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AN APPROACH TO THE SPECTRAL DECOMPOSITION OF J-POSITIZABLE OPERATORS

J. BOGNÁR

INTRODUCTION

We consider a Hilbert space $\mathfrak{H}$ with inner product (f, g), and a linear operator J on $\mathfrak{H}$ such that

$$J^* = J, \quad J^2 = I, \quad J \neq \pm I. \tag{1}$$

The bounded or unbounded linear operator A in $\mathfrak{H}$ is said to be *J-selfadjoint* if

$$(JA)^* = JA; \tag{2}$$

it is said to be *J-positive* if $(JAf, f) \geqslant 0$ for every f in the domain of A.

We say that A is *J-positizable* (in another context the term used in the literature is "definitizable" or "positizable") if

1) the spectrum $\sigma(A)$ of A does not fill the complex plane,

2) A is J-selfadjoint, and

3) there exists a non-zero real polynomial φ such that $\varphi(A)$ is J-positive. In this case we say A is J-positizable by φ.

By a J-selfadjoint projection we mean a bounded J-selfadjoint operator E satisfying $E^2 = E$.

Roughly speaking, a spectral decomposition of a J-selfadjoint operator A is a description of A in terms of J-selfadjoint projections.

In the case where one of the two eigenspaces of J has finite dimension, a spectral decomposition for J-positizable operators was proved by M. G. Kreĭn and H. Langer [6]. In the general case the theorem is due to Langer [7] (see the exposition [8] where motivation for the study of this class of operators is also given).

T. Ando [1] reduced Langer's theorem to the classical spectral theorem in the case where A is bounded and $\varphi(\lambda) = \lambda$ for all numbers λ. In [3] we applied a method related to that of Ando, and were able to exclude complex variables as well as auxiliary spaces.

In the present note we extend our procedure to the case where $\varphi(\lambda) = \gamma(\lambda - \alpha)^k$ with $\gamma \neq 0$, α and γ real, k a positive integer. The extension is non-trivial, but the tools remain elementary, and a new insight into the theorem is achieved.

We do not reduce the problem to the simpler-looking case $\varphi(\lambda) = \lambda^k$ since this would not make things much easier and may cause inconvenience when further generalization is attempted.

Though Langer's theorem is best stated in terms of indefinite inner product spaces, we stick to Hilbert space terminology. First, our methods are those of Hilbert space. Second, we try to keep the paper self-contained.

We only rely on the spectral theory of bounded selfadjoint operators on Hilbert space (see [10], for instance), and recall, partly with proofs, the few additional facts (Lemmas 1—4) we need. Occasionally, to avoid all unnecessary tools, we use redundant assumptions and statements (for instance, boundedness of an everywhere defined J-selfadjoint operator).

STATEMENT OF THE SPECTRAL THEOREM FOR BOUNDED OPERATORS J-POSITIZABLE BY A POLYNOMIAL WITH A SINGLE ZERO. SOME LEMMAS

The theorem we prove is the following special case of Langer's result [7], [8].

Let A be a bounded J-selfadjoint operator on $\mathfrak{H}$. Assume that

$$J\varphi(A) \geqslant 0, \tag{3}$$

where

$$\varphi(\lambda) = \gamma(\lambda - \alpha)^k \quad (-\infty < \lambda < \infty) \tag{4}$$

for some real number α, non-zero real number γ, and positive integer k. Then there is a function $\lambda \mapsto E_\lambda$ from the set of real numbers $\lambda \neq \alpha$ into the set of J-selfadjoint projections on $\mathfrak{H}$ with the following properties:

1. $E_\lambda E_\mu = E_{\min(\lambda,\mu)}$.

2. *If $\varphi(v) > 0$ for $\lambda \leqslant v \leqslant \mu$, then $JE_\lambda \leqslant JE_\mu$; if $\varphi(v) < 0$ for $\lambda \leqslant v \leqslant \mu$, then $JE_\lambda \geqslant JE_\mu$.*

3. *If $\lambda < -\|A\|$ then $E_\lambda = 0$; if $\lambda > \|A\|$ then $E_\lambda = I$.*

4. *The function $\lambda \mapsto E_\lambda$ is right-continuous in the strong operator topology.*

5. *If T is a bounded linear operator on $\mathfrak{H}$ such that $TA = AT$, then $TE_\lambda = E_\lambda T$ for every $\lambda \neq \alpha$.*

6. *The spectrum $\sigma(A|E_\lambda\mathfrak{H})$ is contained in the interval $(-\infty, \lambda]$, while $\sigma(A|(I-E_\lambda)\mathfrak{H})$ is contained in $[\lambda, \infty)$.*

These properties determine the function $\lambda \mapsto E_\lambda$ uniquely.

Moreover,

$$\int_{-\|A\|-0}^{\|A\|} \varphi(\nu)\, \mathrm{d}E_\nu \tag{5}$$

is a strongly convergent improper integral with singular point α, and

$$S := \varphi(A) - \int_{-\|A\|-0}^{\|A\|} \varphi(\nu)\, \mathrm{d}E_\nu \tag{6}$$

is a bounded J-positive operator such that $S^2 = 0$, $(A - \alpha I)S = 0$, and $SE_\lambda = E_\lambda S = 0$ or S according as $\lambda < \alpha$ or $\lambda > \alpha$.

Finally,

$$AE_\lambda = \int_{-\|A\|-0}^{\lambda} \nu\, \mathrm{d}E_\nu \quad \text{if } \lambda < \alpha, \tag{7}$$

$$A(I - E_\lambda) = \int_{\lambda}^{\|A\|} \nu\, \mathrm{d}E_\nu \quad \text{if } \lambda > \alpha. \tag{8}$$

Before starting with our construction, we state and prove some well-known facts.

LEMMA 1. (M. G. Kreĭn [5], also W. T. Reid [9]). *Let B, K be bounded linear operators on the Hilbert space $\mathfrak{H}$. Assume that B is positive and BK is self-adjoint. Then*

$$\|B^{1/2}Kf\| \leqslant \|K\|\,\|B^{1/2}f\| \quad (f \in \mathfrak{H}).$$

Proof. We may assume that $\|B\| = \|K\| = 1$. Setting

$$(f, g)_B = (Bf, g), \quad \|f\|_B = (f, f)_B^{1/2},$$

for $n = 1, 2, \ldots$ we have

$$\|K^n f\|_B^2 = (K^{n-1}f, K^{n+1}f)_B \leqslant$$

$$\leqslant \|K^{n-1}f\|_B \|K^{n+1}f\|_B.$$

In particular, if there is an $n_0 \geqslant 0$ with $\|K^{n_0}f\|_B = 0$ then $\|B^{1/2}Kf\| = \|Kf\|_B = 0$ and the statement holds. Otherwise we can write

$$\frac{\|K^{n+1}f\|_B}{\|K^n f\|_B} \geqslant \frac{\|K^n f\|_B}{\|K^{n-1}f\|_B}.$$

Hence, by recursion,

$$\frac{\|K^n f\|_B}{\|K^{n-1} f\|_B} \geqslant \frac{\|Kf\|_B}{\|f\|_B}$$

and consequently

$$\frac{\|K^n f\|_B}{\|f\|_B} \geqslant \left(\frac{\|Kf\|_B}{\|f\|_B}\right)^n,$$

$$\|Kf\|_B \leqslant \|f\|_B \left(\frac{\|K^n f\|_B}{\|f\|_B}\right)^{1/n}.$$

But, because of $\|B\| = \|K\| = 1$,

$$\|K^n f\|_B \leqslant \|K^n f\| \leqslant \|f\|.$$

So, $\|Kf\|_B \leqslant \|f\|_B$ as required.

LEMMA 2. (see, for instance, [4], Problem 61). *If U and V are bounded linear operators on $\mathfrak{H}$, then*

$$\sigma(VU) \subset \sigma(UV) \cup \{0\}.$$

Proof. It is to be proved that if, for some $\lambda \neq 0$, $UV - \lambda I$ has a bounded inverse, then also $VU - \lambda I$ has a bounded inverse. Replacing V by $(1/\lambda)V$ we may assume that $\lambda = 1$. Now, by a simple calculation, if $(UV - I)^{-1} = T$ then $(VU - I)^{-1} = VTU - I$.

LEMMA 3. *Let Q, T be bounded linear operators on $\mathfrak{H}$. Assume that $Q^2 = Q$ and $QT = TQ$. Then $Q\mathfrak{H}$ and $(I - Q)\mathfrak{H}$ are closed complementary subspaces invariant under T, and*

$$\sigma(T|Q\mathfrak{H}) \cup \sigma(T|(I - Q)\mathfrak{H}) = \sigma(T).$$

In particular,

$$\sigma(T|Q\mathfrak{H}) \subset \sigma(T).$$

The proof is straightforward.

LEMMA 4. (see for instance [2], Corollary IV.8.2 and the discussion preceding it). *Let E be a J-selfadjoint projection. Assume that the operator JE is positive. Then the restriction $J|E\mathfrak{H}$ is also positive, and*

$$\|f\|_J := (Jf, f)^{1/2} \quad (f \in E\mathfrak{H})$$

is a norm equivalent to the original norm on $E\mathfrak{H}$. In particular, $E\mathfrak{H}$ is a Hilbert space with inner product

$$(f, g)_J := (Jf, g) \quad (f, g \in E\mathfrak{H}).$$

Proof. If $E^2 = E, JE \geqslant 0$ and $h \in \mathfrak{H}$, then $(JEh, Eh) = (h, JE^2h) = (h, JEh) = (JEh, h) \geqslant 0$. Thus $J|E\mathfrak{H}$ is positive. By definition, E is bounded; so $E\mathfrak{H}$ is closed. Denoting by Q the usual, selfadjoint projection to $E\mathfrak{H}$ and letting

$$G = QJ|E\mathfrak{H}$$

we have

$$(Jf, g) = (Gf, g) \quad \text{for } f, g \in E\mathfrak{H},$$

and G is a bounded selfadjoint operator on the Hilbert space $E\mathfrak{H}$. Further, if $h \in \mathfrak{H}$ and $g \in E\mathfrak{H}$ then

$$(QJh, g) = (Jh, g) = (Jh, Eg) = (h, JEg) =$$

$$= (JEh, g) = (QJEh, g);$$

hence $QJ = QJE$. In particular, $G(E\mathfrak{H}) = E\mathfrak{H}$. Since G is selfadjoint, it follows that G^{-1} exists, G^{-1} is defined on all of $E\mathfrak{H}$ and, by an easy application of the spectral decomposition of G (or by the closed graph principle), G^{-1} is bounded. As a result,

$$\|f\|_J^2 = (Jf, f) = (Gf, f) \geqslant \beta\|f\|^2$$

for some $\beta > 0$ and all $f \in E\mathfrak{H}$.

CONSTRUCTION OF AN AUXILIARY OPERATOR C

Now we turn to the construction of E_λ appearing in the spectral theorem stated above.

The bounded linear operator

(9) $$B := \varphi(A)J$$

is positive; in fact, by (9), (1) and (3),

$$(Bf, f) = (J\varphi(A)Jf, Jf) \geqslant 0$$

for every f in $\mathfrak{H}$. It follows that the positive square root $B^{1/2}$ exists and is unique.

Introduce the notation

(10) $$\mathfrak{N} := \{f \in \mathfrak{H} : B^{1/2}f = 0\} \text{ and } \mathfrak{R} := \{B^{1/2}f : f \in \mathfrak{H}\}.$$

Then

(11) $$\mathfrak{N} \perp \mathfrak{R} \quad \text{and} \quad \mathfrak{H} = \mathfrak{N} \oplus \overline{\mathfrak{R}},$$

where bar denotes closure.

Let P be the selfadjoint projection of $\mathfrak{H}$ to $\overline{\mathfrak{R}}$. Then, by (11),

(12) $$Pf \in \overline{\mathfrak{R}}, \quad (I-P)f \in \mathfrak{N} \quad (f \in \mathfrak{H})$$

and, by (10),

(13) $$B^{1/2}P = PB^{1/2} = B^{1/2}.$$

Let $B^{(-1/2)}$ be the (possibly unbounded) linear operator defined on the (possibly non-dense) domain

(14) $$\mathfrak{D}_{B^{(-1/2)}} := \mathfrak{R}$$

by the relation

(15) $$B^{(-1/2)}B^{1/2}f = Pf \quad (f \in \mathfrak{H}).$$

The definition of $B^{(-1/2)}$ is correct, since $B^{1/2}f_1 = B^{1/2}f_2$ implies $Pf_1 = Pf_2$. We also note that

(16) $$B^{1/2}B^{(-1/2)}g = g \quad (g \in \mathfrak{R});$$

in fact, from (15) and (13) we obtain

$$B^{1/2}B^{(-1/2)}B^{1/2}f = B^{1/2}Pf = B^{1/2}f \quad (f \in \mathfrak{H}).$$

The next step, the heart of the construction, is to define a linear operator C with domain

(17) $$\mathfrak{D}_C := \mathfrak{N} \dotplus \mathfrak{R}$$

by the relation

(18) $$Cg := B^{(-1/2)}AB^{1/2}g \quad (g \in \mathfrak{D}_C).$$

PROPOSITION 1. *The definition* (17) — (18) *of* C *is correct,* $\mathfrak{D}_C$ *is dense, and*

(19) $$C\mathfrak{N} = \{0\}, \quad C\mathfrak{R} \subset \mathfrak{R}.$$

Proof. According to (17) and (11), $\mathfrak{D}_C$ is dense. Let $g_0 \in \mathfrak{N}$, $f \in \mathfrak{H}$. Then by (10), (9), (1) and (14)

$$AB^{1/2}(g_0 + B^{1/2}f) = ABf = A\varphi(A)Jf = \varphi(A)AJf =$$

$$= BJAJf = B^{1/2}(B^{1/2}JAJf) \in \mathfrak{D}_{B^{(-1/2)}}.$$

Hence, in view of (15) and (13),

$$C(g_0 + B^{1/2}f) = PB^{1/2}JAJf = B^{1/2}JAJf.$$

REMARK. From the proof of Proposition 1 we see that

$$AB = BJAJ \tag{20}$$

and

$$CB^{1/2} = B^{1/2}JAJ. \tag{21}$$

Further, $(BJAJ)^* = J^*(JA)^*B^* = J(JA)B = AB$; hence, in view of (20),

$$(AB)^* = AB. \tag{22}$$

PROPOSITION 2. *The operator C is symmetric.*

Proof. From (21), (20) and (22) it follows that

$$(B^{1/2}CB^{1/2})^* = B^{1/2}CB^{1/2}. \tag{23}$$

Consequently, for every f and g in $\mathfrak{H}$,

$$(CB^{1/2}f, B^{1/2}g) = (B^{1/2}CB^{1/2}f, g) =$$

$$= (f, B^{1/2}CB^{1/2}g) = (B^{1/2}f, CB^{1/2}g).$$

Thus the restriction $C|\mathfrak{R}$ is symmetric. On the other hand, according to (19), $C|\mathfrak{N}$ is zero, hence symmetric. Now the proposition follows with the help of (17), (11) and (19).

PROPOSITION 3. *The operator C is bounded on its domain. More precisely,*

$$\|Cg\| \leqslant \|A\| \, \|g\| \quad (g \in \mathfrak{D}_C). \tag{24}$$

Proof. We know that the operator $B := \varphi(A)J$ (see (9)) is positive. Further, if $K := JAJ$ then by (20) and (22) BK is selfadjoint. Applying Lemma 1 and considering relation (1) we obtain that, for every f in $\mathfrak{H}$,

$$\|B^{1/2}JAJf\| \leqslant \|A\| \, \|B^{1/2}f\|$$

or, using (21),

$$\|CB^{1/2}f\| \leqslant \|A\| \, \|B^{1/2}f\|.$$

Hence

$$\|Cg\| \leqslant \|A\| \, \|g\| \quad \text{if } g \in \mathfrak{R}.$$

On the other hand, in view of (19) we have

$$\|Cg\| = 0 \leqslant \|A\| \, \|g\| \quad \text{if } g \in \mathfrak{N}.$$

Since from (19) we also know that $C\mathfrak{N} \subset \mathfrak{N}$ and $C\mathfrak{R} \subset \mathfrak{R}$, relation (24) follows with the help of (17) and (11).

PROPOSITION 4. *C has a unique bounded linear extension $\bar{C}$ defined on all of $\mathfrak{H}$; moreover,*

$$(\bar{C})^* = \bar{C} \tag{25}$$

and

$$\|\bar{C}\| \leq \|A\|. \tag{26}$$

Proof. By Proposition 1, the domain of C is dense in $\mathfrak{H}$. It remains to apply Propositions 2 and 3.

PROPOSITION 5. *The projection P, defined by* (12), *commutes with $\bar{C}$:*

$$P\bar{C} = \bar{C}P. \tag{27}$$

Proof. From (11), (12) and (19) it follows that $\bar{C}Pf = \bar{C}f - \bar{C}(I-P)f = \bar{C}$ and also $P\bar{C}f = \bar{C}f$.

PROPOSITION 6. *For $m = 0, 1, 2, \ldots,$ we have*

$$\mathfrak{D}_C \subset \mathfrak{D}_{B^{(-1/2)}A^m B^{1/2}} \tag{28}$$

and

$$P(\bar{C})^m = \overline{(B^{(-1/2)}A^m B^{1/2}|\mathfrak{D}_C)}, \tag{29}$$

where P is the projection defined by (12).

Proof. We proceed by recursion. Since, according to (15), $B^{(-1/2)}A^0B^{1/2} = B^{(-1/2)}B^{1/2} = P$, relations (28) and (29) are valid for $m = 0$. We next assume that (28) and (29) hold for some m. Let $g \in \mathfrak{D}_C$. Then, by Proposition 1 and relation (14), $AB^{1/2}g \in \mathfrak{R}$ and, by (19) and (17), $Cg \in \mathfrak{D}_C$. Therefore, in view of the assumption and (16),

$$P(\bar{C})^{m+1}g = P(\bar{C})^m\bar{C}g = P(\bar{C})^m Cg =$$

$$= \overline{(B^{(-1/2)}A^m B^{1/2}|\mathfrak{D}_C)}Cg = B^{(-1/2)}A^m B^{1/2}Cg =$$

$$= B^{(-1/2)}A^m B^{1/2}B^{(-1/2)}AB^{1/2}g = B^{(-1/2)}A^{m+1}B^{1/2}g.$$

We have obtained that $B^{(-1/2)}A^{m+1}B^{1/2}g$ makes sense and

$$P(\bar{C})^{m+1}|\mathfrak{D}_C = B^{(-1/2)}A^{m+1}B^{1/2}|\mathfrak{D}_C.$$

The proof of Proposition 6 is complete.

PROPOSITION 7. *We have*

$$P\varphi(\bar{C}) = B^{1/2}JB^{1/2}, \tag{30}$$

where φ *is the polynomial appearing in* (3) *and* (4) *(see the statement of the Theorem).*

Proof. From Proposition 6 and relations (9), (15), (13) it follows that

$$P\varphi(\bar{C}) = \overline{(B^{(-1/2)}\varphi(A)B^{1/2}|\mathfrak{D}_C)} =$$

$$= \overline{(B^{(-1/2)}BJB^{1/2}|\mathfrak{D}_C)} = \overline{(PB^{1/2}JB^{1/2}|\mathfrak{D}_C)} =$$

$$= \overline{(B^{1/2}JB^{1/2}|\mathfrak{D}_C)} = B^{1/2}JB^{1/2}.$$

CONSTRUCTION OF E_λ

By Proposition 4, $\bar{C}$ is a bounded selfadjoint operator, and its spectral decomposition can be written in the form (cf. (26))

$$\bar{C} = \int_{-\|A\|-0}^{\|A\|} \nu \, \mathrm{d}F_\nu, \tag{31}$$

where $\{F_\nu\}_{\nu=-\infty}^{\infty}$ denotes the right-continuous spectral function of $\bar{C}$ (for the properties of spectral decompositions of bounded selfadjoint operators in Hilbert space to be used in the sequel, see for instance [10], pp. 18—25).

We set

$$D_\lambda := \varphi(\bar{C})|F_\lambda\mathfrak{H} \quad \text{for } \lambda < \alpha, \tag{32}$$

$$D_\lambda := \varphi(\bar{C})|(I - F_\lambda)\mathfrak{H} \quad \text{for } \lambda > \alpha \tag{33}$$

and

$$E_\lambda := B^{1/2}D_\lambda^{-1}F_\lambda B^{1/2}J \quad \text{for } \lambda < \alpha, \tag{34}$$

$$I - E_\lambda := B^{1/2}D_\lambda^{-1}(I - F_\lambda)B^{1/2}J \quad \text{for } \lambda > \alpha \tag{35}$$

or, more explicitly,

$$E_\lambda := B^{1/2} \int_{-\|A\|-0}^{\lambda} \frac{1}{\varphi(\nu)} \, \mathrm{d}F_\nu \cdot B^{1/2}J \quad \text{for } \lambda < \alpha, \tag{36}$$

$$I - E_\lambda := B^{1/2} \int_{\lambda}^{\|A\|} \frac{1}{\varphi(\nu)} \, \mathrm{d}F_\nu \cdot B^{1/2}J \quad \text{for } \lambda > \alpha. \tag{37}$$

In the rest of the paper we verify that the function $\lambda \mapsto E_\lambda$ enjoys the properties stated in the Theorem.

PROOF OF PROPERTY 3 AND THAT E_λ IS BOUNDED AND J-SELFADJOINT.

By (31),

$$F_\lambda = 0 \quad \text{for } \lambda < -\|A\|. \tag{38}$$

Now, if $\lambda < -\|A\|$ and $\lambda < \alpha$ then (36) and (38) yield $E_\lambda = 0$, independently of whether $\alpha < -\|A\|$ (this possibility has not been excluded) or not.

Next let $\lambda < -\|A\|$ and $\lambda > \alpha$. Then, in view of (4), the integrand in (37) is bounded and continuous, and with the help of (38) and (31) we obtain

$$I - E_\lambda = B^{1/2} \int\limits_{-\|A\|-0}^{\|A\|} \frac{1}{\varphi(\nu)}\, dF_\nu \cdot B^{1/2}J =$$

$$= B^{1/2}(\varphi(\bar{C}))^{-1}B^{1/2}J,$$

where $(\varphi(\bar{C}))^{-1}$ is bounded. On the other hand, the number $\alpha < -\|A\|$ cannot belong to the spectrum of A; hence, according to (4), the number $\varphi(\alpha) = 0$ does not belong to the spectrum of $\varphi(A)$, which by (9) and (1) implies that B has a bounded inverse. In particular (see (10) and (12)), $P = I$. Using these facts and relation (30) we further obtain

$$I - E_\lambda = B^{1/2}(B^{1/2}JB^{1/2})^{-1}B^{1/2}J =$$

$$= B^{1/2}B^{-1/2}JB^{-1/2}B^{1/2}J = J^2 = I,$$

and therefore $E_\lambda = 0$ also in this case.

Similarly, from the relation

$$F_\lambda = I \quad \text{for } \lambda > \|A\| \tag{39}$$

it follows that $E_\lambda = I$ if $\lambda > \|A\|$.

Finally, let $-\|A\| \leqslant \lambda \leqslant \|A\|$ and $\lambda \neq \alpha$. Then the integrands in (36) and (37) are bounded continuous functions, so E_λ is bounded. It is also clear from (36)—(37) that $(JE_\lambda)^* = JE_\lambda$.

REMARKS. 1. The projection property $E_\lambda^2 = E_\lambda$ is a consequence of Property 1 to be proved below.

2. If $\alpha < -\|A\|$ or $\alpha > \|A\|$, then Properties 2—3 together with relations (4) and (1) lead to a contradiction. As a result, the condition $J \neq \pm I$ assures that $-\|A\| \leqslant \alpha \leqslant \|A\|$. Our proof, however, does not make use of this fact.

Proof of Property 1. Let $\lambda \leqslant \mu < \alpha$. Then, in view of (34), (30), (32), (27) and (13),

$$E_\lambda E_\mu = B^{1/2}D_\lambda^{-1}F_\lambda B^{1/2}JB^{1/2}D_\mu^{-1}F_\mu B^{1/2}J =$$

$$= B^{1/2}D_\lambda^{-1}F_\lambda P\varphi(\bar{C})D_\mu^{-1}F_\mu B^{1/2}J =$$

$$= B^{1/2}D_\lambda^{-1}F_\lambda PF_\mu B^{1/2}J = B^{1/2}D_\lambda^{-1}F_\lambda F_\mu PB^{1/2}J =$$

$$= B^{1/2}D_\lambda^{-1}F_\lambda B^{1/2}J = E_\lambda$$

and, similarly, $E_\mu E_\lambda = E_\lambda$.

Next let $\alpha < \lambda \leqslant \mu$. Then, by a similar argument,

$$(I - E_\lambda)(I - E_\mu) = I - E_\mu = (I - E_\mu)(I - E_\lambda)$$

and therefore

$$E_\lambda E_\mu = E_\lambda = E_\mu E_\lambda.$$

Finally, let $\lambda < \alpha < \mu$. Then, performing the same steps as above, we obtain

$$E_\lambda(I - E_\mu) = 0 = (I - E_\mu)E_\lambda,$$

which again leads to the desired conclusion.

Proof of Property 2. Since $\varphi(\alpha) = 0$, if $\varphi(\nu)$ has constant positive or negative sign for $\lambda \leqslant \nu \leqslant \mu$ then either $\alpha < \lambda$ or $\mu < \alpha$. In the first case, for every f in $\mathfrak{H}$,

$$(J(E_\mu - E_\lambda)f, f) =$$

$$= (JB^{1/2}(D_\mu^{-1}F_\mu - D_\lambda^{-1}F_\lambda)B^{1/2}Jf,\ f) =$$

$$= ((D_\mu^{-1}F_\mu - D_\lambda^{-1}F_\lambda)B^{1/2}Jf,\ B^{1/2}Jf) =$$

$$= \int_\lambda^\mu \frac{1}{\varphi(\nu)}\,\mathrm{d}(F_\nu B^{1/2}Jf,\ B^{1/2}Jf),$$

while in the second case the expression

$$(J[(I - E_\lambda) - (I - E_\mu)]f,\ f)$$

can be given the same form. Since the weight functions are increasing, the conclusion follows.

Proof of Property 4. From (36) for $\lambda < \mu < \alpha$, and from (37) also for $\alpha < \lambda < \mu$, we obtain

$$\|(E_\mu - E_\lambda)f\|^2 = \left\| B^{1/2} \int_\lambda^\mu \frac{1}{\varphi(\nu)} \, dF_\nu B^{1/2} J f \right\|^2 \leqslant$$

$$\leqslant \|B\| \int_\lambda^\mu \frac{1}{|\varphi(\nu)|^2} \, d\|F_\nu B^{1/2} J f\|^2 \leqslant$$

$$\leqslant \|B\| \max_{\lambda \leqslant \nu \leqslant \mu} \frac{1}{|\varphi(\nu)|^2} \|(F_\mu - F_\lambda) B^{1/2} J f\|^2 \to 0$$

if $\mu \to \lambda + 0$.

Proof of Property 5. Let T be a bounded linear operator which commutes with A, that is, $TA = AT$. By Property 3, in proving the relation $TE_\lambda = E_\lambda T$ we may assume that $-\|A\| \leqslant \lambda \leqslant \|A\|$.

Considering the case $\lambda < \alpha$ first, let $\{p_n\}_1^\infty$ be a sequence of polynomials which is bounded on $[-\|A\|, \|A\|]$ and satisfies the condition

$$\lim_{n\to\infty} p_n(\nu) = \begin{cases} \dfrac{1}{\varphi(\nu)} & \text{if } -\|A\| \leqslant \nu \leqslant \lambda, \\ 0 & \text{if } \lambda < \nu \leqslant \|A\|. \end{cases}$$

Then, according to (36) and (31), $B^{1/2} p_n(\bar{C}) B^{1/2} J \to E_\lambda$ strongly. Hence it is sufficient to prove that T commutes with $B^{1/2}(\bar{C})^m B^{1/2} J$ for $m = 0, 1, 2, \ldots$. But from relations (13), (29), (17) and (10) we obtain

$$B^{1/2}(\bar{C})^m B^{1/2} J = B^{1/2} P(\check{C})^m B^{1/2} J =$$

$$= B^{1/2}(\overline{B^{(-1/2)} A^m B^{1/2} | \mathfrak{D}_C}) B^{1/2} J =$$

$$= B^{1/2} B^{(-1/2)} A^m B^{1/2} B^{1/2} J$$

or, in view of (16), (28), (14) and (9),

$$B^{1/2}(\bar{C})^m B^{1/2} J = A^m \varphi(A) \quad (m = 0, 1, 2, \ldots), \tag{40}$$

so T commutes with these operators.

For $\lambda > \alpha$ the proof is similar.

Proof of the relations (7) *and* (8). According to (21),

$$\bar{C}B^{1/2} = B^{1/2}(JA)J.$$

Hence, taking adjoints,

$$B^{1/2}\bar{C} = AB^{1/2}. \tag{41}$$

Now let $\lambda < \alpha$. Then with the help of (34), (41), (31), (32), (38), (39) and (36) we obtain:

$$AE_\lambda = AB^{1/2}D_\lambda^{-1}F_\lambda B^{1/2}J = B^{1/2}\bar{C}D_\lambda^{-1}F_\lambda B^{1/2}J =$$

$$= B^{1/2}\int_{-\|A\|-0}^{\|A\|} \nu\,\mathrm{d}F_\nu \int_{-\|A\|-0}^{\lambda} \frac{1}{\varphi(\nu)}\,\mathrm{d}F_\nu \cdot B^{1/2}J =$$

$$= B^{1/2}\int_{-\|A\|-0}^{\lambda} \nu\frac{1}{\varphi(\nu)}\,\mathrm{d}F_\nu \cdot B^{1/2}J =$$

$$= \int_{-\|A\|-0}^{\lambda} \nu\frac{1}{\varphi(\nu)}\,\mathrm{d}(B^{1/2}F_\nu B^{1/2}J) = \int_{-\|A\|-0}^{\lambda} \nu\,\mathrm{d}E_\nu .$$

The proof of (8) is similar.

A relation for the spectrum of $\varphi(A)$. The next lines are a preparation for proving Property 6. We first observe that

$$\varphi(A)E_\lambda = B^{1/2}F_\lambda B^{1/2}J \quad \text{for every } \lambda \neq \alpha. \tag{42}$$

In fact, if $\lambda < \alpha$ then using the relations (9), (1), (34), (30), (13) and (32) we obtain

$$\varphi(A)E_\lambda = BJB^{1/2}D_\lambda^{-1}F_\lambda B^{1/2}J =$$

$$= B^{1/2}P\varphi(\bar{C})D_\lambda^{-1}F_\lambda B^{1/2}J = B^{1/2}F_\lambda B^{1/2}J$$

and, therefore,

$$\varphi(A)(I - E_\lambda) = BJ - B^{1/2}F_\lambda B^{1/2}J = B^{1/2}(I - F_\lambda)B^{1/2}J,$$

whereas in the case $\lambda > \alpha$ we first obtain the expression for $\varphi(A)(I - E_\lambda)$ and then pass to $\varphi(A)E_\lambda$.

Now let $\lambda < \mu$ be real numbers different from α. Then, in view of Property 1, $E_\mu - E_\lambda$ is a J-selfadjoint projection which, by Property 5, commutes with $\varphi(A)(E_\mu - E_\lambda)$. Thus from Lemma 3, relation (42), Lemma 2, relation (30) and relation (27) it follows that

$$\sigma(\varphi(A) \mid (E_\mu - E_\lambda)\mathfrak{H}) =$$

$$= \sigma(\varphi(A)(E_\mu - E_\lambda) \mid (E_\mu - E_\lambda)\mathfrak{H}) \subset$$

$$\subset \sigma(\varphi(A)(E_\mu - E_\lambda)) = \sigma(B^{1/2}(F_\mu - F_\lambda)B^{1/2}J) \subset$$

$$\subset \{0\} \cup \sigma((F_\mu - F_\lambda)B^{1/2}JB^{1/2}) =$$

$$= \{0\} \cup \sigma((F_\mu - F_\lambda)\varphi(\bar{C})P).$$

But P appearing in the last member may be suppressed; this can be seen by first applying Lemma 3 to $T = (F_\mu - F_\lambda)\varphi(\bar{C})P$ and $Q = P$ (recall (31)), then using the projection property of P, and finally applying Lemma 3 to $T = (F_\mu - F_\lambda)\varphi(\bar{C})$ and $Q = P$. So, by (31), (38) and (39),

$$\sigma(\varphi(A)|(E_\mu - E_\lambda)\mathfrak{H}) \subset \{0\} \cup \varphi([\lambda, \mu]) \quad \text{if } \lambda < \mu, \tag{43}$$

where $[\lambda, \mu] = \{\nu : \lambda \leqslant \nu \leqslant \mu\}$.

Since $\varphi(\alpha) = 0$ (see (4)), in the special case $\lambda < \alpha < \mu$ relation (43) reduces to

$$\sigma(\varphi(A) \mid (E_\mu - E_\lambda)\mathfrak{H}) \subset \varphi([\lambda, \mu]) \quad \text{if } \lambda < \alpha < \mu. \tag{44}$$

Proof of Property 6. Let λ, μ be real numbers such that either $\lambda < \mu < \alpha$ or $\alpha < \lambda < \mu$. According to (4), Property 2 and Lemma 4, the subspace $(E_\mu - E_\lambda)\mathfrak{H}$ is a Hilbert space $\mathfrak{H}_{\lambda,\mu}$ with respect to one of the inner products $(f, g)_J = (Jf, g)$ and $(f, g)_{-J} = -(Jf, g)$, and the corresponding norm on this space is equivalent to the original one. On the other hand, by (7), (8) and Property 3 we have

$$A(E_\mu - E_\lambda) = \int_\lambda^\mu \nu \, dE_\nu \quad \text{if } \alpha \notin [\lambda, \mu]. \tag{45}$$

Since the E_ν are J-selfadjoint as well as $(-J)$-selfadjoint, (45) restricted to $(E_\mu - E_\lambda)\mathfrak{H}$ is a spectral decomposition in the Hilbert space $\mathfrak{H}_{\lambda,\mu}$. It follows that

$$\sigma(A \mid (E_\mu - E_\lambda)\mathfrak{H}) \subset [\lambda, \mu] \quad \text{if } \alpha \notin [\lambda, \mu]. \tag{46}$$

Now, using (44) and (46), we shall prove the relation

(47) $$\sigma(A \mid (E_\mu - E_\lambda)\mathfrak{H}) \subset [\lambda, \mu] \quad \text{if } \lambda < \mu \quad (\lambda, \mu \neq \alpha)$$

which, in view of Property 3, implies Property 6.

If $\alpha \notin [\lambda, \mu]$ then (47) coincides with (46). So let $\lambda < \alpha < \mu$, and let β be a real number not in $[\lambda, \mu]$.

Since $\varphi(\beta) \neq 0$, $\varphi(\alpha) = 0$, and φ is continuous (see (4)), there exist two numbers λ_1, μ_1 such that

(48) $$\lambda < \lambda_1 < \alpha < \mu_1 < \mu$$

and

(49) $$\varphi(\beta) \notin \varphi([\lambda_1, \mu_1]).$$

From (44) and (49) we obtain

$$\varphi(\beta) \notin \sigma(\varphi(A) \mid (E_{\mu_1} - E_{\lambda_1})\mathfrak{H})$$

and hence, by an elementary special case of the spectral mapping theorem (namely that $A - \beta I$ divides $\varphi(A) - \varphi(\beta)I$ from both the right and the left),

(50) $$\beta \notin \sigma(A \mid (E_{\mu_1} - E_{\lambda_1})\mathfrak{H}).$$

Further, according to (46),

(51) $$\beta \notin \sigma(A \mid (E_{\lambda_1} - E_\lambda)\mathfrak{H})$$

and

(52) $$\beta \notin \sigma(A \mid (E_\mu - E_{\mu_1})\mathfrak{H}).$$

Relations (48), (50), (51), (52) and Lemma 3 imply that

$$\beta \notin \sigma(A \mid (E_\mu - E_\lambda)\mathfrak{H}).$$

This proves (47).

Proof of uniqueness. Assume that $\lambda \mapsto E_\lambda$ and $\lambda \mapsto E'_\lambda$ (λ real, $\lambda \neq \alpha$; E_λ and E'_λ some J-selfadjoint projections on $\mathfrak{H}$) are two functions with the properties 1—6.

Let $\lambda < \mu < \alpha$. Obviously,

$$E'_\lambda = E'_\lambda E_\mu + E'_\lambda(I - E_\mu).$$

By Property 5, E'_λ and E_μ commute with A and each other. In particular, $E'_\lambda(I - E_\mu)$ is a J-selfadjoint projection which commutes with A. Applying Lemma 3 in the Hilbert space $E'_\lambda\mathfrak{H}$ to $T = A \mid E'_\lambda\mathfrak{H}$ and $Q = (I - E_\mu) \mid E'_\lambda\mathfrak{H}$ as well as in the

Hilbert space $(I-E_\mu)\mathfrak{H}$ to $T=A\mid(I-E_\mu)\mathfrak{H}$ and $Q=E'_\lambda\mid(I-E_\mu)\mathfrak{H}$, then using Property 6, we obtain:

$$\sigma(A\mid E'_\lambda(I-E_\mu)\mathfrak{H})\subset$$
$$\subset\sigma(A\mid E'_\lambda\mathfrak{H})\cap\sigma(A\mid(I-E_\mu)\mathfrak{H})\subset$$
$$\subset(-\infty,\lambda]\cap[\mu,\infty)=\emptyset.$$

On the other hand we have

(53) $$E'_\lambda(I-E_\mu)\mathfrak{H}\subset E'_\lambda\mathfrak{H},$$

and from $\lambda<\alpha$, relation (4), Property 2 and Lemma 4 it follows that $E'_\lambda\mathfrak{H}$ is a Hilbert space with one of the inner products $(f,g)_J=(Jf,g)$ and $(f,g)_{-J}=-(Jf,g)$, while the corresponding norm is equivalent to the original one; moreover, in view of (2), the operator A is selfadjoint on this Hilbert space (as well as on its subspace $E'_\lambda(I-E_\mu)\mathfrak{H}$).

But the spectrum of a bounded selfadjoint operator on a Hilbert space can be empty only if the space is zero. So $E'_\lambda(I-E_\mu)=0$, that is,

$$E'_\lambda=E'_\lambda E_\mu \quad \text{if } \lambda<\mu<\alpha.$$

Letting $\mu\to\lambda+0$ and using Property 4 we conclude that $E'_\lambda=E'_\lambda E_\lambda$. Similarly, $E_\lambda=E_\lambda E'_\lambda$. Therefore, by Property 5, $E'_\lambda=E_\lambda$.

In the case $\alpha<\lambda<\mu$ we consider, instead of (53), the relation

$$E'_\lambda(I-E_\mu)\mathfrak{H}\subset(I-E_\mu)\mathfrak{H}$$

and argue that $(I-E_\mu)\mathfrak{H}$ can be turned into a Hilbert space on which A is selfadjoint; in other respects the reasoning is the same.

Proof of the convergence of (5). We first assume that $\alpha>-\|A\|$ and choose $\varepsilon>0$ satisfying $\alpha-\varepsilon>-\|A\|$. Then, observing that D_λ is a restriction of D_μ if $\lambda<\mu<\alpha$ (see (32)) and making use of (34), (31) and (32) we obtain the formal relations

$$\int_{-\|A\|-0}^{\alpha-\varepsilon}\varphi(\nu)\,\mathrm{d}E_\nu=\int_{-\|A\|-0}^{\alpha-\varepsilon}\varphi(\nu)\,\mathrm{d}(B^{1/2}D_\nu^{-1}F_\nu B^{1/2}J)=$$
$$=B^{1/2}D_{\alpha-\varepsilon}^{-1}\int_{-\|A\|-0}^{\alpha-\varepsilon}\varphi(\nu)\,\mathrm{d}F_\nu\cdot B^{1/2}J=$$
$$=B^{1/2}D_{\alpha-\varepsilon}^{-1}\varphi(\bar{C})F_{\alpha-\varepsilon}B^{1/2}J=B^{1/2}F_{\alpha-\varepsilon}B^{1/2}J.$$

Reading these relations from the right to the left it follows that the integral on the left-hand side exists in the strong operator topology. Moreover, letting $\varepsilon \to 0$ we obtain that also the limit

$$\int_{-\|A\|-0}^{\alpha-0} \varphi(\nu)\, \mathrm{d}E_\nu = B^{1/2} F_{\alpha-0} B^{1/2} J \tag{54}$$

exists in the strong operator topology. By (38) and Property 3, relation (54) holds for $\alpha \leqslant -\|A\|$ as well.

A similar evaluation yields:

$$\int_{\alpha+0}^{\|A\|} \varphi(\nu)\, \mathrm{d}E_\nu = B^{1/2}(I - F_\alpha) B^{1/2} J. \tag{55}$$

By (54) and (55), the assertion concerning (5) is true.

Proof of the properties of S. According to (6), (9), (1), (54) and (55),

$$S = BJ - B^{1/2} F_{\alpha-0} B^{1/2} J - B^{1/2}(I - F_\alpha) B^{1/2} J,$$

that is,

$$S = B^{1/2}(F_\alpha - F_{\alpha-0}) B^{1/2} J. \tag{56}$$

Clearly, S is bounded and J-positive. Further, by (56) and (30),

$$S^2 = B^{1/2}(F_\alpha - F_{\alpha-0}) P \varphi(\bar{C}) (F_\alpha - F_{\alpha-0}) B^{1/2} J.$$

But, in view of (31) and (4),

$$\varphi(\bar{C})(F_\alpha - F_{\alpha-0}) = \int_{\alpha-0}^{\alpha} \varphi(\nu)\, \mathrm{d}F_\nu =$$

$$= \varphi(\alpha)(F_\alpha - F_{\alpha-0}) = 0,$$

so that $S^2 = 0$.

The relation $(A - \alpha I)S = 0$ follows from (56), (41) and (31):

$$AS = B^{1/2} \bar{C} (F_\alpha - F_{\alpha-0}) B^{1/2} J =$$

$$= B^{1/2} \int_{\alpha-0}^{\alpha} \nu\, \mathrm{d}F_\nu \cdot B^{1/2} J =$$

$$= B^{1/2} \cdot \alpha (F_\alpha - F_{\alpha-0}) B^{1/2} J = \alpha S.$$

Now let $\lambda < \alpha$. Then by (56), (34), (30) and (27),

$$SE_\lambda = B^{1/2}(F_\alpha - F_{\alpha-0})\varphi(\bar{C})PD_\lambda^{-1}F_\lambda B^{1/2}J.$$

But, as above,

$$(F_\alpha - F_{\alpha-0})\varphi(\bar{C}) = \varphi(\alpha)(F_\alpha - F_{\alpha-0}) = 0.$$

As a result, $SE_\lambda = 0$.

If $\lambda > \alpha$, the relation $S(I - E_\lambda) = 0$ follows similarly.

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J. BOGNÁR
Mathematical Institute of the
Hungarian Academy of Sciences,
Reáltanoda u. 13, H-1053 Budapest,
Hungary.

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