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SUFFICIENT CONDITIONS FOR MEMBERSHIP IN THE CLASSES A AND $A_{\aleph_0}$

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INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$.

This paper deals with some recent aspects of the theory of ultraweakly closed subalgebras of $\mathscr{L}(\mathscr{H})$ (dual algebras). We shall give some new criteria for membership in the classes A and $A_{\aleph_0}$, appearing in the theory of dual algebras. These classes of operators, as well as the classes A_n, $n \geqslant 1$, were introduced in [2] and studied in an extensive literature; the paper [3] is a standard reference for the situation until 1985. The class $A(\mathscr{H})$ is related to the invariant subspace problem for those contractions T in $\mathscr{L}(\mathscr{H})$ for which $\sigma(T) \supset \mathbf{T}$.

In [2] it was conjectured that $A = A_1$ and there is a hope that this conjecture is true. Thus it is important to find criteria for membership in the class $A(\mathscr{H})$.

On the other hand, operators in the class $A_{\aleph_0}$ have some remarkable properties (for example, they are reflexive and have a rich invariant subspace lattice (see [3])). Various criteria for membership in the class $A_{\aleph_0}$ were established, but few criteria that an absolutely continuous contraction belongs to $A(\mathscr{H})$ are known.

The content of the paper is the following.

In the first section we recall some notations and basic definitions from the theory of dual algebras.

In the second section a certain growth condition on the resolvent of a contraction is reviewed and it is shown that this condition ensures the membership in the class $A(\mathscr{H})$. This result improves similar ones obtained in [1] and [5].

In the last section, a new criterium that a dual algebra has property $(A_{\aleph_0})$ is given. As a corollary, we obtain an improvement of [3, Lemma 7.6].

1. PRELIMINARIES

In this section we shall recall some definitions from the theory of dual algebras.

Let $C_1(\mathscr{H})$ denote the Banach space of trace-class operators on $\mathscr{H}$ equipped with the trace-norm $\|\ \|_1$. Then $\mathscr{L}(\mathscr{H})$ is identified with the dual space of $C_1(\mathscr{H})$, via the bilinear map

$$\langle T, L\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{L}(\mathscr{H}),\ L \in C_1(\mathscr{H}).$$

A weak*-closed subalgebra of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ is called a *dual algebra*. If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra, then $\mathscr{A}$ is identified with the dual space of the Banach space $Q_{\mathscr{A}} = C_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, via the bilinear map

$$\langle T, [L]\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{A},\ [L] \in Q_{\mathscr{A}}.$$

If x and y are vectors from $\mathscr{H}$, then $x\otimes y$ denotes the rank-one operator

$$(x\otimes y)(z) = (z, y)x \quad z \in \mathscr{H}.$$

Suppose that $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and n is a cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$. Then $\mathscr{A}$ is said to have property (A_n) if for every system $\{[L_{ij}];\ 0 \leqslant i, j < n\}$ of elements from $Q_{\mathscr{A}}$ there exist vectors $\{x_i, y_j;\ 0 \leqslant i, j < n\}$ in $\mathscr{H}$ such that

$$[L_{ij}] = [x_i\otimes y_j], \quad 0 \leqslant i, j < n.$$

Let $\mathbf{D} = \{z \in \mathbf{C};\ |z| < 1\}$ and $\mathbf{T} = \partial\mathbf{D}$. A subset $S \subset \mathbf{D}$ is said to be *dominating* (for $\mathbf{T}$) if almost every point of $\mathbf{T}$ is a nontangential limit of a sequence of points from S.

As usual, we denote by H^∞ the Banach algebra of all bounded analytic functions on $\mathbf{D}$.

For an absolutely continuous contraction T in $\mathscr{L}(\mathscr{H})$, the dual algebra generated by T in $\mathscr{L}(\mathscr{H})$ will be denoted by $\mathscr{A}_T$ and the predual of $\mathscr{A}_T$ will be denoted by Q_T.

The class $\mathbf{A}(\mathscr{H})$ consists of all absolutely continuous contractions in $\mathscr{L}(\mathscr{H})$ for which the Sz.-Nagy—Foiaş functional calculus Φ_T is an isometry. For such T, Φ_T is a weak* homeomorphism between H^∞ and $\mathscr{A}_T$ and for every $\lambda \in \mathbf{D}$, there exists a unique element $[C_\lambda]$ in Q_T such that

$$\langle \Phi_T(f), [C_\lambda]\rangle = f(\lambda) \quad \forall f \in H^\infty.$$

If n is a cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$ then $\mathbf{A}_n$ denotes the class of all $T \in \mathbf{A}$ for which $\mathscr{A}_T$ has property (A_n).

2. THE CLASS $A(\mathscr{H})$

If T is a contraction in $\mathscr{L}(\mathscr{H})$ and $0 < \theta < 1$, then we put

$$\zeta_\theta(T) = (\mathbf{D} \cap \sigma(T)) \cup \left\{\lambda \in \mathbf{D} \setminus \sigma(T) : \theta\|(T-\lambda)^{-1}\| > \frac{1}{1-|\lambda|}\right\}.$$

These sets were introduced in [1], where it was shown that if T is completely nonunitary and if all these sets are dominating for $\mathbf{T}$, then $T \in \mathbf{A}$.

More recently, it was shown in [5] that if there exists some θ, satisfying $0 < \theta < 1/2$, such that $\zeta_\theta(T)$ is dominating for $\mathbf{T}$, then $T \in \mathbf{A}$. Our improvement of these results is the following:

THEOREM 2.1. *Suppose T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ such that for some θ satisfying $0 < \theta < 1$, the set $\zeta_\theta(T)$ is dominating for* $\mathbf{T}$. *Then* $T \in \mathbf{A}$.

Proof. Fix a nonconstant f in H^∞ such that $\|f\|_\infty = 1$ and take a sequence $(\lambda_n)_{n=1}^\infty \subset \zeta_\theta(T)$ such that $\lim_{n\to\infty} |f(\lambda_n)| = \|f\|_\infty = 1$.

If $\lambda_n \in \sigma(T) \cap \mathbf{D}$ for all $n \geqslant 1$, then one knows from [6] that $f(\lambda_n) \in \sigma(f(T))$ for all $n \geqslant 1$, hence $1 = \lim_{n\to\infty} |f(\lambda_n)| \leqslant \|f(T)\| \leqslant \|f\|_\infty = 1$. Thus we may suppose, by taking a subsequence, that $\lambda_n \in \zeta_\theta(T) \setminus \sigma(T)$ and $f(\lambda_n) \in \mathbf{D} \setminus \sigma(f(T))$, for all $n \geqslant 1$.

It follows from the invariant form of Schwartz's lemma ([7], Chapter I) that we have

$$\left|\frac{f(z) - f(\lambda_n)}{1 - \overline{f(\lambda_n)}f(z)}\right| \leqslant \left|\frac{z - \lambda_n}{1 - \bar{\lambda}_n z}\right|,$$

for all $n \geqslant 1$ and $z \in \mathbf{D}$. Hence there exist functions $h_n \in H^\infty$, $\|h_n\|_\infty \leqslant 1$, such that

$$\frac{f(z) - f(\lambda_n)}{1 - \overline{f(\lambda_n)}f(z)} = h_n(z) \frac{z - \lambda_n}{1 - \bar{\lambda}_n z}, \qquad z \in \mathbf{D},\ n \geqslant 1.$$

Since the functional calculus is a norm-decreasing algebra homomorphism, it follows easily that

$$(f(T) - f(\lambda_n))^{-1}(I - \overline{f(\lambda_n)}f(T))h_n(T) = (T - \lambda_n)^{-1}(I - \bar{\lambda}_n T).$$

By a short calculation (see [9], p. 263), we obtain

$$\frac{1}{\theta} < \|(T-\lambda_n)^{-1}\|(1-|\lambda_n|) \leqslant \|(T-\lambda_n)^{-1}(I-\bar{\lambda}_n T)\| \leqslant$$

$$\leqslant \|(f(T)-f(\lambda_n))^{-1}(I-\overline{f(\lambda_n)}f(T))\| \leqslant$$

$$\leqslant 1 + 2(1-|f(\lambda_n)|)\|(f(T)-f(\lambda_n))^{-1}\|$$

hence

$$\|(f(T)-f(\lambda_n))^{-1}\| > \frac{1-\theta}{2\theta(1-|f(\lambda_n)|)}, \quad \text{for all } n \geqslant 1.$$

Thus $\sigma(f(T)) \cap \mathbf{T} \neq \emptyset$ and the proof is complete.

REMARK. The proof works equally if we replace $\mathscr{L}(\mathscr{H})$ by an arbitrary unital Banach algebra $\mathscr{B}$ and $f \to f(T)$ by a norm contractive unital homomorphism $\varphi: H^\infty \to \mathscr{B}$ with the property that for some $0 < \theta < 1$, the set:

$$\zeta_\theta(\varphi) = (\mathbf{D} \cap \sigma(x)) \cup \left\{\lambda \in \mathbf{D}\setminus\sigma(x)\,;\, \theta\|(x-\lambda)^{-1}\| > \frac{1}{1-|\lambda|}\right\}$$

is dominating for $\mathbf{T}$. (Here $x = \varphi(z)$.)

Now suppose that T is a given completely non-unitary contraction in $\mathscr{L}(\mathscr{H})$, write $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$ and define the subspaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ of $\mathscr{H}$ to be the closures of the ranges of D_T and D_{T^*} respectively.

The analytic function Θ_T defined on $\mathbf{D}$ by

$$\Theta_T(\lambda) = (-T + \lambda D_{T^*}(I - \lambda T^*)^{-1}D_T)\,|\,\mathscr{D}_T, \quad \lambda \in \mathbf{D}$$

satisfies $\|\Theta_T\|_\infty \leqslant 1$ (cf. [9], p. 238) and the contractive analytic function $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T\}$ is called the characteristic function of T.

THEOREM 2.2. *Suppose T is a completely non-unitary contraction in $\mathscr{L}(\mathscr{H})$ and $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T\}$ is its characteristic function. If there exists θ, satisfying $0 < \theta < 1$, such that the following set*

$$\zeta'_\theta(T) = (\mathbf{D} \cap \sigma(T)) \cup \{\lambda \in \mathbf{D}\setminus\sigma(T)\,;\, \theta\|\Theta_T(\lambda)^{-1}\| > 1\}$$

is dominating for $\mathbf{T}$, then $T \in \mathbf{A}$.

Proof. It follows from ([9], p. 259) that $\lambda \in \mathbf{D}\setminus\sigma(T)$ if and only if $\Theta_T(\lambda)$ is boundedly invertible, hence the definition of the set $\zeta_0'(T)$ is consistent. Now, if $\lambda \in \zeta_0'(T)\setminus\sigma(T)$ then we have (cf. [9], p. 263):

$$\|\Theta_T(\lambda)^{-1}\| = \|(T-\lambda)^{-1}(I-\bar{\lambda}T)\|$$

and the proof goes like in the above theorem.

This last result offers a method to construct various examples of contractions in **A**, by using the functional model associated with a given contractive analytic function.

3. SUFFICIENT CONDITIONS FOR MEMBERSHIP IN $A_{\aleph_0}$

In this section we shall give a new sufficient condition that a given dual algebra $\mathscr{A}$ have property $(A_{\aleph_0})$.

The main result is the following:

THEOREM 3.1. *Suppose $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and that there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathscr{H}$ such that*

$$\alpha)\qquad \sup_{n>N} |(Tx_n, y_n)| = \|T\|, \quad T\in\mathscr{A},\ N\in\mathbf{N},$$

and

$$\beta)\qquad \|[x_n\otimes x_m]\|,\ \|[y_n\otimes y_m]\|,\ \|[x_n\otimes y_m]\| < \frac{1}{2^{n+m}}, \quad n, m\in\mathbf{N},\ n\neq m.$$

Suppose also that $\{[L_{ij}]\}_{i,j\geq 1}$ is a given doubly indexed family of elements of $Q_{\mathscr{A}}$ and $\varepsilon > 0$. Then there exist sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ from $\mathscr{H}$ satisfying

$$\langle T, [L_{ij}]\rangle = \langle T, [u_i\otimes v_j]\rangle, \quad T\in A,\ 1\leq i, j<\infty.$$

If, moreover, $\sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty$, $j\geq 1$, and $\sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty$, $i\geq 1$, then the above sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ can be chosen to satisfy

$$\|u_i\| < \left(\sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} + \varepsilon\right), \quad 1\leq i<\infty$$

and

$$\|v_j\| < \left(\sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} + \varepsilon\right), \quad 1\leq j<\infty.$$

In particular, $\mathscr{A}$ has property $(A_{\aleph_0})$.

Before proving this theorem, we need two lemmas that are similar with ([8], Lemmas 3.9 and 3.10). If $\mathcal{S} \subset \mathcal{H}$, we denote by span($\mathcal{S}$) the set of all finite linear combinations of elements from $\mathcal{S}$.

LEMMA 3.2. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra with properties α) and β). Let $N > 0$ and suppose $u_i \in \mathrm{span}\{x_n ;\ n \geqslant 1\}$, $v_j \in \mathrm{span}\{y_n ;\ n \geqslant 1\}$ and $\{[L_{ij}]\}_{1 \leqslant i, j \leqslant N} \subset Q_{\mathcal{A}}$.*
Assume that

$$\|[u_i \otimes v_j] - [L_{ij}]\| < \varepsilon_{ij}, \quad 1 \leqslant i, j \leqslant N.$$

Let $1 \leqslant i_0, j_0 \leqslant N$ and let $0 < \delta < \varepsilon_{i_0 j_0}$. Then there exist $u'_{i_0} \in \mathrm{span}\{x_n ; n \geqslant 1\}$ and $v'_{j_0} \in \mathrm{span}\{y_n ;\ n \geqslant 1\}$ such that

A) $\|[u'_{i_0} \otimes v'_{j_0}] - [L_{i_0 j_0}]\| < \delta$,

B) $\|[u_i \otimes v'_{j_0}] - [L_{i j_0}]\| < \varepsilon_{i j_0}$ *for each* i,

C) $\|[u'_{i_0} \otimes v_j] - [L_{i_0 j}]\| < \varepsilon_{i_0 j}$ *for each* j,

D) $\|u'_{i_0} - u_{i_0}\|^2 < \varepsilon_{i_0 j_0}$,

E) $\|v'_{j_0} - v_{j_0}\|^2 < \varepsilon_{i_0 j_0}$.

Proof. Let $[K] = [L] - [u_{i_0} \otimes v_{j_0}]$. Set $d = \|[K]\|$.

We may assume that $d > 0$ since otherwise we can simply take $u'_{i_0} = u_{i_0}$ and $v'_{j_0} = v_{j_0}$.

Let

$$0 < \rho < \min_{1 \leqslant i, j \leqslant N} \left(\frac{\delta}{4}, \ \varepsilon_{ij} - \|[L_{ij}] - [u_i \otimes v_j]\| \right).$$

It follows from property β) of $\mathcal{A}$, that we may may choose $m \in \mathbf{Z}^+$ large enough such that

$$1 < 2^m \rho \quad \text{and} \quad \max_{1 \leqslant i, j \leqslant N} (\|[x_n \otimes v_j]\|, \|[u_i \otimes y_n]\|) < \frac{\rho}{2^{n-m}}$$

for all $n > m$.

Since $\mathcal{A}$ has also property α), it follows from ([3], Proposition 1.21) that the closed absolutely convex hull of the set $\{[x_n \otimes y_n];\ n > m\}$ equals the closed unit ball in $Q_{\mathcal{A}}$. We may therefore choose $k \in \mathbf{Z}^+, \alpha_1, \ldots, \alpha_k \in \mathbf{C}$ such that $\sum_{i=1}^{k} |\alpha_i| \leqslant 1$

and

$$\left\| d^{-1}[K] - \sum_{i=1}^{k} \alpha_i [x_{m+i} \otimes y_{m+i}] \right\| < \frac{\delta}{4d}.$$

Choose $\gamma_1, \ldots, \gamma_k \in \mathbf{C}$ such that $\gamma_i^2 = \alpha_i d$, $1 \leqslant i \leqslant k$. Set $s = \sum_{i=1}^{k} \gamma_i x_{m+i}$ and $t = \sum_{i=1}^{k} \gamma_i y_{m+i}$. We claim that we may take

$$u'_{i_0} = u_{i_0} + s \quad \text{and} \quad v'_{j_0} = v_{j_0} + t.$$

First, observe that

$$\|(u_{i_0} + s) - u_{i_0}\|^2 = \sum_{i=1}^{k} \left\| \gamma_i x_{m+i} \right\|^2 \leqslant$$

$$\leqslant \sum_{i=1} |\gamma_i|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} |\gamma_i \gamma_j| \, |(x_{m+i}, x_{m+j})| < d + \rho < \varepsilon_{i_0 j_0}$$

and likewise $\|(v_{j_0} + t) - v_{j_0}\|^2 < \varepsilon_{i_0 j_0}$.

Next, we show that we may satisfy condition A. We have:

$$\|[(u_{i_0} + s) \otimes (v_{j_0} + t)] - [L_{i_0 j_0}]\| \leqslant$$

$$\leqslant \|[u_{i_0} \otimes v_{j_0}] - [L_{i_0 j_0}] + [s \otimes t]\| + \|[s \otimes v_{j_0}]\| + \|[u_{i_0} \otimes t]\| \leqslant$$

$$\leqslant \left\| \left[\sum_{i=1}^{k} \gamma_i x_{m+i} \otimes \sum_{j=1}^{k} \bar{\gamma}_j y_{m+j} \right] - [K] \right\| + \|[s \otimes v_{j_0}]\| + \|[u_{i_0} \otimes t]\| \leqslant$$

$$\leqslant \left\| \sum_{i=1}^{k} \gamma_i^2 [x_{m+i} \otimes y_{m+i}] - [K] \right\| + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} |\gamma_i \gamma_j| \, \|[x_{m+i} \otimes y_{m+j}]\| +$$

$$+ \sum_{i=1}^{k} |\gamma_i| \, \|[x_{m+i} \otimes y_{j_0}]\| + \sum_{i=1}^{k} |\gamma_i| \, \|[u_{i_0} \otimes y_{m+i}]\| < \frac{\delta}{4} + 3\rho < \delta.$$

Finally, note that

$$\|[(u_{i_0} + s) \otimes v_j] - [L_{i_0 j}]\| \leqslant \|[u_{i_0} \otimes v_j] - [L_{i_0 j}]\| +$$

$$+ \sum_{i=1}^{k} |\gamma_i| \, \|[x_{m+i} \otimes v_j]\| < \varepsilon_{i_0 j} \quad \text{for each } j$$

and similarly

$$\|[u_i \otimes (v_{j_0} + t)] - [L_{ij_0}]\| < \varepsilon_{ij_0} \quad \text{for each } i.$$

The proof is complete.

By N^2 successive applications of the preceding lemma we immediately obtain the following result:

LEMMA 3.3. *Let $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ be a dual algebra which satisfies the hypothesis of Theorem* 4.1.

Let $N > 0$, $u_i \in \operatorname{span}\{x_n ; n \geqslant 1\}$, $v_j \in \operatorname{span}\{y_n : n \geqslant 1\}$ *and* $\{[L_{ij}]\}_{1 \leqslant i,j \leqslant N} \subset Q_{\mathscr{A}}$. *Assume that*

$$\|[u_i \otimes v_j] - [L_{ij}]\| < \varepsilon_{ij} \quad 1 \leqslant i, j \leqslant N$$

and let $0 < \delta_{ij} < \varepsilon_{ij}$ *for* $1 \leqslant i, j \leqslant N$.

Then there exist $u_i' \in \operatorname{span}\{x_n ; n \geqslant 1\}$ *and* $v_j' \in \operatorname{span}\{y_n ; n \geqslant 1\}$ *such that*

A′) $\|[u_i' \otimes v_j'] - [L_{ij}]\| < \delta_{ij}$ *for all i, j,*

B′) $\|u_i' - u_i\| < \sum_{j=1}^{N} (\varepsilon_{ij})^{1/2}$ *for all i,*

C′) $\|v_j' - v_j\| < \sum_{i=1}^{N} (\varepsilon_{ij})^{1/2}$ *for all j.*

Proof of Theorem 3.1. First, we suppose that $\{[L_{ij}]\} \subset Q_{\mathscr{A}}$, $i,j \geqslant 1$ satisfy

$$\sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty \quad \text{for all } j \geqslant 1$$

and

$$\sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} < \infty \quad \text{for all } i \geqslant 1.$$

Set $u_i^{(0)} = v_j^{(0)} = 0$ for all $i, j \geqslant 1$.

We choose by induction sequences $\{u_i^{(k)}\}_{k=0}^{\infty}$ and $\{v_j^{(k)}\}_{k=0}^{\infty}$ $(i,j \geqslant 1)$ in $\mathscr{H}$, such that

a) $\|[u_i^{(k)} \otimes v_j^{(k)}] - [L_{ij}]\| < \dfrac{1}{k^2 \cdot 2^{2(k+2)}}$ $\quad 1 \leqslant i, j \leqslant k,$

b) $u_i^{(k)} = v_j^{(k)} = 0$ for $i > k$ and $j > k$,

c) $\|u_i^{(k)} - u_i^{(k-1)}\| < \|[L_{ik}]\|^{1/2} + \dfrac{1}{2^k}$ for $1 \leqslant i < k$,

d) $\|u_k^{(k)} - u_k^{(k-1)}\| < \sum_{j=1}^{k} \|[L_{kj}]\|^{1/2} + \frac{1}{2^k}$,

e) $\|v_j^{(k)} - v_j^{(k-1)}\| < \|[L_{kj}]\|^{1/2} + \frac{1}{2^k}$ for $1 \leqslant j < k$,

f) $\|v_k^{(k)} - v_k^{(k-1)}\| < \sum_{i=1}^{k} \|[L_{ik}]\|^{1/2} + \frac{1}{2^k}$.

Suppose for the moment that these sequences $\{u_i^{(k)}\}$ and $\{v_j^{(k)}\}$ $(i, j \geqslant 1)$ have been constructed to satisfy the above inequalities.

Let $i \geqslant 1$. Then:

$$\sum_{k=1}^{\infty} \|u_i^{(k)} - u_i^{(k-1)}\| = \sum_{k=1}^{i-1} \|u_i^{(k)} - u_i^{(k-1)}\| + \|u_i^{(i-1)} - u_i^{(i)}\| +$$

$$+ \sum_{k=i+1}^{\infty} \|u_i^{(k)} - u_i^{(k-1)}\| < 0 + \sum_{j=1}^{i} \|[L_{ij}]\|^{1/2} + \frac{1}{2^i} +$$

$$+ \sum_{k=i+1}^{\infty} \left(\|[L_{ik}]\|^{1/2} + \frac{1}{2^k} \right) = \sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} + \frac{1}{2^{i-1}}.$$

It follows that the sequence $\{u_i^{(k)}\}_{k=0}^{\infty}$ is norm convergent to some $u_i \in \mathscr{H}$, with

$$\|u_i\| < \sum_{j=1}^{\infty} \|[L_{ij}]\|^{1/2} + \frac{1}{2^{i-1}}.$$

Similarly, for each j, the sequence $\{v_j^{(k)}\}_{k=0}^{\infty}$ converges in norm to some $v_j \in \mathscr{H}$ and $\|v_j\| < \sum_{i=1}^{\infty} \|[L_{ij}]\|^{1/2} + 1/2^{j-1}$. Finally, from a) we deduce that

$$[u_i \otimes v_j] = [L_{ij}] \quad i, j \geqslant 1.$$

Thus, to complete the proof in this case, it suffices to construct, by induction, the sequences $\{u_i^{(k)}\}_{k=0}^{\infty}$ and $\{v_j^{(k)}\}_{k=0}^{\infty}$.

Let $n \geqslant 0$. Suppose that $\{u_i^{(0)}, \dots, u_i^{(n)}\}$ and $\{v_j^{(0)} \dots, v_j^{(n)}\}$ have been constructed for every $i, j \geqslant 1$ to satisfy a) — f) for the appropriate values of k, and the further induction hypothesis that each $u_i^{(k)}$ for $0 \leqslant k \leqslant n$ and $i \geqslant 1$ belongs to $\text{span}\{x_n : n \geqslant 1\}$ (i.e., is some finite linear combination of vectors in the sequence $\{x_n\}$) and similarly for the $v_j^{(k)}$ relative to the sequence $\{y_n\}$. By Lemma 3.3 (with $N = n + 1$), there exist $u_i^{(n+1)} \in \text{span}\{x_k, k \geqslant 1\}$ and $v_j^{(n+1)} \in \text{span}\{y_k, k \geqslant 1\}$ $(1 \leqslant i, j \leqslant n + 1)$, such that

$$\|[u_i^{(n+1)} \otimes v_j^{(n+1)}] - [L_{ij}]\| < \frac{1}{(n+1)^2 \cdot 2^{2(n+3)}},$$

$$\|u_i^{(n+1)} - u_i^{(n)}\| < \sum_{j=1}^{n+1} \|[u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}]\|^{1/2} + \frac{1}{2^{n+2}},$$

$$\|v_j^{(n+1)} - v_j^{(n)}\| < \sum_{i=1}^{n+1} \|[u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}]\|^{1/2} + \frac{1}{2^{n+2}}.$$

Let $1 \leqslant i < n+1$. Then we have:

$$\|u_i^{(n+1)} - u_i^{(n)}\| \leqslant \sum_{j=1}^{n+1} \|[u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}]\|^{1/2} + \frac{1}{2^{n+2}} \leqslant$$

$$\leqslant \sum_{j=1}^{n} \frac{1}{n \cdot 2^{n+2}} + \|[L_{i,n+1}]\|^{1/2} + \frac{1}{2^{n+2}} \leqslant \|[L_{i,n+1}]\|^{1/2} + \frac{1}{2^{n+1}}$$

and similarly

$$\|v_j^{(n+1)} - v_j^{(n)}\| < \|[L_{n+1,j}]\|^{1/2} + \frac{1}{2^{n+1}} \quad \text{for } 1 \leqslant j < n+1.$$

Since $u_{n+1}^{(n)} = v_{n+1}^{(n)} = 0$, we obtain

$$\|u_{n+1}^{(n+1)} - u_{n+1}^{(n)}\| < \sum_{j=1}^{n+1} \|[u_{n+1}^{(n)} \otimes v_j^{(n)}] - [L_{n+1,j}]\|^{1/2} + \frac{1}{2^{n+2}} \leqslant$$

$$\leqslant \sum_{j=1}^{n+1} \|[L_{n+1,j}]\|^{1/2} + \frac{1}{2^{n+1}}$$

and

$$\|v_{n+1}^{(n+1)} - v_{n+1}^{(n)}\| < \sum_{i=1}^{n+1} \|[L_{i,n+1}]\|^{1/2} + \frac{1}{2^{n+1}}.$$

Now, set $u_i^{(n+1)} = v_j^{(n+1)} = 0$ for all $i, j > n+1$. Thus we have constructed by induction the required sequences.

If $\{[L_{ij}]\}_{i,j \geqslant 1} \subset Q_{\mathcal{A}}$ do not satisfy the above conditions, we can proceed like in the proof of ([8], Theorem 3.14) and the proof is complete.

COROLLARY 4.4. *Suppose $T \in \mathbf{A}$, and that there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathscr{H}$ such that*

$$\alpha_1) \qquad \sup_{n>N} |(f(T)x_n, y_n)| = \|f\|_{\infty}, \quad f \in H^{\infty}, \ N \in \mathbf{N}$$

and

$$\beta_1) \qquad \|[x_n \otimes x_m]\|, \ \|[y_n \otimes y_m]\|, \ \|[x_n \otimes y_m]\| < \frac{1}{2^{n+m}}$$

for all $n, m \in \mathbf{N}$, $n \neq m$.

Then $T \in \mathbf{A}_{\aleph_0}$.

Proof. Just apply Theorem 3.1, with $\mathscr{A} = \mathscr{A}_T$.

REMARK. Corollary 3.4 generalizes a similar result obtained in [3], Lemma 7.6.

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