geqslant \|x\|$ and

$$\|x+\alpha y\| \geqslant \|\alpha y\| - \|x\| = |\alpha| - \|x\|.$$

If we add these two inequalities we get

$$\|x+\alpha y\| \frac{2-\varepsilon_2}{1-\varepsilon_2} \geqslant |\alpha|,$$

$$\|x+\alpha y\| \geqslant |\alpha|(1-\varepsilon_2)/2.$$

Denote by $\mathscr{P}_n$ the set of all monic polynomials of degree n and let $\mathscr{P} = \bigcup_{n=1}^{\infty} \mathscr{P}_n$. Similarly, $\mathscr{P}'_n$ denotes the set of all monic polynomials of degree n with roots $\alpha_1, \ldots, \alpha_n$, $|\alpha_i| \leqslant 2$ $(i = 1, \ldots, n)$ and $\mathscr{P}' = \bigcup_{n=1}^{\infty} \mathscr{P}'_n$.

LEMMA 4. *Let* $p \in \mathscr{P}'_n$ *and let* $z_1, z_2 \in \mathbf{C}$, $|z_1|, |z_2| \leqslant 1$. *Then* $|p(z_1) - p(z_2)| \leqslant \leqslant n\, 3^{n-1}|z_1 - z_2|$.

Proof. Let $p(z) = (z - \alpha_1) \ldots (z - \alpha_n)$, $|\alpha_i| \leqslant 2$ $(i = 1, \ldots, n)$. Then

$$|p'(z)| = \left|\sum_{i=1}^{n} \frac{p(z)}{z - \alpha_i}\right| \leqslant n\, 3^{n-1} \quad \text{for every } z \in \mathbf{C},\ |z| \leqslant 1.$$

Hence $|p(z_1) - p(z_2)| \leqslant |z_1 - z_2|\, n3^{n-1}$.

The following lemma shows that we may restrict ourselves to the (compact) set of polynomials $\mathscr{P}'_n$.

LEMMA 5. *Let* $T \in B(X)$, $\|T\| \leqslant 1$, $x \in X$. *Let n be a positive integer. Then* $\inf_{p \in \mathscr{P}_n} \|p(T)x\| = \inf_{p \in \mathscr{P}'_n} \|p(T)x\|$.

Proof. Let $p \in \mathscr{P}_n$, $p(z) = (z - \alpha_1) \ldots (z - \alpha_n)$. Suppose $|\alpha_1| > 2$. Denote $y = (T - \alpha_2) \ldots (T - \alpha_n)x$. Then

$$\|p(T)x\| = \|(T - \alpha_1)y\| \geqslant |\alpha_1|\, \|y\| - \|Ty\| \geqslant 2\|y\| - \|y\| =$$

$$= \|y\| \geqslant \|Ty\| = \|T(T - \alpha_2) \ldots (T - \alpha_n)x\| = \|p_1(T)x\|$$

where $p_1(z) = z(z - \alpha_2) \ldots (z - \alpha_n)$.

If we replace all terms $z - \alpha_i$ with $|\alpha_i| > 2$ by the term z we get a polynomia $q(z) \in \mathscr{P}'_n$ such that $\|q(T)x\| \leqslant \|p(T)x\|$. Hence

$$\inf_{p \in \mathscr{P}_n} \|p(T)x\| \geqslant \inf_{p \in \mathscr{P}'_n} \|p(T)x\|$$

and the second inequality is trivial.

LEMMA 6. *Let* $T \in B(X)$, $\|T\| = 1$, $2/3 \geqslant \varepsilon > 0$. *Then there exists an ε-net in* $\partial\hat{\sigma}_e(T)$ *with at most* $25/\varepsilon^2$ *points.*

Proof. Put $S_0 = \{k_1\delta + ik_2\delta,\ k_1, k_2 = 0, \pm 1, \ldots, \pm [(1 + \delta)/\delta]\}$ where $\delta = \varepsilon/\sqrt{2}$. Then

$$\text{card } S_0 = \left(2\left[\frac{1+\delta}{\delta}\right] + 1\right)^2 \leqslant (3 + 2/\delta)^2 = (3 + 2\sqrt{2}/\varepsilon)^2 \leqslant 25/\varepsilon^2.$$

Clearly, $\operatorname{dist}\{z, S_0\} \leqslant \varepsilon/2$ for every $z \in \mathbf{C}$, $|z| \leqslant 1$. Let $S_1 = \{s \in S_0,$ $\operatorname{dist}\{s, \partial\hat{\sigma}_e(T)\} \leqslant \varepsilon/2\}$. For every $s \in S_1$ choose $t_s \in \partial\hat{\sigma}_e(T)$ with $|s - t_s| \leqslant \varepsilon/2$. Put $S_2 = \{t_s, s \in S_1\}$. Then $\operatorname{card} S_2 \leqslant 25/\varepsilon^2$ and $\operatorname{dist}\{\lambda, S_2\} \leqslant \varepsilon$ for every $\lambda \in \partial\hat{\sigma}_e(T)$.

LEMMA 7. *Let $T \in B(X)$, $\|T\| = 1$, let $\lambda \in \mathbf{C}$, $|\lambda| \leqslant 1$ and let $p \in \mathscr{P}'_n$. Then*

$$\|(p(T) - p(\lambda))x\| \leqslant 3^n n \|(T - \lambda)x\|.$$

Proof. For $i = 1, \ldots, n$ we have

$$\|(T^i - \lambda^i)x\| = \|(T^{i-1} + \lambda T^{i-2} + \ldots + \lambda^{i-1})(T - \lambda)x\| \leqslant$$

$$\leqslant i\|(T - \lambda)x\| \leqslant n\|(T - \lambda)x\|.$$

Let $p \in \mathscr{P}'_n$. Then $p(z) = \sum_{i=0}^{n} \alpha_i z^i$ with $\sum_{i=0}^{n} |\alpha_i| \leqslant 3^n$. Hence

$$\|(p(T) - p(\lambda))x\| = \left\| \sum_{i=0}^{n} \alpha_i (T^i - \lambda^i)x \right\| \leqslant 3^n n \|(T - \lambda)x\|.$$

PROPOSITION 8. *Let $T \in B(X)$ be such that $\|T\| = 1$, $\operatorname{cap}\sigma(T) = r$ and let n be a positive integer. Let $E \subset X$ be a finite dimensional subspace. Let also $r^n/56 > \delta > 0$. Then there exists $y \in X$, $\|y\| = 1$, such that*

(i) $\|e + p(T)y\| \geqslant \|e\|(1 - \delta) - 3\delta \quad \left(e \in E, p \in \bigcup_{j=1}^{n} \mathscr{P}'_j\right)$

(ii) $\|q(T)y\| \geqslant s^n \quad (q \in \mathscr{P}'_n)$

where $s = r^3/450$ (note that s does not depend on n).

Proof. Put $\varepsilon = 2r^n/n3^n$. By Lemma 6 there is a finite ε-net $L = \{\lambda_1, \ldots, \lambda_k\}$ in $\partial\hat{\sigma}_e(T)$ with k points, $k \leqslant \dfrac{25}{\varepsilon^2} = \dfrac{25}{4}\dfrac{n^2 3^{2n}}{r^{2n}}$.

By Proposition 3 we can find inductively points $y_1, \ldots, y_k \in X$ such that $\|y_1\| = \ldots = \|y_k\| = 1$,

(1) $$\|(T - \lambda_i)y_i\| \leqslant \frac{\delta}{2^i n 3^n}$$

(2) $$\|x + \alpha y_i\| \geqslant \|x\|(1 - \delta)^{1/k} \quad (x \in E \vee \{y_1, \ldots, y_{i-1}\},\ \alpha \in \mathbf{C})$$

(3) $$\|x + \alpha y_i\| \geqslant (1/2)\,|\alpha|(1 - \delta)^{1/k} \quad (x \in E \vee \{y_1, \ldots, y_{i-1}\},\ \alpha \in \mathbf{C}).$$

Let $e \in E, \alpha_1, \ldots, \alpha_k \in \mathbf{C}, j \in \{1, \ldots, k\}$. Then

$$\left\| e + \sum_{i=1}^{k} \alpha_i y_i \right\| \geqslant \left\| e + \sum_{i=1}^{k-1} \alpha_i y_i \right\| (1-\delta)^{1/k} \geqslant \ldots$$

$$\ldots \geqslant \left\| e + \sum_{i=1}^{j} \alpha_i y_i \right\| (1-\delta)^{(k-j)/k} \geqslant \frac{1}{2} |\alpha_j| (1-\delta)^{(k-j+1)/k} \geqslant \frac{1}{2} |\alpha_j| (1-\delta).$$

Thus

$$\left\| e + \sum_{i=1}^{k} \alpha_i y_i \right\| \geqslant \frac{1}{2}(1-\delta) \max\{|\alpha_i|, i = 1, \ldots, k\}. \tag{4}$$

Similarly,

$$\left\| e + \sum_{i=1}^{k} \alpha_i y_i \right\| \geqslant (1-\delta)\|e\|. \tag{5}$$

Put $y = \left(\sum_{i=1}^{k} y_i \right) \Big/ \left\| \sum_{i=1}^{k} y_i \right\|$. Clearly, $\|y\| = 1$ (the vectors $y_1, \ldots, y_k$ are linearly independent by (4)).

Let $p \in \bigcup_{j=1}^{n} \mathscr{P}'_j$. Then by Lemma 7

$$\|p(T)y_i - p(\lambda_i)y_i\| \leqslant \delta\, 2^{-i} \quad (i = 1, \ldots, k),$$

so

$$\left\| \sum_{i=1}^{k} (p(T) - p(\lambda_i)) y_i \right\| \leqslant \delta. \tag{6}$$

Let $e \in E$, $p \in \bigcup_{j=1}^{n} \mathscr{P}'_j$. Then by (6), (5) and (4)

$$\|e + p(T)y\| = \left\| e + \frac{\sum_{i=1}^{k} p(T)y_i}{\left\| \sum_{i=1}^{k} y_i \right\|} \right\| \geqslant \left\| e + \frac{\sum_{i=1}^{k} p(\lambda_i)y_i}{\left\| \sum_{j=1}^{k} y_i \right\|} \right\| -$$

$$- \frac{\delta}{\left\| \sum_{i=1}^{k} y_i \right\|} \geqslant \|e\|(1-\delta) - \frac{2\delta}{1-\delta} \geqslant \|e\|(1-\delta) - 3\delta,$$

hence (i).

Let $q \in \mathscr{P}_n'$. Then by Lemma 4

$$\max_{z \in L} q(z) \geqslant \max_{z \in \partial \hat{\sigma}_e(T)} |q(z)| - n\,3^{n-1}\varepsilon \geqslant r^n/3. \tag{7}$$

Further,

$$\|q(T)y\| = \frac{1}{\left\|\sum_{i=1}^{k} y_i\right\|}\left\|\sum_{i=1}^{k} q(T)y_i\right\| \geqslant$$

$$\geqslant k^{-1}\left\|\sum_{i=1}^{k} q(T)y_i\right\| \geqslant k^{-1}\left(\left\|\sum_{i=1}^{k} q(\lambda_i)y_i\right\| - \delta\right).$$

Using (4) and (7) we get

$$\|q(T)y\| \geqslant k^{-1}\left(\frac{1-\delta}{2}\max_{\lambda_i \in L} |q(\lambda_i)| - \delta\right) \geqslant$$

$$\geqslant k^{-1}\left(\frac{(1-\delta)r^n}{6} - \delta\right) \geqslant \frac{r^n}{8k} \geqslant \frac{r^{3n}}{50\,n^2 3^{2n}} \geqslant s^n$$

where $s = r^3/450$.

REMARK. Let y be the vector from the previous theorem. If we replace E by the subspace $E^{(1)} = E \vee \{y, Ty, \ldots, T^n y\}$, Proposition 8 proves the existence of a vector $y^{(1)} \in X$ such that

a) $\|e + p(T)y^{(1)}\| \geqslant \|e\|(1-\delta) - 3\delta \quad \left(e \in E^{(1)},\ p \in \bigcup_{j=1}^{n} \mathscr{P}_j'\right)$

b) $\|q(T)y^{(1)}\| \geqslant s^n \quad (q \in \mathscr{P}_n')$.

Using this construction repeatedly we get:

PROPOSITION 9. *With the assumptions of Proposition* 8, *there exists a sequence* $y^{(1)}, y^{(2)}, \ldots \in X$, $\|y^{(1)}\| = \|y^{(2)}\| = \ldots = 1$ *such that*

i) $\|e + p(T)y^{(i)}\| \geqslant \|e\|(1-\delta) - 3\delta \quad \left(e \in E \vee \bigvee_{i'=1}^{i-1} \bigvee_{j=0}^{n} T^j y^{(i')},\ p \in \bigcup_{j=1}^{n} \mathscr{P}_j',\ i = 1, 2, \ldots\right)$

ii) $\|q(T)y^{(i)}\| \geqslant s^n \quad (q \in \mathscr{P}_n',\ i = 1, 2, \ldots)$.

PROPOSITION 10. *Let* $T \in B(X)$, $\|T\| = 1$, $\operatorname{cap} \sigma(T) = r$, $s = r^3/450$, $s/2 < s_1 < s_2$, $x \in X$, *n a positive integer. Let* $\|p(T)x\| \geqslant s_1^{\deg p}$ *for every* $p \in \bigcup_{j=1}^{n} \mathscr{P}'_j$. *Then there exists* $z \in X$, $\|z - x\| \leqslant 2^{-n}$ *and*

$$\|p(T)z\| \geqslant s_2^{\deg p} \quad \left(p \in \bigcup_{j=1}^{n+1} \mathscr{P}'_j\right).$$

Proof. Put $E = \{x, Tx, \ldots, T^n x\}$ and choose δ sufficiently small $\left(\delta \leqslant \min_{j=1,\ldots,n+1} \frac{s_1^j - s_2^j}{3 + s_1^j}, \text{i.e. } s_1^j(1-\delta) - 3\delta \geqslant s_2^j \ (j=1, 2, \ldots, n+1)\right)$. Let $y^{(1)}, y^{(2)}, \ldots$ be the sequence the existence of which was proved in Proposition 9. Let $y^{(i)}$ be an arbitrary member of this sequence and let $p \in \bigcup_{j=1}^{n} \mathscr{P}'_j$. Put $z^{(i)} = x + y^{(i)} 2^{-n}$. Then $\|z^{(i)} - x\| \leqslant 2^{-n}$ and

$$\|p(T)z^{(i)}\| = \|p(T)x + 2^{-n} p(T)y^{(i)}\| \geqslant \|p(T)x\|(1-\delta) - 3\delta \geqslant$$
$$\geqslant s_1^{\deg p}(1-\delta) - 3\delta \geqslant s_2^{\deg p}.$$

Suppose there exists $i \in \{1, 2, \ldots\}$ such that $\inf_{q \in \mathscr{P}'_{n-1}} \|q(T)z^{(i)}\| \geqslant s_2^{n+1}$. Then $z = z^{(i)}$ satisfies all of the conditions of Proposition 10. Suppose on the contrary that for every $i = 1, 2, \ldots$ there exists $q_i \in \mathscr{P}'_{n+1}$ such that $\|q_i(T)z^{(i)}\| < s_2^{n+1}$. The set $\{q(T), q \in \mathscr{P}'_{n+1}\}$ is compact. Therefore, there exist integers i, j, $i < j$, such that

$$\|q_i(T) - q_j(T)\| \leqslant \frac{\delta}{\|x\| + 2^{-n}}.$$

We have

$$\begin{aligned} \|q_i(T)z^{(j)}\| &\leqslant \|q_j(T)z^{(j)}\| + \|(q_i - q_j)(T)z^{(j)}\| \leqslant \\ &\leqslant s_2^{n+1} + \frac{\delta}{\|x\| + 2^{-n}} \|z^{(j)}\| \leqslant s_2^{n+1} + \delta. \end{aligned} \tag{8}$$

Further (from Proposition 9(i))

$$\begin{aligned} \|q_i(T)(z^{(i)} - z^{(j)})\| &= 2^{-n}\|q_i(T)y^{(i)} - q_i(T)y^{(j)}\| \geqslant \\ &\geqslant 2^{-n}[\|q_i(T)y^{(i)}\|(1-\delta) - 3\delta] \geqslant 2^{-n}[s^{n+1}(1-\delta) - 3\delta] \geqslant \\ &\geqslant 2^{-n}[2^{n+1}s_1^{n+1}(1-\delta) - 3\delta] \geqslant 2s_2^{n+1} + 3\delta. \end{aligned} \tag{9}$$

On the other hand

$$\|q_i(T)(z^{(i)} - z^{(j)})\| \leqslant \|q_i(T)\, z^{(i)}\| + \|q_i(T)z^{(j)}\| \leqslant$$
$$\leqslant s_2^{n+1} + s_2^{n+1} + \delta = 2s_2^{n+1} + \delta,$$

a contradiction.

THEOREM 11. *Let* $T \in B(X)$, $\|T\| = 1$, $\operatorname{cap}\sigma(T) = r$, $s = r^3/450$. *Then there exists* $x \in X$ *such that*

$$\inf_{p \in \mathscr{P}} \|p(T)x\|^{1/\deg p} \geqslant \frac{s}{3}.$$

Proof. Choose a sequence $s_1, s_2, \ldots$ of positive numbers such that $s/2 > > s_1 > s_2 > \ldots > s/3$. By Proposition 8 there exists $x_1 \in X$, $\|x_1\| = 1$ such that $\|p(T)x_1\| \geqslant s_1$ for every $p \in \mathscr{P}'_1$. Using Proposition 10 repeatedly, we get a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X, $\|x_n - x_{n+1}\| \leqslant 2^{-n}$, such that $\|p(T)x_n\| \geqslant s_n^{\deg p}$ $(p \in \bigcup_{j=1}^{n} \mathscr{P}'_j)$.

Let x be the limit of the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$. We have

$$\|p(T)x\| = \lim_{n\to\infty} \|p(T)x_n\| \geqslant \lim_{n\to\infty} s_n^{\deg p} \geqslant (s/3)^{\deg p}$$

for every $p \in \mathscr{P}'$. Hence

$$\inf_{p \in \mathscr{P}} \|p(T)x\|^{1/\deg p} = \inf_{p \in \mathscr{P}'} \|p(T)x\|^{1/\deg p} \geqslant s/3.$$

COROLLARY. *An operator* $T \in B(X)$ *is locally quasialgebraic if and only if it is quasialgebraic.*

REMARK. The estimates in Propositions 8 and 10 are not the best possible. They can be essentially improved, especially in case of an operator on a Hilbert space. We do not know, however, if the following is true:

If we denote $\operatorname{cap}(T, x) = \inf_{p \in \mathscr{P}} \|p(T)x\|^{1/\deg p}$ *(we may call this number the local capacity of* T *at the point* x*), is it true that* $\sup_{\substack{x \in X \\ x \neq 0}} \operatorname{cap}(T, x) = \operatorname{cap} T$?

CONCLUDING REMARKS. The definition of locally quasialgebraic operators we use in this paper is not the only possible. Another possibility was suggested, already in the original paper [3]: We may call T locally quasialgebraic if $\lim_{n\to\infty} \operatorname{cap}_n(T, x)^{1/n}$ exists and equals 0 for every $x \in X$ where $\operatorname{cap}_n(T, x) = \inf_{p \in \mathscr{P}_n} \|p(T)x\|$ (in general this limit does not exist, so it is better to say equivalently

$\limsup_{n\to\infty} \operatorname{cap}_n(T, x)^{1/n} = 0$). The following characterization of quasialgebraic elements in a Banach algebra (see [3]) provides other possibilities. The following statements are equivalent:

1) $a \in A$ is quasialgebraic.

2) $\inf(r(p(a)))^{1/\deg p} = 0$ where $r(x)$ denotes the spectral radius of $x \in A$.

3) $\operatorname{cap} \sigma(a) = 0$.

For local version of Condition 2 we can use the local spectral radius $r(T, x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$, then define the n-th spectral capacity $\operatorname{cap}'_n(T, x) = \inf_{p\in\mathscr{P}_n} r(p(T), x)$ and call an operator locally quasialgebraic if $\limsup \operatorname{cap}_n(T, x)^{1/n} = 0$ for eveey $x \in X$. This definition was used by Vasilescu in [7].

Instead of "lim sup" in the previous definition we can again use "inf".

The local version of Condition 3 was studied also in [7] where instead of $\sigma(T)$ one can take the local spectrum $\sigma_T(x)$ (or $\gamma_T(x)$; for details see [6]).

In [8] P. Vrbová, proved the existence of a large set of elements $x \in X$ with extremal local spectrum $\sigma_T(x) = \sigma(T)$. As a consequence she proved the version of Kaplansky's theorem for this definition of locally quasialgebraic elements.

All these possible definitions of locally quasialgebraic operators are equivalent:

THEOREM 12. *Let X be a Banach space and $T \in B(X)$. Then the following statements are equivalent:*

1) *T is quasialgebraic.*

2) $\inf_{p\in\mathscr{P}} \|p(T)x\|^{1/\deg p} = 0$ *for every* $x \in X$.

3) $\limsup_{n\to\infty} \operatorname{cap}_n(T, x)^{1/n} = 0, \quad (x \in X)$.

4) $\inf_{p\in\mathscr{P}} r(p(T), x)^{1/\deg p} = 0, \quad (x \in X)$.

5) $\limsup_{n\to\infty} \operatorname{cap}'_n(T, x)^{1/n} = 0 \quad (x \in X)$.

6) $\operatorname{cap} \sigma_T(x) = 0 \quad (x \in X)$.

7) $\operatorname{cap} \gamma_T(x) = 0 \quad (x \in X)$.

Proof. 1) implies easily all the remaining conditions. The implications 6) $\Rightarrow$ 1) and 7) $\Rightarrow$ 1) were proved in [8]. This together with Theorem 3 of [7] gives the implication 5) $\Rightarrow$ 1). The remaining implications 2) $\Rightarrow$ 1), 3) $\Rightarrow$ 1) and 4) $\Rightarrow$ 1) follow from Theorem 11 of the present paper.

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