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UNITARY EXTENSIONS AND POLAR DECOMPOSITIONS IN A C^*-ALGEBRA

GERT K. PEDERSEN

INTRODUCTION

If T is a bounded operator on some Hilbert space $\mathscr{H}$, it has a polar decomposition $T=V|T|$ in $\mathbf{B}(\mathscr{H})$. The partial isometry V relates the operators $|T|$ $(=(T^*T)^{1/2})$ and $|T^*|$ $(=(TT^*)^{1/2}=V|T|V^*)$, and in particular it implements the Murray-von Neumann equivalence between the spectral projections $E(\Lambda)$ and $F(\Lambda)$ for $|T|$ and $|T^*|$, respectively, corresponding to any Borel set Λ in $\mathbf{R}_+$ not containing zero. For the set $\{0\}$ the operator V gives no information, and we have to use other means, such as the index of T, to relate the kernels of T and T^*.

If now Λ is taken to be an interval $[0, \delta]$, then clearly the spectral projections $E(\Lambda)$ and $F(\Lambda)$ will have a greater chance of becoming equivalent, the greater δ is. Indeed, $E(\Lambda) = F(\Lambda) = I$ if $\delta \geqslant \|T\|$. It was shown by Olsen in [4, Section 3] that if T is an element in a von Neumann algebra $\mathfrak{A}$ of operators on $\mathscr{H}$, then the infimum of numbers δ such that

$$(*) \qquad E([0, \delta]) \sim F([0, \delta])$$

(equivalence in $\mathfrak{A}$) is precisely the distance from T to the group $\mathrm{GL}(\mathfrak{A})$ of invertible elements in $\mathfrak{A}$. Equivalently, $\mathrm{dist}(T, \mathrm{GL}(\mathfrak{A}))$ is the infimum of numbers δ for which the partial isometry $VE(]\delta, \|T\|])$ has a unitary extension in $\mathfrak{A}$, i.e.

$$(**) \qquad UE(]\delta, \|T\|]) = VE(]\delta, \|T\|]),$$

for some unitary U in $\mathfrak{A}$. To see that the first condition implies the seemingly stronger second condition, note that if $E([0, \delta])$ and $F([0, \delta])$ are equivalent by a partial isometry W in $\mathfrak{A}$, then

$$U = W + VE(]\delta, \|T\|])$$

will be a unitary in $\mathfrak{A}$ satisfying $(**)$.

The object of this note is to establish Olsen's result in a C^*-algebraic setting. This statement requires careful explanation; because if T is an element of a C^*-algebra $\mathfrak{A}$ of operators on $\mathscr{H}$, we do not, in general, expect either the partial isometry V from its polar decomposition, or the spectral projections from $|T|$ and $|T^*|$ to belong to $\mathfrak{A}$. To be precise we shall therefore say that a partial isometry V_0 in $\mathbf{B}(\mathscr{H})$ has a *unitary extension in* $\mathfrak{A}$, if

$$V_0 = UV_0^*V_0 \ (= V_0V_0^*U)$$

for some unitary U in $\mathfrak{A}$. In this strong form the second condition, above, makes sense; and we propose to show that $\operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$ is precisely the infimum of δ's for which $(**)$ is satisfied.

The key to our proof is an ingenious result by Rørdam, reproduced here for the reader's convenience as Lemma 1. Rørdam uses it in [7] to complete the theory of convex combinations of unitary operators, started in [3] and developed with special emphasis on von Neumann algebras in [5]. He shows that the formulas and results from [5] can be carried over to any C^*-algebra. The only modifications arise from the fact that in von Neumann algebras one has the miraculous appearance of unitary approximations, established by Olsen in [4]. Thus some formulas, which in a von Neumann algebra are stated with a "$\leqslant$", will in a C^*-algebra only be valid with a "$<$".

NOTATION

Let T be an element in a unital C^*-algebra $\mathfrak{A}$ of operators on some Hilbert space $\mathscr{H}$. We have the polar decomposition $T = V|T|$, where $|T| = (T^*T)^{1/2} \in \mathfrak{A}$ and $V \in \mathfrak{A}''$. In general, $V \notin \mathfrak{A}$, but we know that $Vf(|T|) \in \mathfrak{A}$ for every f in $C(\mathrm{sp}(|T|))$ with $f(0) = 0$ (cf. [1, Lemma 2.1]). Let $E(\lambda)$ and $F(\lambda)$ denote the spectral measures for $|T|$ and $|T^*|$, respectively, and for notational convenience put

$$E_\lambda = E(]\lambda, \|T\|]), \quad F_\lambda = F(]\lambda, \|T\|]).$$

Note that $VE_\lambda = F_\lambda V$ for every $\lambda > 0$ and, more generally, $Vf(|T|) = f(|T^*|)V$ for every Borel function f on $\mathbf{R}$.

For fixed numbers β and γ, $0 < \gamma < \beta$, we shall need the continuous functions f and g on $\mathbf{R}_+$ given by

$$f(t) = \begin{cases} 0 & 0 \leqslant t \leqslant \gamma \\ (t-\gamma)(\beta-\gamma)^{-1}, & \gamma \leqslant t \leqslant \beta; \\ 1 & \beta \leqslant t \end{cases} \quad g(t) = \begin{cases} \gamma^{-1}, & 0 \leqslant t \leqslant \gamma \\ t^{-1}, & \gamma \leqslant t \end{cases}.$$

RESULTS

LEMMA 1. (Rørdam). *If* $A \in \mathrm{GL}(\mathfrak{A})$ *with* $\|T - A\| < \gamma$, *then the element*

$$B = g^{-1}(|T^*|)A^{*-1}(1-f)(|T|) + Vf(|T|)$$

belongs to $\mathrm{GL}(\mathfrak{A})$ *and satisfies* $BE_\beta = VE_\beta$.

Proof. Clearly $B \in \mathfrak{A}$, and since $f(t) = 1$ for $t \geqslant \beta$ it follows that $BE_\beta = VE_\beta$. To show that B is invertible, consider the element

$$C = f(|T|) - A^*Vg(|T|)f(|T|).$$

Noting that $f(|T|) = |T|V^*Vg(|T|)f(|T|)$, we see that

$$C = (|T|V^* - A^*)Vg(|T|)f(|T|) = (T^* - A^*)V(fg)(|T|).$$

Consequently

$$\|C\| \leqslant \|T^* - A^*\| \, \|fg\|_\infty \leqslant \|T - A\|\gamma^{-1} < 1.$$

It follows that $I - C \in \mathrm{GL}(\mathfrak{A})$. Finally,

$$A^*g(|T^*|)B = I - f(|T|) + A^*g(|T^*|)\,Vf(|T|) =$$

$$= I - f(|T|) + A^*V(fg)(|T|) = I - C;$$

which proves that B, being a quotient of invertible elements, is itself invertible.

LEMMA 2. *If* $BE_\beta = VE_\beta$ *for some* B *in* $\mathrm{GL}(\mathfrak{A})$, *then* $F_\beta B^{*-1} = F_\beta V\ (= VE_\beta)$.

Proof. We have $E_\beta = B^{-1}VE_\beta$, whence

$$B^{-1}F_\beta = B^{-1}F_\beta VV^* = B^{-1}VE_\beta V^* = E_\beta V^*.$$

Thus $F_\beta B^{*-1} = VE_\beta = F_\beta V$.

LEMMA 3. *If* $\beta > \gamma > \mathrm{dist}(T, \mathrm{GL}(\mathfrak{A}))$, *and if* $A \in \mathrm{GL}(\mathfrak{A})$ *with* $\|T - A\| < \gamma$; *then with* B *as in Lemma* 1 *and with* $h(t) = (t - \beta) \vee 0$ *we have for every* $\varepsilon > 0$ *that*

$$A_0 = B(h + \varepsilon)(|T|) \in \mathrm{GL}(\mathfrak{A}),$$

and $\|T - A_0\| \leqslant \beta + \varepsilon\|B\|$.

Proof. Since $B \in GL(\mathfrak{A})$ and $h \geqslant 0$ it follows that $A_0 \in GL(\mathfrak{A})$. Moreover, as $h(|T|) = E_\beta h(|T|)$, we have

$$T - A_0 = V|T| - Bh(|T|) - B\varepsilon =$$

$$= V|T| - Vh(|T|) - \varepsilon B = V(|T| - h(|T|)) - \varepsilon B,$$

whence $\|T - A_0\| \leqslant \beta + \varepsilon\|B\|$.

LEMMA 4. *For every* $\delta > \operatorname{dist}(T, GL(\mathfrak{A}))$ *there is an element* B_0 *in* $GL(\mathfrak{A})$ *such that*

$$B_0 E_\delta = F_\delta B_0 = F_\delta V = V E_\delta .$$

Proof. Choose β and γ with $\operatorname{dist}(T, GL(\mathfrak{A})) < \beta < \gamma < \delta$. Then take A_0 as in Lemma 3 for an appropriate choice of A in $GL(\mathfrak{A})$ and $\varepsilon > 0$, such that $\|T - A_0\| < \delta$. Now use Lemma 1 with A_0 in place of A, and with some new functions f_0 and g_0 arising from numbers β_0 and γ_0, such that $\|T - A_0\| < \gamma_0 < \beta_0 < \delta$. Call the resulting element B_0. Thus $B_0 \in GL(\mathfrak{A})$, and $B_0 E_\delta = V E_\delta$ since $\delta > \beta_0$. From the formulas in Lemma 1 and Lemma 3 it follows that

$$B_0 - Vf_0(|T|) = g_0^{-1}(|T^*|)A_0^{*-1}(1 - f_0)(|T|) =$$

$$= g_0^{-1}(|T^*|)B^{*-1}(h + \varepsilon)^{-1}(|T|)(1 - f_0)(|T|).$$

Invoking Lemma 2 this implies that

$$F_\delta B_0 - F_\delta V = F_\delta(B_0 - Vf_0(|T|)) =$$

$$= g_0^{-1}(|T^*|)F_\delta B^{*-1}(h + \varepsilon)^{-1}(|T|)(1 - f_0)(|T|) =$$

$$= g_0^{-1}(|T^*|)F_\delta V(h + \varepsilon)^{-1}(|T|)(1 - f_0)(|T|) =$$

$$= V(g_0(h + \varepsilon))^{-1}(|T|)E_\delta(1 - f_0)(|T|) = 0,$$

since $f_0(t) = 1$ for $t \geqslant \delta$. This completes the proof.

THEOREM 5. *If* $T \in \mathfrak{A}$ *there is for every* $\delta > \operatorname{dist}(T, GL(\mathfrak{A}))$ *a unitary* U *in* $\mathfrak{A}$ *such that* $UE_\delta = VE_\delta$.

Proof. Take B_0 as in Lemma 4, and let $B_0 = U|B_0|$ be its polar decomposition. Then U is unitary in $\mathfrak{A}$, and since

$$B_0^* B_0 E_\delta = B_0^* V E_\delta = B_0^* F_\delta V =$$

$$= (F_\delta B_0)^* V = (F_\delta V)^* V = V^* F_\delta V = E_\delta ,$$

it follows that $|B_0|E_\delta = (B_0^*B_0)^{1/2}E_\delta = E_\delta$; whence

$$UE_\delta = U|B_0|E_\delta = B_0E_\delta = VE_\delta,$$

as desired.

COROLLARY 6. *For every T in $\mathfrak{A}$, $\operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$ is the infimum of numbers δ for which the partial isometry VE_δ has a unitary extension in $\mathfrak{A}$.*

Proof. From the theorem we see that it suffices to show that if $VE_\delta = UE_\delta$ for some unitary U in $\mathfrak{A}$, then $\delta \geqslant \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$. But with $h(t) = (t - \delta) \vee 0$ and $\varepsilon > 0$ we see as in Lemma 3 that $A_0 = U(h + \varepsilon)(|T|)$ belongs to $\mathrm{GL}(\mathfrak{A})$ and that $\|T - A_0\| \leqslant \delta + \varepsilon$. As ε is arbitrary, the conclusion follows.

COROLLARY 7. *For each T in $\mathfrak{A}$, the spectral projections for $|T|$ and $|T^*|$, corresponding to the interval $[0, \delta]$, are unitarily equivalent in $\mathfrak{A}$, whenever $\delta > \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$.*

COROLLARY 8. *If $T \in \mathfrak{A}$ with polar decomposition $T = V|T|$, then the element $Vh(|T|)$ admits a unitary polar decomposition in $\mathfrak{A}$ whenever $h \in C(\mathbf{R})$ and $h(t) = 0$ for all $t \leqslant \delta$ for some $\delta > \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$.*

PROPOSITION 9. *If $T \in \mathfrak{A}$ with polar decomposition $T = V|T|$, and if the partial isometry VE_δ admits a unitary extension in $\mathfrak{A}$, i.e., $VE_\delta = UE_\delta$, then*

$$\|T - U\| \leqslant (\|T\| - 1) \vee (\delta + 1).$$

Proof. Since $UE_\delta = F_\delta U$ and $U(I - E_\delta) = (I - F_\delta)U$, it follows that

$$\|T - U\| = \|(V|T| - U)(E_\delta + I - E_\delta)\| =$$

$$= \|F_\delta(V|T| - U)E_\delta + (I - F_\delta)(V|T| - U)(I - E_\delta)\| =$$

$$= \|F_\delta(V|T| - V)E_\delta\| \vee \|(I - F_\delta)(V|T| - U)(I - E_\delta)\| \leqslant$$

$$\leqslant \| |T| - I\| \vee (\| |T| (I - E_\delta)\| + \|U\|) \leqslant (\|T\| - 1) \vee (\delta + 1).$$

Using Theorem 5 we can now give a proof of Rørdam's result, [7, Theorem 2.7], which is quite close to Olsen's original von Neumann algebra argument from [4].

THEOREM 10. *If $\mathfrak{A}$ is a unital C^*-algebra with unitary group $\mathcal{U}(\mathfrak{A})$, and T is a non-invertible element in $\mathfrak{A}$ with $\alpha = \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$, then*

$$\operatorname{dist}(T, \mathcal{U}(\mathfrak{A})) = (\|T\| - 1) \vee (\alpha + 1).$$

Proof. As observed by Olsen in Section 3 of [4], we have $\|T - W\| \geqslant \|T\| - 1$ and

$$\|T - W\| = \|I - W^*T\| \geqslant r(I - W^*T) \geqslant 1 + \alpha,$$

for every W in $\mathscr{U}(\mathfrak{A})$. Here r denotes the spectral radius, and the last inequality stems from the fact that $\mathrm{sp}(W^*T)$ contains a disk with center 0 and radius α, cf. [4, Corollary 2.10]. Thus

$$\mathrm{dist}(T, \mathscr{U}(\mathfrak{A})) \geqslant (\|T\| - 1) \vee (\alpha + 1).$$

The converse inequality is established in analogy with [4, Lemma 3.3], by combining Theorem 5 and Proposition 9.

COROLLARY 11. *If* $T \in \mathfrak{A}$ *such that*

$$\alpha = \mathrm{dist}(T, \mathrm{GL}(\mathfrak{A})) < \|T\| - 2,$$

then T admits a unitary approximant U, viz. a unitary extension of the partial isometry VE_δ, where $\alpha < \delta < \|T\| - 2$.

Proof. By Theorem 5 we can find U in $\mathscr{U}(\mathfrak{A})$ with $UE_\delta = VE_\delta$. Thus by Proposition 9 and Theorem 10 we have

$$\|T - U\| \leqslant (\|T\| - 1) \vee (\delta + 1) =$$
$$= (\|T\| - 1) \vee (\alpha + 1) = \mathrm{dist}(T, \mathscr{U}(\mathfrak{A})).$$

REMARK 12. We see from Proposition 9 and Theorem 10 that if we can take $\delta = \alpha$ in Theorem 5, i.e. if we can find U in $\mathscr{U}(\mathfrak{A})$ such that $UE_\alpha = VE_\alpha$, then U is an approximant to T in $\mathscr{U}(\mathfrak{A})$. If $\mathfrak{A}$ is a von Neumann algebra, this case can be equivalently expressed by asking that the element $T_\alpha = Vh_\alpha(|T|)$, where $h_\alpha(t) = (t - \alpha) \vee 0$, has index zero in $\mathfrak{A}$. Now we know from [4] that unitary approximants do exist (at least when $\alpha > 0$), so one might wonder whether T_α (the canonical approximant to T in $\mathrm{GL}(\mathfrak{A})^=$) always has index zero. Unfortunately this is not the case. To obtain a counterexample it suffices to take $\mathfrak{A} = \mathbf{B}(\mathscr{H})$, V a non-unitary isometry, and A a positive operator without eigenvalues but with $\mathrm{sp}(A) = [0, 1]$. Then with $T = V(A + I)$ we have

$$\mathrm{dist}(T, \mathrm{GL}(\mathscr{H})) = 1$$

by Theorem 5, since all non-zero spectral projections of A are infinite-dimensional. Thus

$$T_1 = Vh_1(|T|) = VA,$$

and clearly T_1 has negative index.

REMARK 13. If T is a normal operator in a von Neumann algebra $\mathfrak{A}$ then $\operatorname{dist}(T, \mathrm{GL}(\mathfrak{A})) = 0$, and we can find a unitary extension in $\mathfrak{A}$ of the partial isometry V in the polar decomposition $T = V|T|$, viz. $U = V + I - V^*V$. Any such U will be an approximant to T in $\mathscr{U}(\mathfrak{A})$, cf. [4, Proposition 3.1].

If $\mathfrak{A}$ is only a C^*-algebra, the normal case is far from trivial. In fact, a normal operator T in $\mathfrak{A}$ need not have the form $T = U|T|$ for some unitary U in $\mathfrak{A}$; it may have $\operatorname{dist}(T, \mathrm{GL}(\mathfrak{A})) > 0$; and it need not have a unitary approximant. To exemplify the difficulties, take $\mathfrak{A} = C(X)$ for some compact Hausdorff space X and consider T in $\mathfrak{A}$. For $\delta > 0$ put

$$X_\delta = \{x \in X \mid |T(x)| \geqslant \delta\},$$

and define the unitary function $V_\delta : X_\delta \to \mathbf{T}$ by $V_\delta(x) = T(x)\,|T(x)|^{-1}$. Considering the problem of extending V_δ from X_δ to X as a unitary function, we see from Theorem 5 that it can be done for $\delta > \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$, but not for $\delta < \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$. It follows from the next example that we may not be able to find a unitary extension for $\delta = \operatorname{dist}(T, \mathrm{GL}(\mathfrak{A}))$, cf. Remark 12.

EXAMPLE 14. There is a compact subset X of $\mathbf{C}$, such that the identical function T given by $T(z) = z$, $z \in X$, does not admit a unitary approximant in $C(X)$.

Proof. Let $\mathbf{D} = \{z \in \mathbf{C} \mid |z| \leqslant 1\}$ and choose a decreasing sequence (r_n) in $\mathbf{R}$ such that $r_1 = 3$ and $r_n \to 1$. Put

$$X = \cup \{z \in \mathbf{C} \mid |z| = r_n\} \cup \mathbf{D}.$$

The function T given by $T(z) = z$ is a generator for $C(X)$ and $\operatorname{sp}(T) = X$. Since for any $\varepsilon > 0$ we can find λ not in X with $|\lambda| < 1 + \varepsilon$, we see that

$$\alpha = \operatorname{dist}(T, \mathrm{GL}(C(X)) \leqslant 1.$$

On the other hand, if $A \in \mathrm{GL}(C(X))$ and $\|T - A\| \leqslant 1 - \varepsilon$ for some $\varepsilon > 0$, then the winding number around 0 for the curve $\{A(z) \mid |z| = r\}$ must be 1 if $r = 1$. But, of course, it must be 0 if $r = 0$; and it must vary continuously with r, a contradiction. Thus no such A exists, and $\alpha = 1$. Obviously $\|T\| = 3\ (= r_1)$, so by Theorem 10 we have

$$\operatorname{dist}(T, \mathscr{U}(C(X)) = \|T\| - 1 = \alpha + 1 = 2.$$

Suppose now that U was a unitary function on X with $\|T - U\| = 2$. Thus $|U(z) - z| \leqslant 2$ for every z in X. This implies that the winding number for the curve $\{U(z) \mid |z| = r\}$ is 1 for every $r = r_n$. By continuity the winding number is therefore also 1 for $r = 1$. But the number must be 0 for $r = 0$ and must vary continuously for $0 \leqslant r \leqslant 1$; a contradiction. Thus no unitary approximant exists.

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GERT K. PEDERSEN
Københavns Universitet,
Matematisk Institut,
Universitetsparken 5,
2100 København Ø,
Denmark.

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