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TYPE CLASSIFICATION OF VON NEUMANN ALGEBRAS IN THE FRAMEWORK OF TOMITA-TAKESAKI THEORY AND ARVESON'S SPECTRAL THEORY

LIANG SEN WU

The theory of traces has occupied a central place in the classical theory of von Neumann algebras, and it still plays a vital role in the modern theory of von Neumann algebras. On the other hand, the advent of Tomita-Takesaki theory changed the scope of the theory of von Neumann algebras significantly. Curiously however, Tomita-Takesaki theory was not used in the type classification of von Neumann algebras, despite its success in the structure analysis of type III von Neumann algebras. Thus, currently one must develop the projection lattice analysis for the type classification and prove the existence of a semi-finite normal trace on a semi-finite von Neumann algebra. Furthermore the existence proof of a trace is non-constructive.

The aim of the paper is to point out that the type classification of von Neumann algebras is entirely possible within the frame of Tomita-Takeski theory with help from Arveson's spectral theory of one parameter automorphism groups. At this point, one should note that Tomita-Takesaki theory can be developed from the double commutation theorem of J. von Neumann and Kaplansky's density theorem together with the standard theory of closed unbounded operators.

The existence of a faithful semi-finite normal trace is characterized by the global innerness of the modular automorphism group and the construction of a trace is explicitly given by Pedersen-Takesaki [4]. Therefore, the only missing link between Tomita-Takesaki theory and the theory of traces is the implication of the type III property in the sense of the projection lattice from the global outerness of the modular automorphism groups.

DEFINITION 1. i) We say that a von Neumann algebra $\mathfrak{M}$ is *semi-finite* if the modular automorphism group $\{\sigma_t^\varphi\}$ of a faithful semi-finite normal weight φ on $\mathfrak{M}$ is implemented by a strongly continuous one parameter unitary group $\{u(t)\}$ in $\mathfrak{M}$.

ii) We say that a von Neumann algebra $\mathfrak{M}$ is *purely infinite* if it has no non-trivial semi-finite reduced algebras.

Note that the above semi-finiteness does not depend on the choice of φ by the Connes' cocycle Radon-Nikodym theorem, which is free from the theory of trace.

We now present our main results.

THEOREM 1. i) *Every von Neumann algebra $\mathfrak{M}$ is a direct sum of a semi-finite von Neumann algebra and a purely infinite von Neumann algebra.*

ii) *If $\mathfrak{M}$ is purely infinite, then there is no non-zero finite projection in $\mathfrak{M}$.*

Assertion i) is known because it simply asserts that a fixing one parameter automorphism group splits into the direct sum of the inner part and the properly outer part, [3]. But we present a proof for completeness.

Proof. i) Represent $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$. For a non-zero $e \in \mathrm{Proj}(\mathfrak{M})$, suposse $\mathfrak{M}_e$ is semi-finite. Then choose a faithful semi-finite normal weight φ on $\mathfrak{M}_e$ and ψ on $\mathfrak{M}_{z-e}$, where z is the central support of e. Let $\rho = \varphi + \psi$ on $\mathfrak{M}_z$. Then $\sigma_t^\rho(e) = e$ and $\sigma_t^\rho \,|\, \mathfrak{M}_e = \sigma_t^\varphi$. We claim that $\{\sigma_t^\rho\}$ is inner on $\mathfrak{M}_z$. Let $\{u(t)\}$ be a one parameter unitary group in $\mathfrak{M}_e$ such that $\sigma_t^\varphi = \mathrm{Ad}(u(t))$, $t \in \mathbb{R}$. Since z is the central support of e, we know that $[\mathfrak{M}e\mathfrak{H}] = z\mathfrak{H}$. Define

$$V(t) \sum_{i=1}^{n} x_i e\xi_i = \sum_{i=1}^{n} \sigma_t^\rho(x_i)u(t)e\xi_i$$

for $x_1, \ldots, x_n \in \mathfrak{M}$ and $\xi_1, \ldots, \xi_n \in \mathfrak{H}$. We then have

$$\begin{aligned}
&\left(V(t) \sum_{i=1}^{m} x_i e\xi_i \,\middle|\, V(t) \sum_{j=1}^{n} y_j e\eta_j \right) = \\
&= \sum_{ij} (\sigma_t^\rho(x_i)u(t)e\xi_i \,|\, \sigma_t^\rho(y_j)u(t)e\eta_j) = \\
&= \sum_{ij} (e\sigma_t^\rho(y_j^* x_i)eu(t)e\xi_i \,|\, u(t)e\eta_j) = \\
&= \sum_{ij} (u(t)^*\sigma_t^\rho(ey_j^* x_i e)u(t)e\xi_i \,|\, e\eta_j) = \\
&= \sum_{ij} (u(t)^*\sigma_t^\varphi(ey_j^* x_i e)u(t)e\xi_i \,|\, e\eta_j) = \\
&= \sum_{ij} (ey_j^* x_i e\xi_i \,|\, e\eta_j) = \left(\sum_{i=1}^{m} x_i e\xi_i \,\middle|\, \sum_{j=1}^{n} y_j e\eta_j \right).
\end{aligned}$$

Here each $V(t)$ is an isometry from $\mathfrak{M}e\mathfrak{H}$ onto $\mathfrak{M}e\mathfrak{H}$, so that it can be extended to a unitary on $z\mathfrak{H}$, which we denote by $V(t)$ again. The one parameter group property of $V(t)$ and the continuity property of $V(t)$ follow now easily. It is straightforward to see that $V(t) \in \mathfrak{M}_z$.

For every $a \in \mathfrak{M}_z$, we have

$$V(t)a \sum_{i=1}^{n} x_i e\xi_i = \sum_{i=1}^{n} \sigma_t^\rho(ax_i)u(t)e\xi_i =$$

$$= \sigma_t^\rho(a) \sum_{i=1}^{n} \sigma_t^\rho(x_i)u(t)e\xi_i = \sigma_t^\rho(a)V(t) \sum_{i=1}^{n} x_i e\xi_i .$$

Thus, $\sigma_t^\rho = \mathrm{Ad}(V(t))$, $t \in \mathbf{R}$. Therefore $\mathfrak{M}_z$ is semi-finite.

Let $\{z_i\}$ be a maximal family of orthogonal central projections in $\mathfrak{M}$ such that $\mathfrak{M}_{z_i}$ is semi-finite. Then with $z = \sum z_i$, $\mathfrak{M}_z$ is semi-finite. By the maximality of $\{z_i\}$ and the above arguments, $\mathfrak{M}_{I-z}$ is purely infinite. Q.E.D.

For $a > 0$, let $E = \chi_{[a,+\infty)}$ be the characteristic function over $[a, +\infty)$. For each $x \in \mathfrak{M}$, let $x = uh = ku$ be the polar decomposition and set $u_a(x) = uE_a(h)$. We then have

$$u_a(x^*) = u^*E_a(k) = E_a(h)u^* = (uE_a(h))^* = u_a(x)^*,$$

so that

$$u_a(x^*) = u_a(x)^*, \quad a > 0.$$

LEMMA 1. *For every $\xi \in \mathfrak{H}$, we have*

$$\|x\xi\|^2 = \int_0^\infty \|u_{\sqrt{a}}(x)\xi\|^2 \, \mathrm{d}a.$$

Proof. We simply compute, with spectral decomposition $|x| = \int_0^\infty \lambda \, \mathrm{d}e(\lambda)$

$$\|x\xi\|^2 = (x^*x\xi \mid \xi) = \int_0^\infty \lambda^2 \, \mathrm{d}\|e(\lambda)\xi\|^2 =$$

$$= \int_0^\infty \lambda \, \mathrm{d}\|e(\sqrt{\lambda})\xi\|^2 = -\int_0^\infty a \mathrm{d}\|E_{\sqrt{a}}(|x|)\xi\|^2 =$$

$$= \int_0^\infty \|E_{\sqrt{a}}(|x|)\xi\|^2 \mathrm{d}a = \int_0^\infty \|uE_{\sqrt{a}}(|x|)\xi\|^2 \, \mathrm{d}a = \int_0^\infty \|u_{\sqrt{a}}\xi\|^2 \mathrm{d}a.$$

Q.E.D.

To prove the absence of a finite non-zero projection in purely infinite von Neumann algebra $\mathfrak{M}$, it suffices to prove that every σ-finite projection is infinite in the sense of the projection lattice. Thus, we may assume that $\mathfrak{M}$ is σ-finite purely infinite von Neumann algebra.

Let φ be a faithful normal state.

LEMMA 2. *For every* $\varepsilon > 0$ *and a non-zero* $e \in \mathrm{Proj}(\mathfrak{M})$ *there exists* $f \in \mathrm{Proj}(\mathfrak{M})$ *such that*

$$e \geqslant f, \quad f \lesssim e - f$$

$$\varphi(f) \geqslant (1 - \varepsilon)\varphi(e).$$

Proof. Let $p \in \mathrm{Proj}(\mathfrak{M})$, $p \neq 0$. Then Arveson's spectral theory shows that the modular operator Δ_{φ_p} for the restriction φ_p of φ on $\mathfrak{M}_p$ must be unbounded. Let $\alpha > 1$, and $\Delta_{\varphi_p} = \int_0^\infty \lambda \mathrm{d}E(\lambda)$ be the spectral decomposition.

If $x\xi_{\varphi_p} \in [1 - E(\alpha)]\,\mathfrak{H}$, then we have

$$\|x^*\xi_{\varphi_p}\|^2 = \|\Delta_{\varphi_p}^{1/2} x\xi_{\varphi_p}\|^2 = \int_0^\infty \lambda \mathrm{d}\|E(\lambda)x\xi_{\varphi_p}\|^2 =$$

$$= \int_\alpha^\infty \lambda \mathrm{d}\|E(\lambda)x\xi_{\varphi_p}\|^2 \geqslant \alpha \int_\alpha^\infty \mathrm{d}\|E(\lambda)x\xi_{\varphi_p}\|^2 = \alpha\|x\xi_{\varphi_p}\|^2.$$

Hence we get $\|x^*\xi_{\varphi_p}\|^2 \geqslant \alpha\|x\xi_{\varphi_p}\|^2$. By Lemma 1, we have

$$\int_0^\infty \|u_{\sqrt{a}}(x^*)\xi_{\varphi_p}\|^2 \,\mathrm{d}a \geqslant \alpha \int_0^\infty \|u_{\sqrt{a}}(x)\xi_{\varphi_p}\|^2 \,\mathrm{d}a.$$

Thus we must have

$$\|u_{\sqrt{a}}(x)^*\xi_{\varphi_p}\|^2 \geqslant \alpha\|u_{\sqrt{a}}(x)\xi_{\varphi_p}\|^2 > 0$$

for some $a > 0$. This means that, with $u = u_{\sqrt{a}}(x) \in \mathfrak{M}_p$, we have

$$\varphi(uu^*) \geqslant \alpha\varphi(u^*u).$$

Therefore, we conclude that for any $\alpha > 1$ and $p \in \mathrm{Proj}(\mathfrak{M})$, $p \neq 0$, there exists a pair q, r of non-zero projections in $\mathfrak{M}_p$ such that $q \sim r$ and $\varphi(q) \geqslant \alpha\varphi(r)$. By the usual exhaustion argument, we get projections s, t in $\mathfrak{M}_e$ such that $s \sim t$, $s \vee t = e$ and $\varphi(t) \geqslant \alpha\varphi(s)$. Let $f = e - s$. Then we have

$$f = s \vee t - s \sim t - s \wedge t \leqslant t \sim s$$

so that $f \lesssim s = e - f$. But

$$\varphi(f) = \varphi(e) - \varphi(s) \geqslant \varphi(e) - \frac{1}{\alpha}\varphi(t) \geqslant \left(1 - \frac{1}{\alpha}\right)\varphi(e).$$

Therefore, with $\alpha = \frac{1}{\varepsilon}$, we conclude that

$$f \lesssim e - f \quad \text{and} \quad \varphi(f) \geqslant (1 - \varepsilon)\varphi(e)$$

Q.E.D.

Proof of Theorem 1 (ii). Fix $0 < \varepsilon < 1$, and choose a projection $e(0)$ in $\mathfrak{M}$ such that $\varphi(e(0)) \geqslant 1 - \varepsilon$ and $e(0) \lesssim 1 - e(0)$. Let $e(1) \leqslant I - e(0)$ be a projection such that $e(0) \sim e(1)$. Choose a partial isometry $u(1)$ such that $u(1)^*u(1) = e(0)$ and $u(1)u(1)^* = e(1)$. Set $u(0) = e(0)$. By induction, we choose a system

$$\{u(s),\ e(s) : s = (s_1, \ldots, s_n) \in \{0, 1\}^n\}$$

of partial isometries and projections such that

a) $\varphi(e(0_n)) \geqslant (1 - \varepsilon^n)\varphi(e(0_{n-1}))$;

b) $e(s) \perp e(t), \quad s \neq t$;

c) $e((s_1, s_2, \ldots, s_{n-1})) \geqslant e(s_1, \ldots, s_{n-1}, 0) + e(s_1, \ldots, s_{n-1}, 1)$;

d) $u(s_1, s_2, \ldots, s_{n-1}, 0) = u(s_1, \ldots, s_{n-1})e(0_n), \quad u(s_1, s_2, \ldots, s_{n-1}, 1) = u(s_1, \ldots, s_{n-1})u(0_{n-1}, 1)$;

e) $u(s)^*u(s) = e(0_n), \quad u(s)u(s)^* = e(s)$.

Lemma 2 guarantees the induction steps. The projection $\{e(0_n) : n \in \mathbf{N}\}$ is decreasing and $\varphi(e(0_n)) \geqslant \prod_{k=1}^{n} (1 - \varepsilon^k)$. Hence we have $e(0_\infty) = \lim_{n\to\infty} e(0_n) \in \mathrm{Proj}(\mathfrak{M})$ and $e(0_\infty) \neq 0$.

For every $s = (s_n) \in \{0, 1\}^{\mathbf{N}}$ such that $s_k = 0$ for $k \geqslant n$, we set

$$e(s) = u(s_1, \ldots, s_n)e(0_\infty)u(s_1, \ldots, s_n)^*.$$

Then $\{e(s)\}$ is an orthogonal family of mutually equivalent projections. Therefore, the identity can not be a finite projection.

Since the above is true for every reduced algebra of $\mathfrak{M}$, $\mathfrak{M}$ has no finite non-zero projection. Q.E.D.

If $\{\sigma_t^\varphi\}$ is inner, there exists a faithful semi-finite normal trace τ on $\mathfrak{M}$, which is explicitly constructed from φ in [4].

Definition. The von Neumann algebra $\mathfrak{M}$ is *properly τ-infinite* if $\tau(z) = +\infty$ for every non-zero $z \in \mathrm{Proj}(\mathfrak{M} \cap \mathfrak{M}')$.

Theorem 2. *If $\mathfrak{M}$ is properly τ-infinite, then $\mathfrak{M}$ is properly infinite in the sense of projection lattice.*

Proof. Without loss of generality, we can suppose that $\mathfrak{M}$ is σ-finite. Because of semi-finiteness of $\mathfrak{M}$, there exists $e \in \mathrm{Proj}(\mathfrak{M})$ such that $\tau(e) < +\infty$, $z(e) = I$.

We choose $\{e_j\}_{j \in J}$ as the maximal family of disjoint mutually equivalent projections of $\mathrm{Proj}(\mathfrak{M})$ such that $e \sim e_j$.

Suppose $\mathfrak{M}$ were not properly-infinite in the sense of projection lattice. Then J is a finite set, say $J = \{1, 2, \ldots, n\}$, otherwise, I will be properly infinite. Let $f = \sum_{j=1}^{n} e_j$. By the maximality of $\{e_j\}_{j \in J}$

$$e \nprec I - f.$$

By comparability theorem, there exists $p \in \mathfrak{Z}$, $p \neq 0$,

$$ep \succsim (I - f)p$$

$$p = \sum_{j=1}^{n} e_j p + (I - f)p,$$

$$\tau(p) = \sum_{j=1}^{n} \tau(e_j p) + \tau((I - f)p) \leqslant$$

$$\leqslant n\tau(e) + \tau(e) < +\infty,$$

which contradicts properly τ-infiniteness of $\mathfrak{M}$. Q.E.D.

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LIANG SEN WU
Department of Mathematics,
East China Normal University,
Shanghai, 200062,
People's Republic of China.

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