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THE STRUCTURE OF LINEAR FUNCTIONALS ON SPACES OF OPERATORS

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1. INTRODUCTION

Throughout this paper $\mathscr{X}$ will denote a complex Banach space, $\mathscr{X}^*$ will denote the dual of $\mathscr{X}$, and $\mathscr{L}(\mathscr{X})$ will denote the algebra of bounded linear transformations on $\mathscr{X}$. We use τ to denote a topology on $\mathscr{L}(\mathscr{X})$ which makes $\mathscr{L}(\mathscr{X})$ into a locally convex topological vector space. $\mathscr{S}$ will denote a linear manifold in $\mathscr{L}(\mathscr{X})$. A *form* on $\mathscr{S}$ is a linear functional on $\mathscr{S}$. For $x \in \mathscr{X}$, $f \in \mathscr{X}^*$, $\omega_{x,f}$ will denote the form defined by $\omega_{x,f}(T) = f(Tx)$, $T \in \mathscr{L}(\mathscr{X})$. An *elementary form* on $\mathscr{S}$ is the restriction to $\mathscr{S}$ of some form $\omega_{x,f}$. We say $\mathscr{S}$ is *τ-elementary* if every τ-continuous form on $\mathscr{S}$ is an elementary form. In this paper, we give necessary and sufficient conditions for $\mathscr{S}$ to be τ-elementary. We use this result to obtain some invariant subspace theorems and a dilation theory for τ-elementary algebras on a Hilbert space.

2. ELEMENTARY FORMS AND REFLEXIVITY

We recall some standard terminology. The *weak topology* (also called the weak operator topology) on $\mathscr{L}(\mathscr{X})$ is the topology induced by the elementary forms on $\mathscr{L}(\mathscr{X})$. A subspace of $\mathscr{X}$ is a norm closed linear manifold in $\mathscr{X}$. For $\mathscr{T}$ a subset of $\mathscr{L}(\mathscr{X})$, $\operatorname{Lat}\mathscr{T}$ denotes $\{\mathscr{M} \subset \mathscr{X} : \mathscr{M}$ is a subspace and $T\mathscr{M} \subset \mathscr{M}$ for all $T \in \mathscr{T}\}$. $\operatorname{Alg}\operatorname{Lat}\mathscr{T}$ denotes $\{S \in \mathscr{L}(\mathscr{X}) : S\mathscr{M} \subset \mathscr{M}$ for all $\mathscr{M} \in \operatorname{Lat}\mathscr{T}\}$. For $x \in \mathscr{X}$, $[\mathscr{T}x]$ denotes the norm closed linear span of $\{Tx : T \in \mathscr{T}\}$. The following pertinent concept is from [3]. $\operatorname{Ref}\mathscr{T}$ denotes $\{S \in \mathscr{L}(\mathscr{X}) : Sx \in [\mathscr{T}x]$ for all $x \in \mathscr{X}\}$. $\mathscr{T}$ is called *reflexive* if $\operatorname{Ref}\mathscr{T} = \mathscr{T}$. We list some easily established properties of $\operatorname{Ref}\mathscr{T}$.

PROPOSITION 1. *Let $\mathscr{T}$, $\mathscr{U}$ and $\mathscr{V}$ be subsets of $\mathscr{L}(\mathscr{X})$.*

(a) $\mathscr{T} \subset \operatorname{Ref}\mathscr{T}$.

(b) $\operatorname{Ref}\mathscr{T}$ *is a weakly closed linear manifold in* $\mathscr{L}(\mathscr{X})$.

(c) $\operatorname{Ref}\mathscr{T}$ *is reflexive.*

(d) *If* $\mathscr{T} \subset \mathscr{U}$, *then* $\operatorname{Ref}\mathscr{T} \subset \operatorname{Ref}\mathscr{U}$.

(e) *The intersection of any family of reflexive sets is reflexive.*

(f) *If* $\mathscr{T} = \mathscr{U} \cap \mathscr{V}$ *and* $\mathscr{V}$ *is reflexive, then* $(\operatorname{Ref}\mathscr{T}) \cap \mathscr{U} = \mathscr{T}$.

(g) *If* $\mathscr{A}$ *is a unital subalgebra of* $\mathscr{L}(\mathscr{X})$, *then* $\operatorname{Ref}\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$.

Proof. We prove only (f). By (a), $\mathscr{T} \subset (\operatorname{Ref}\mathscr{T}) \cap \mathscr{U}$. By (d), $\operatorname{Ref}\mathscr{T} = \operatorname{Ref}(\mathscr{U} \cap \mathscr{V}) \subset (\operatorname{Ref}\mathscr{U}) \cap (\operatorname{Ref}\mathscr{V})$. Thus $(\operatorname{Ref}\mathscr{T}) \cap \mathscr{U} \subset (\operatorname{Ref}\mathscr{U}) \cap (\operatorname{Ref}\mathscr{V}) \cap \mathscr{U} = \mathscr{U} \cap \mathscr{V} = \mathscr{T}$, by (a) and the reflexivity of $\mathscr{V}$.

The following proposition characterizes the reflexive subsets of $\mathscr{L}(\mathscr{X})$. For a form φ on $\mathscr{S}$, $\operatorname{Ker}\varphi$, the *kernel* of φ, denotes $\{S \in \mathscr{S} : \varphi(S) = 0\}$. For $\mathscr{E}$ a subset of $\mathscr{X}$, $\mathscr{E}^{a}$ denotes $\{f \in \mathscr{X}^* : f(x) = 0 \text{ for all } x \text{ in } \mathscr{X}\}$.

PROPOSITION 2. *Let* $\mathscr{T}$ *be a subset of* $\mathscr{L}(\mathscr{X})$.

(a) *For* $x \in \mathscr{X}$, $f \in \mathscr{X}^*$, $\operatorname{Ker}\omega_{x,f}$ *is reflexive.*

(b) $\operatorname{Ref}\mathscr{T} = \bigcap \operatorname{Ker}\omega_{x,f}$ *where the intersection is over all* $\omega_{x,f}$ *such that* $\mathscr{T} \subset \operatorname{Ker}\omega_{x,f}$.

(c) $\mathscr{T}$ *is reflexive if and only if* $\mathscr{T}$ *is the intersection of kernels of elementary forms on* $\mathscr{L}(\mathscr{X})$.

Proof. (a) is straightforward. (c) follows from (a), (b) and Proposition 1(e). We prove (b). Suppose $\mathscr{T} \subset \operatorname{Ker}\omega_{x,f}$. Then $f \in [\mathscr{T}x]^{a}$. If $S \in \operatorname{Ref}\mathscr{T}$, then $Sx \in [\mathscr{T}x]$. So $0 = f(Sx) = \omega_{x,f}(S)$. We have $\operatorname{Ref}\mathscr{T} \subset \bigcap \operatorname{Ker}\omega_{x,f}$. Now suppose $S \in \bigcap \operatorname{Ker}\omega_{x,f}$. Fix $x \in \mathscr{X}$. If $f \in [\mathscr{T}x]^{a}$, then $\mathscr{T} \subset \operatorname{Ker}\omega_{x,f}$. So $0 = \omega_{x,f}(S) = f(Sx)$. We have $[\mathscr{T}x]^{a} \subset \{Sx\}^{a}$, and so $Sx \in [\mathscr{T}x]$. Since x was arbitrary, $S \in \operatorname{Ref}\mathscr{T}$.

The next result gives necessary and sufficient conditions for a form on $\mathscr{S}$ to be an elementary form.

LEMMA 3. *Let* φ *be a form on* $\mathscr{S}$. *The following are equivalent.*

(a) φ *is an elementary form.*

(b) $\operatorname{Ker}\varphi = (\operatorname{Ref}\operatorname{Ker}\varphi) \cap \mathscr{S}$.

(c) *There is a vector* $x \in \mathscr{X}$ *such that* $|\varphi(S)| \leqslant \|Sx\|$ *for all* $S \in \mathscr{S}$.

Proof. The lemma is trivial if $\varphi = 0$. So assume $\varphi \neq 0$. We show (a) implies (b). By hypothesis there exist $x \in \mathscr{X}$, $f \in \mathscr{X}^*$ such that $\varphi(S) = f(Sx)$, $S \in \mathscr{S}$. Clearly $\operatorname{Ker}\varphi = (\operatorname{Ker}\omega_{x,f}) \cap \mathscr{S}$, and $\operatorname{Ker}\omega_{x,f}$ is reflexive by Proposition 2(a). Thus, by Proposition 1(f), $(\operatorname{Ref}\operatorname{Ker}\varphi) \cap \mathscr{S} = \operatorname{Ker}\varphi$.

To show (b) implies (c), we choose $T \in \mathscr{S}$ such that $\varphi(T) = 1$. By the hypothesis, $T \notin \operatorname{Ref}\operatorname{Ker}\varphi$. So there is an $x \in \mathscr{X}$ such that $1 = \varphi(T) \leqslant \|(T - A)x\|$ for all $A \in \operatorname{Ker}\varphi$. For $S \in \mathscr{S}$, $S = \varphi(S)T - (\varphi(S)T - S)$ and $\varphi(S)T - S \in \operatorname{Ker}\varphi$. Thus

$$|\varphi(S)| = |\varphi(S)|\varphi(T) \leqslant \|(\varphi(S)T - (\varphi(T) - S))x\| = \|Sx\|.$$

Finally, we show (c) implies (a). Let $f(Sx) = \varphi(S)$. Then $|f(Sx)| \leq \|Sx\|$ for $S \in \mathscr{S}$, so f is well-defined. Furthermore, f extends to a bounded linear functional on $\mathscr{X}$ by the Hahn-Banach theorem. So there is a $g \in \mathscr{X}^*$ such that

$$\varphi(S) = f(Sx) = g(Sx) = \omega_{x,g}(S), \quad S \in \mathscr{S}.$$

We now give necessary and sufficient conditions for $\mathscr{S}$ to be τ-elementary. This theorem sharpens [3, Theorem 3.2].

THEOREM 4. *$\mathscr{S}$ is τ-elementary if and only if for every relatively τ-closed linear manifold $\mathscr{R}$ in $\mathscr{S}$, $(\mathrm{Ref}\,\mathscr{R}) \cap \mathscr{S} = \mathscr{R}$.*

Proof. We first prove sufficiency. If φ is a τ-continuous form on $\mathscr{S}$, then $\mathrm{Ker}\,\varphi$ is a relatively τ-closed linear manifold in $\mathscr{S}$. By hypothesis, $\mathrm{Ker}\,\varphi = (\mathrm{Ref\,Ker}\,\varphi) \cap \mathscr{S}$. By Lemma 3, φ is an elementary form.

To prove necessity, let $\mathscr{R}$ be a relatively τ-closed linear manifold in $\mathscr{S}$. By the geometric Hahn-Banach theorem, $\mathscr{R} = \bigcap \mathrm{Ker}\,\varphi$, where the intersection is over all τ-continuous forms φ such that $\mathscr{R} \subset \mathrm{Ker}\,\varphi$. But each φ is an elementary form, so $\mathrm{Ker}\,\varphi = (\mathrm{Ref\,Ker}\,\varphi) \cap \mathscr{S}$ by Lemma 3. Thus, $\mathscr{R} = \bigcap((\mathrm{Ref\,Ker}\,\varphi) \cap \cap \mathscr{S}) = (\bigcap \mathrm{Ref\,Ker}\,\varphi) \cap \mathscr{S}$. By Proposition 1(c), $\mathrm{Ref\,Ker}\,\varphi$ is reflexive for each φ. By Proposition 1(e), $\bigcap \mathrm{Ref\,Ker}\,\varphi$ is reflexive. Thus by Proposition 1(f), $(\mathrm{Ref}\,\mathscr{R}) \cap \mathscr{S} = \mathscr{R}$.

3. ELEMENTARY FORMS AND INVARIANT SUBSPACES

If $\mathscr{T}$ is a subset of $\mathscr{L}(\mathscr{X})$, then a subspace $\mathscr{M}$ of $\mathscr{X}$ is a *non-trivial invariant subspace* (n.i.s.) for $\mathscr{T}$ if $\mathscr{M} \in \mathrm{Lat}\,\mathscr{T}$ and $\{0\} \neq \mathscr{M} \neq \mathscr{X}$. $\mathscr{M}$ is a *non-trivial hyperinvariant subspace* (n.h.s.) for $\mathscr{T}$ if $\mathscr{M}$ is a n.i.s. for $\mathscr{T}' = \{S \in \mathscr{L}(\mathscr{X}) : ST = TS$ for all $T \in \mathscr{T}\}$. $\mathscr{A}$ will denote a unital subalgebra of $\mathscr{L}(\mathscr{X})$. It is natural to ask if every τ-elementary algebra $\mathscr{A}$ has a n.i.s. The following results give some partial answers.

PROPOSITION 5. *If $\mathscr{A}$ is τ-elementary and $\mathscr{J}$ is a proper relatively τ-closed left ideal of $\mathscr{A}$, then $\mathscr{A}$ has* a n.i.s. *or* a n.h.s.

Proof. Since $\mathscr{J}$ is proper, there is a $T \in \mathscr{A}$ such that $T \notin \mathscr{J}$. Since $\mathscr{A}$ is τ-elementary, $T \notin (\mathrm{Ref}\,\mathscr{J}) \cap \mathscr{A}$. So there is an $x \in \mathscr{X}$ such that $Tx \notin [\mathscr{J}x]$. This implies $x \neq 0$ and $[\mathscr{J}x] \neq \mathscr{X}$. If $[\mathscr{J}x] = \{0\}$, then there is an $A \in \mathscr{J}$ such that $A \neq 0$ and $Ax = 0$. It follows that $\{y \in \mathscr{X} : Ay = 0\}$ is a n.h.s. for $\mathscr{A}$. If $[\mathscr{J}x] \neq \{0\}$, then $[\mathscr{J}x]$ is a n.i.s. for $\mathscr{A}$ because $\mathscr{J}$ is a left ideal.

We use $\sigma(T)$ to denote the spectrum of T in $\mathscr{L}(\mathscr{X})$. For $T \in \mathscr{A}$, $\sigma_{\mathscr{A}}(T)$ denotes $\{\lambda \in \mathbf{C} : T - \lambda$ has no inverse in $\mathscr{A}\}$.

PROPOSITION 6. *Suppose both left and right multiplication by a fixed operator are τ-continuous operations on $\mathscr{L}(\mathscr{X})$. Let $\mathscr{A}$ be τ-closed and τ-elementary. If there is an $A \in \mathscr{A}$ such that $\sigma_{\mathscr{A}}(A)\setminus\sigma(A)$ is non-empty, then $\mathscr{A}$ has a* n.i.s.

Proof. Suppose $\lambda \in \sigma_{\mathscr{A}}(A)\setminus\sigma(A)$. Let $\mathscr{I}$ be the τ-closure of $\{S(A-\lambda) : S \in \mathscr{A}\}$. Then $\mathscr{I}$ is a τ-closed left ideal of $\mathscr{A}$. Since $A-\lambda$ is invertible, $\mathscr{I} \neq \{0\}$. If $\mathscr{I} = \mathscr{A}$, then there is a net $\{S_i\}_{i\in I}$ in $\mathscr{A}$ such that $S_i(A-\lambda)$ converges to the identity operator 1 in the τ topology. Thus $S_i = S_i(A-\lambda)(A-\lambda)^{-1}$ converges to $(A-\lambda)^{-1}$ in the τ topology. This contradicts the fact that $\lambda \in \sigma_{\mathscr{A}}(A)$. So $\mathscr{I}$ is proper. As in the proof of Proposition 5, there is an $x \in \mathscr{X}$ such that $x \neq 0$ and $[\mathscr{I}x] \neq \mathscr{X}$. $[\mathscr{I}x] \neq \{0\}$, because $A-\lambda$ is an invertible element of $\mathscr{A}$. Since $\mathscr{I}$ is a left ideal, $[\mathscr{I}x]$ is a n.i.s.

When $\mathscr{X}$ is finite dimensional, $\operatorname{Lat}\mathscr{A} = \operatorname{Lat}\mathscr{L}(\mathscr{X})$ implies $\mathscr{A} = \mathscr{L}(\mathscr{X})$ by Burnside's Theorem [4, p. 142]. We have the following analogue for τ-elementary algebras.

Proposition 7. *Let $\mathscr{A}$ be a τ-elementary algebra. If $\mathscr{B}$ is a relatively τ-closed unital subalgebra of $\mathscr{A}$ with* $\operatorname{Lat}\mathscr{B} = \operatorname{Lat}\mathscr{A}$, *then $\mathscr{B} = \mathscr{A}$.*

Proof. $\operatorname{Lat}\mathscr{B} = \operatorname{Lat}\mathscr{A}$ implies $\operatorname{Ref}\mathscr{B} = \operatorname{Ref}\mathscr{A}$, by Proposition 1(g). By Theorem 4 and Proposition 1(a), $\mathscr{B} = (\operatorname{Ref}\mathscr{B}) \cap \mathscr{A} = (\operatorname{Ref}\mathscr{A}) \cap \mathscr{A} = \mathscr{A}$.

4. ELEMENTARY FORMS AND DILATION THEORY

In this section, we restrict our attention to the case where $\mathscr{X}$ is a separable Hilbert space, which we denote by $\mathscr{H}$. First, we obtain a more convenient way to represent elementary forms.

PROPOSITION 8. *If φ is a τ-elementary form on a linear submanifold $\mathscr{S}$ in $\mathscr{L}(\mathscr{H})$, then there exist vectors x and y in $\mathscr{H}$ such that $\varphi(S) = (Sx, y)$, $S \in \mathscr{S}$.*

Proof. By definition, there exist $x \in \mathscr{H}$ and $f \in \mathscr{H}^*$ such that $\varphi(S) = f(Sx)$, $S \in \mathscr{S}$. Let J denote the canonical conjugate linear isomorphism from $\mathscr{H}^*$ to $\mathscr{H}$. Let $y = Jf$. Then $\varphi(S) = f(Sx) = (Sx, y)$, $S \in \mathscr{S}$.

A subspace $\mathscr{L}$ of $\mathscr{H}$ is *semi-invariant* for $\mathscr{A}$ if $P_{\mathscr{L}}AP_{\mathscr{L}}BP_{\mathscr{L}} = P_{\mathscr{L}}ABP_{\mathscr{L}}$, $A, B \in \mathscr{A}$, where $P_{\mathscr{L}}$ denotes the orthogonal projection onto $\mathscr{L}$. For subspaces $\mathscr{M}, \mathscr{N}$ in $\mathscr{H}$ with $\mathscr{N} \subset \mathscr{M}$, $\mathscr{M} \ominus \mathscr{N}$ denotes $\mathscr{M} \cap \mathscr{N}^{\perp}$. If $\mathscr{A}$ and $\mathscr{B}$ are subalgebras of $\mathscr{L}(\mathscr{H})$, we say $\mathscr{A}$ is a *dilation* of $\mathscr{B}$ (or $\mathscr{B}$ is a *compression* of $\mathscr{A}$) if there is a semi-invariant subspace $\mathscr{L}$ for $\mathscr{A}$ such that

$$P_{\mathscr{L}}\mathscr{A}P_{\mathscr{L}} = \{P_{\mathscr{L}}AP_{\mathscr{L}} : A \in \mathscr{A}\} = \mathscr{B}.$$

For τ-elementary algebras the homomorphisms satisfying a certain continuity condition yield information about the compressions of the algebra. In particular, we will obtain information about compressions to semi-invariant subspaces generated by two vectors. Some useful properties of these subspaces are listed below. We use $\mathscr{A}^*$ to denote $\{A^* : A \in \mathscr{A}\}$.

LEMMA 9. *Let* $x, y \in \mathscr{H}$. *Let* P *denote the orthogonal projection onto* $[\mathscr{A}x] \ominus \ominus ([\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp)$.

(a) $P\mathscr{H}$ *is semi-invariant for* $\mathscr{A}$.

(b) $(PAPu, v) = (APu, v) = (Au, v)$, $A \in \mathscr{A}$, $u \in [Ax]$, $v \in [\mathscr{A}^*y]$.

(c) $\{PAPx : A \in \mathscr{A}\}$ *is dense in* $P\mathscr{H}$.

Proof. Since $[\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp \subset [\mathscr{A}x]$ and both subspaces are in Lat$\mathscr{A}$, a tedious calculation shows that $P\mathscr{H}$ is semi-invariant.

For (b) we write $u = Pu + w$, with $w \in [\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp$. Then

$$(Au, v) = (APu, v) + (w, A^*v) = (APu, v), \quad A \in \mathscr{A},$$

because $A^*v \in [\mathscr{A}^*y]$. Also $Pu \in [\mathscr{A}x]$ implies $APu \in [\mathscr{A}x]$, $A \in \mathscr{A}$. So we can write $APu = PAPu + z$ with $z \in [\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp$ and dependent on $A \in \mathscr{A}$. Then

$$(APu, v) = (PAPu, v) + (z, v) = (PAPu, v), \quad A \in \mathscr{A},$$

because $v \in [\mathscr{A}^*y]$.

To prove (c) we first show $PAPx = PAx$, $A \in \mathscr{A}$. Again, write $x = Px + w$ with $w \in [\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp$. Then

$$PAx = PAPx + PAw, \quad A \in \mathscr{A}.$$

But $Aw \in [\mathscr{A}x] \cap [\mathscr{A}^*y]^\perp$, $A \in \mathscr{A}$. So $PAw = 0$.

Now we prove the density statement. Suppose $v \in P\mathscr{H}$ and $(PAPx, v) = 0$, $A \in \mathscr{A}$. We show $v = 0$. By the above,

$$0 = (PAPx, v) = (PAx, v) = (Ax, v), \quad A \in \mathscr{A}.$$

So $v \in [\mathscr{A}x]^\perp$. But $v \in [\mathscr{A}x]$. Thus $v = 0$.

We now establish a dilation theorem for τ-elementary algebras. $\mathscr{K}$ will denote a separable, complex Hilbert space, and $\mathscr{B}$ will denote a unital subalgebra of $\mathscr{L}(\mathscr{K})$. For a linear operator X, $\mathscr{D}(X)$ will denote the domain of X. This theorem generalizes [1, Theorem 4.12] in the case $n = 1$.

THEOREM 10. *Let $\mathscr{A}$ be τ-elementary. Suppose $\Phi : (\mathscr{A}, \tau) \to (\mathscr{B}, \text{weak})$ is a continuous, surjective homomorphism. Then for each pair $u, v \in \mathscr{H}$, there exist $\mathscr{M}$, $\mathscr{N} \in \operatorname{Lat} \mathscr{A}$ with $\mathscr{N} \subset \mathscr{M}$ and a closed, injective linear transformation $X : \mathscr{D}(X) \to \to \mathscr{M} \ominus \mathscr{N}$ such that*

(a) *the linear manifold $\mathscr{D}(X)$ is dense in*

$$[\mathscr{B}u] \ominus ([\mathscr{B}u] \cap [\mathscr{B}^* v]^{\perp}),$$

(b) *the range of X is dense in $\mathscr{M} \ominus \mathscr{N}$,*

(c) *$PAPXz = XQ\Phi(A)Qz$, $z \in \mathscr{D}(X)$, $A \in \mathscr{A}$, where P is the orthogonal projection onto $\mathscr{M} \ominus \mathscr{N}$ and Q is the orthogonal projection onto $[\mathscr{B}u] \ominus ([\mathscr{B}u] \cap [\mathscr{B}^* v]^{\perp})$, and*

(d) *v is in the range of X^*.*

Proof. Set $\varphi(A) = (\Phi(A)u, v)$, $A \in \mathscr{A}$. Then φ is a τ-continuous form on $\mathscr{A}$. So there exist $x, y \in \mathscr{H}$ such that $(Ax, y) = (\Phi(A)u, v)$, $A \in \mathscr{A}$. Let $\mathscr{M} = [\mathscr{A}x]$ and $\mathscr{N} = [\mathscr{A}x] \cap [\mathscr{A}^* y]^{\perp}$. By Lemma 9(a), $P\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$ and $Q\mathscr{H} = [\mathscr{B}u] \ominus \ominus ([\mathscr{B}u] \cap [\mathscr{B}^* v]^{\perp})$ are semi-invariant for $\mathscr{A}$ and $\mathscr{B}$, respectively. Define a linear transformation X_0 with $\mathscr{D}(X_0) = \{Q\Phi(A)Qu : A \in \mathscr{A}\}$ by $X_0(Q\Phi(A)Qu) = PAPx$, $A \in \mathscr{A}$. We show X_0 is well-defined and injective. Since $\Phi(A)Qu \in [\mathscr{B}u]$, $Q\Phi(A)Qu = = 0$ if and only if $(\Phi(A)Qu, B^* v) = 0$ for all $B \in \mathscr{B}$. Since Φ is surjective, this is equivalent to $(\Phi(A)Qu, \Phi(S)^* v) = 0$ for all $S \in \mathscr{A}$. Now for all $S \in \mathscr{A}$, using Lemma 9(b) we have:

$$(\Phi(A)Qu, \Phi(S)^* v) = (\Phi(SA)Qu, v) = (\Phi(SA)u, v) =$$

$$= (SAx, y) = (SAPx, y) = (APx, S^* y).$$

But $(APx, S^* y) = 0$ for all $S \in \mathscr{A}$ if and only if $PAPx = 0$. To show X_0 is closable, and its closure, which we call X, is injective, it suffices to show that if $Q\Phi(A_n)Qu \to \to u'$ and $PA_nPx \to x'$, then $u' = 0$ if and only if $x' = 0$. This follows from a straightforward application of the above argument.

Since $\mathscr{D}(X_0) = \{Q\Phi(A)Qu : A \in \mathscr{A}\}$, $\mathscr{D}(X_0)$ is dense in $[\mathscr{B}u] \ominus ([\mathscr{B}u] \cap \cap [\mathscr{B}^* v]^{\perp})$ by Lemma 9(c). Thus $\mathscr{D}(X)$ is dense in $[\mathscr{B}u] \ominus ([\mathscr{B}u] \cap [\mathscr{B}^* v]^{\perp})$ because X is the closure of X_0. Similarly the range of X_0 is $\{PAPx : A \in \mathscr{A}\}$, which is dense in $\mathscr{M} \ominus \mathscr{N}$ by Lemma 9(c). So the range of X is dense in $\mathscr{M} \ominus \mathscr{N}$.

To show (c) note that for $A, S \in \mathscr{A}$,

$$PAPX_0 Q\Phi(S)Qu = PAPPSPx = PASPx = X_0 Q\Phi(AS)Qu = X_0 Q\Phi(A)QQ\Phi(S)Qu.$$

Thus

$$PAPX_0 z = X_0 Q\Phi(A)Qz, \quad A \in \mathscr{A}, \ z \in \mathscr{D}(X_0).$$

The assertion now follows from the fact that X is the closure of X_0.

Finally we show (d). It suffices to show that for every $A \in \mathscr{A}$, $(XQ\Phi(A)Qu, y) = (Q\Phi(A)Qu, v)$. Using Lemma 9(b), we have

$$(XQ\Phi(A)Qu, y) = (PAPx, y) =$$
$$= (Ax, y) = (\Phi(A)u, v) = (Q\Phi(A)Qu, v).$$

We use Theorem 10 to obtain necessary and sufficient conditions for an algebra to be weakly elementary. For each positive integer n, $\mathscr{H}^{(n)}$ will denote the direct sum of n copies of $\mathscr{H}$. For $A \in \mathscr{A}$, $A^{(n)}$ will denote the direct sum of n copies of A. $\mathscr{A}^{(n)}$ will denote $\{A^{(n)} : A \in \mathscr{A}\}$.

THEOREM 11. *$\mathscr{A}$ is weakly elementary if and only if for every positive integer n and for every pair $\bar{x}, \bar{y} \in \mathscr{H}^{(n)}$, there exist $\mathscr{M}, \mathscr{N} \in \operatorname{Lat}\mathscr{A}$ with $\mathscr{N} \subset \mathscr{M}$ and a linear transformation $X : \mathscr{D}(X) \to \mathscr{M} \ominus \mathscr{N}$ such that*

(a) *$PAPXQ\bar{x} = XQA^{(n)}Q\bar{x}$, $A \in \mathscr{A}$, where P is the orthogonal projection onto $\mathscr{M} \ominus \mathscr{N}$ and Q is the orthogonal projecton onto $[\mathscr{A}^{(n)}\bar{x}] \ominus ([\mathscr{A}^{(n)}\bar{x}] \cap [(\mathscr{A}^{(n)})^{*}y]^{\perp})$, and*

(b) *$\bar{y}$ is in the range of X^{*}.*

Proof. For each positive integer n, define $\Phi_n : (\mathscr{A}, \text{weak}) \to (\mathscr{A}^{(n)}, \text{weak})$ by $\Phi_n(A) = A^{(n)}$. It is easy to see that each Φ_n is a homeomorphic algebra isomorphism. Necessity now follows from Theorem 10.

We show sufficiency. Given a weakly continuous form φ on $\mathscr{A}$, there exist $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in $\mathscr{H}$ such that $\varphi(A) = \sum_{i=1}^{n} (Ax_i, y_i)$, $A \in \mathscr{A}$ (cf. [2, p. 42]). Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$. Then $\varphi(A) = (A^{(n)}\bar{x}, \bar{y})$, $A \in \mathscr{A}$. By hypothesis there is a linear transformation satisfying (a) and (b). Let $x = XQ\bar{x}$ and choose y such that $\bar{y} = X^{*}y$. Then

$$\varphi(A) = (A^{(n)}\bar{x}, \bar{y}) = (QA^{(n)}Q\bar{x}, \bar{y}) =$$
$$= (QA^{(n)}Q\bar{x}, X^{*}y) = (XQA^{(n)}Q\bar{x}, y) = (PAPx, y) = (APx, Py), \quad A \in \mathscr{A}.$$

Thus φ is an elementary form on $\mathscr{A}$.

The *ultraweak topology* on $\mathscr{L}(\mathscr{H})$ is the topology induced by the forms $\sum_{i=1}^{\infty} \omega_{x_i, y_i}$ with $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$ (cf. [2]). We now use Theorem 10 to obtain necessary and sufficient conditions for an algebra to be ultraweakly elementary. $\tilde{\mathscr{H}}$ will denote the direct sum of a countably infinite number of copies of $\mathscr{H}$. For $A \in \mathscr{A}$, $\tilde{A}$ will denote the direct sum of a countably infinite number of copies of A. $\tilde{\mathscr{A}}$ will denote $\{\tilde{A} : A \in \mathscr{A}\}$.

THEOREM 12. *$\mathscr{A}$ is ultraweakly elementary if and only if for every $\tilde{x}, \tilde{y} \in \tilde{\mathscr{H}}$ there exist $\mathscr{M}, \mathscr{N} \in \mathrm{Lat}\,\mathscr{A}$ with $\mathscr{N} \subset \mathscr{M}$ and a linear transformation $X : \mathscr{D}(X) \to \to \mathscr{M} \ominus \mathscr{N}$ such that*

(a) $PAPXQ\tilde{x} = XQ\tilde{A}Q\tilde{x}$, $A \in \mathscr{A}$, where P is the orthogonal projection onto $\mathscr{M} \ominus \mathscr{N}$ and Q is the orthogonal projection onto $[\tilde{\mathscr{A}}\tilde{x}] \ominus ([\tilde{\mathscr{A}}\tilde{x}] \cap [(\tilde{\mathscr{A}})^*\tilde{y}]^\perp)$, and

(b) $\tilde{y}$ is in the range of X^*.

Proof. It is easy to verify that the map $\Phi : (\mathscr{A}$, ultraweak$) \to (\tilde{\mathscr{A}}$, weak$)$ is a homeomorphic algebra isomorphism. Necessity follows from Theorem 10.

If φ is an ultraweakly continuous form on $\mathscr{A}$, there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$ and $\varphi(A) = \sum_{i=1}^{\infty} (Ax_i, y_i)$ (cf. [2, p. 42]). Set $\tilde{x} = (x_1, x_2, \ldots)$ and $\tilde{y} = (y_1, y_2, \ldots)$. Then $\tilde{x}, \tilde{y} \in \tilde{\mathscr{H}}$ and $\varphi(A) = (\tilde{A}\tilde{x}, \tilde{y})$, $A \in \mathscr{A}$. Sufficiency is now proved as in Theorem 11.

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