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STABLE RANK FOR A CERTAIN CLASS OF TYPE I C^*-ALGEBRAS

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In [11] M. A. Riefell introduced the notion of topological stable rank of a C^*-algebra A as the least integer n such that any n-tuple $(x_1, \ldots, x_n) \in A^n$ can be approximated arbitrarily close by an n-tuple of elements of $\tilde{A}$ ($\tilde{A}$ denotes the algebra $A + \mathbf{C}1$) which generate $\tilde{A}$ as a left ideal (if no such integer exists we take the topological stable rank of A to be ∞). One of the reasons to study the topological stable rank is that it can be used to obtain cancellation theorems for projective modules as done in [12, 14, 15]. As shown in [3] the topological stable rank and the Bass stable rank coincide for C^*-algebras. We shall denote their common value for a C^*-algebra A by $\mathrm{sr}(A)$ (the stable rank of A).

It is known [1] that for a separable type I C^*-algebra A there exists a composition series with continuous trace subquotients. We shall find the value of the stable rank of A for a separable C^*-algebra with a finite such composition series with locally trivial quotients (Theorem 7). This result generalises results from [8, 11, 14]. It also clarifies and simplifies the proof of [8]. We improve a theorem of [11] concerning the value of $\mathrm{sr}(A)$ in terms of $\mathrm{sr}(I)$ and of $\mathrm{sr}(A/I)$ for I a certain continuous trace ideal and show that $\mathrm{sr}(A \otimes B) \leqslant \mathrm{sr}(A) + \mathrm{sr}(B)$ for certain separable C^*-algebras of type I, see also [8].

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The following facts can be found in [1]. Let I be a C^*-algebra. We shall denote by $\hat{I}$ the spectrum of I and by $m(I)$ the linear span of the set of those $x \in I_+$ such that the function $\pi \to \mathrm{tr}\,\pi(x)$ is continuous on $\hat{I}$ (3.1.5, 4.5.2). One says that I is of continuous trace if $m(I)$ (which is an ideal) is dense in I (4.5.3). In this case $\hat{I}$ is separated and I is isomorphic to a C^*-algebra corresponding to a continuous field $\mathscr{A} = ((I_t)_{t \in \hat{I}}, \Gamma)$ of elementary C^*-algebras on $\hat{I}$. Moreover $\mathscr{A}$ satisfies Fell's condition (10.5.4, 10.5.7, 10.5.8).

Let $M(I)$ be the algebra of multipliers of I ([9]). If $b \in M(I)$ then it can be identified with a certain function $t \to b(t) \in M(I_t)$ on $\hat{I}$.

We recall that for a topological space T the covering dimension, $\dim(T)$, is the least integer n, such that each open cover of T has an open rafinement such that each point is contained in at most $n+1$ sets. If no such integer exists $\dim(T) = \infty$. If T is a compact metric space then all definitions of dimension are equivalent (see [6]).

We shall suppose that T, the spectrum of I, has finite covering dimension.

Let $t \to a(t) \in (I_t)_+$, $t \to b(t) \in M(I_t)_+$ be two positive elements of I and of $M(I)$, respectively. We shall suppose that $b(t)$ is not of finite rank for any $t \in T$.

We shall denote by χ_A the characteristic function of the set A.

LEMMA 1. *Under the above hypothesis there exists a function $t \to v(t) \in I_t$ defined on T, for T compact, which gives an element of I satisfying:*

$$\chi_{[1/2,\infty)}(a(t)) \leqslant v^*(t)v(t) \leqslant \chi_{(0,\infty)}(a(t)) = s(a(t)) \tag{1.1}$$

$$v(t)v(t)^* \leqslant \chi_{(0,\infty)}(b(t)) = s(b(t)) \tag{1.2}$$

for any $t \in T$.

Proof. The assumptions and Lemma 10.7.11 of [1] give for I and $T = \hat{I}$:

(i) a finite open cover $(\mathring{T}_1, \dots, \mathring{T}_n)$ of T, with T_j closed;

(ii) for any $j \in \{1, \dots, n\}$ a continuous field $((\mathscr{H}_j(t))_{t\in T_j}, \Gamma_j)$ of Hilbert spaces and isomorphisms h_j from $\mathscr{A}/T_j$ onto $\mathscr{A}(\mathscr{H}_j)$ — the CCR-C^*-algebra induced by $\mathscr{H}_j$ ([1], 10.7.2);

(iii) for any $i, j \in \{1, \dots, n\}$ an isomorphism $g_{ij}(t) : \mathscr{H}_j(t) \to \mathscr{H}_i(t)$ for $t \in T_{ij} = T_i \cap T_j$ which induces $h_i h_j^{-1}$ from $\mathscr{A}(\mathscr{H}_j/T_{ij})$ onto $\mathscr{A}(\mathscr{H}_i/T_{ij})$;

(iv) For any $j \in \{1, \dots, n\}$ two numbers $0 < a_j < b_j < 1/2$ such that $(a_j, b_j) \cap \sigma(a(t)) = \emptyset$ on T_j.

Denote by $c_j = (a_j + b_j)/2$ and by $p_j(t_j) = h\chi_{[c_j,\infty]}(a(t)))$ which belongs to $\mathscr{A}(\mathscr{H}_j)$ due to (iv). Let us denote by $b_j(t) = h_j(b(t))$, $t \in T_j$.

We shall solve the following technical problem:

Problem (P). Construct for any $j \in \{1, \dots, n\}$ a continuous function $t \to u_j(t) \in \mathscr{K}(\mathscr{H}_j(t))$ which gives a partial isometry in $\mathscr{A}(\mathscr{H}_j)$ with the properties:

(a) $u_j^*(t)u_j(t) = p_j(t)$ on T_j,

(b) $u_i(t)^* g_{ij}(t) u_j(t) = 0$ on T_{ij} for $i \neq j$,

(c) $u_j(t)u_j^*(t) \leqslant s(b_j(t))$ on T_j.

Let us observe that if we can solve Problem (P) then we can solve the corresponding problem with p_j replaced in (a) by q_j such that $0 \leqslant q_j \leqslant p_j$, for $u_j' = u_j q_j$

will satisfy (a), (b), (c) in this new form. We may suppose then that p_j defines a trivial vector bundle of rank r_j on T_j. Then Problem (P) is equivalent to :

Problem (P_1). Construct continuous sections $\xi_i^j \in \Gamma_j$ for $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, r_j\}$ such that

$$\text{(a')} \qquad (\xi_i^j(t), g_{jk}(t)\xi_e^k(t)) = \delta_{ie}\delta_{jk} \quad \text{on } T_{jk},$$

$$\text{(b')} \qquad \xi_i^j(t) \in \overline{b(t)\mathscr{H}_j(t)} \quad \text{on } T_j.$$

We shall solve now Problem (P_1).

Let us suppose that we have defined the sections ξ_i^k for $k < j$ and $1 \leqslant i \leqslant r_k$ and that we have extended the sections $\zeta_i^k = g_{jk}\xi_i^k$, $k < m$, $1 \leqslant i \leqslant r_k$ from T_{jk} to all of T_j such that $(\zeta_i^k(t), \zeta_e^{m'}(t)) = \delta_{km'}\delta_{ie}$ for $t \in T$. Let $p(t)$ be the orthogonal projection onto the linear span of the vectors $\zeta_i^k(t)$ for $k < m$, $1 \leqslant i \leqslant r_k$. Then $\overline{(1 - p(t))b(t)\mathscr{H}_j(t)}$ defines a continuous field of Hilbert spaces on T_j of infinite dimension in each point. The proof of 10.8.7 of [1] shows, using Michael's theorem [4], that we can extend $\zeta_1^m, \ldots, \zeta_{r_m}^m$ to T_j, or, if $m = j$, that we can find sections $\xi_1^j, \ldots, \xi_{r_j}^j$ with the desired properties. Problem (P_1) is trus solved.

To obtain the function v we shall choose a partition of unity $(\varphi_j)_{j=1}^n$ subordinated to the cover $(T_1, \ldots, T_n)$. Then $\varphi_j^{1/2}h_j^{-1}(u_j)$ are well defined elements of I and their sum v satisfies our requirements.

Let $0 \to I \to A \to B \to 0$ be a short exact sequence of C^*-algebras, when I is as above.

To any point $t \in T = I$ corresponds an ideal $J_t \subset B$ in the following way : the representation t has a unique (up to equivalence) extension to a representation of A on $\mathscr{H}_t$ (the Hilbert space of t). The kernel of the induced map $B \to \mathscr{L}(\mathscr{H}_t)/\mathscr{K}(\mathscr{H}_t)$ will be denoted by J_t (remember that $t(I) \subset \mathscr{K}(\mathscr{H}_t)$ because any C^*-algebra of continuous trace is a CCR-C^*-algebra). See also [10], Definition 1.7.

LEMMA 2. *Let I be a closed two-sided ideal of A with continuous trace. We shall suppose that* $\dim(\mathscr{H}_t) = \infty$ *for any* $t \in \hat{T} = I$. *Then* $\mathrm{sr}(A) \leqslant \max\{\mathrm{sr}(A/I), 2\}$.

Proof. Let us suppose that $s = \max\{\mathrm{sr}(A/I), 2\} < \infty$; otherwise the lemma is obvious. Also, we may suppose that A has unit.

Let $x_1, \ldots, x_s \in A$, π the quotient map $A \to A/I$, $\varepsilon > 0$. We may suppose that, after a small perturbation, $\pi(x_1), \ldots, \pi(x_s)$ generate A/I as a left ideal. We want to show that there exist $x_1', \ldots, x_s'$ which generate A as a left ideal and such that $\|x_j - x_j'\| < \varepsilon$ for any $j \in \{1, \ldots, s\}$. This will show that $\mathrm{sr}(A) \leqslant s$.

Let $x = [x_1, \ldots, x_s]^t \in M_{s,1}(A)$, $y = x^*x = x_1^*x_1 + \ldots + x_s^*x_s$.

By the assumption there exists $\eta > 0$ such that $\pi(y) \geqslant \eta$. Let $f : [0, \infty) \to \to [0, 1]$ be a continuous function, $f(t) = 1$ for $t \in [0, \eta/2]$, $\operatorname{supp} f \subset [0, \eta]$, $z = f(y) \in I$. The set of points $t \in T$ on which $\|z(t)\| < \eta/4$ has a compact complement K_1 in T. Let K be a compact neighborhood of K_1, φ a continuous function with values in $[0, 1]$, $\varphi = 1$ on K_1, $\varphi = 0$ off K.

Let $\delta, \gamma > 0$ to be specified later and let $g : [0, \infty) \to [0, 1]$ be a continuous function vanishing off $[0, \delta]$ such that $g(0) = 1$. We want to apply Lemma 1 for $M_s(I)\,|\,K$, $a = z\,|\,K$, $b = g(xx^*)\,|\,K$ to obtain a v such that

$$\chi_{[1/2,1]}(z(t)) \leqslant v^*(t)v(t) \tag{2.1}$$

and if h is a continuous function on $[0, \infty)$ with values in $[0, 1]$ such that $[0, \delta] \subset \subset h^{-1}(\{1\})$, $[2\delta, \infty) \subset h^{-1}(\{0\})$ then

$$h(xx^*)(t)v(t) = v(t) \tag{2.2}$$

(we have denoted by $z(t)$ $(h(xx^*)(t))$ the image of $z(h(xx^*))$ in I_t $(M(I_t))$). All we have to check is that $b(t)$ is nowhere of finite rank. Let us suppose that $b(t)$ is of finite rank for some $t \in K$. Let B denote A/I, T the operator $x(t)$ and $[\mathscr{H}]$ the orthogonal projection onto the closure of the space $\mathscr{H}$. If $b(t)$ is of finite rank then $b(t) \geqslant \geqslant 1 - [\operatorname{Ran} T]$ and $\ker T$ is finite dimensional from the assumption that $\pi(y) \geqslant \eta > 0$. This means that T is a Fredholm operator. Since we have an injection $B/J_t \to \mathscr{L}(\mathscr{H}_t)/\mathscr{K}(\mathscr{H}_t)$ by the very definition of J_t, we obtain that the image of $x(t)$ in $M_{s,1}(B/J_t)$ is invertible. Since $s \geqslant 2$ this means that $M_s(B/J_t)$ contain two isometries with orthogonal ranges. $\mathscr{H}_t$ is infinite dimensional and B has unit, hence $J_t \neq B$. Proposition 6.5 of [11] shows that $\operatorname{sr}(M_s(B/J_t)) = \infty$ and hence ([11], Theorems 6.1 and 4.3) $\operatorname{sr}(B) = \infty$, contradicting our assumption.

Denote by $u = \varphi v \in I$, $x' = x + \gamma u = [x_1' \ldots x_s']^t$.

For $t \notin K_1$ we obtain using (2.1)

$$\begin{aligned}(x'^*x')(t) &= (x^*x)(t) + \gamma(u^*x + x^*u + \gamma u^*u) \geqslant \\ &\geqslant (\eta/2 - z(t)) - 2\gamma(\|x\| + 1) \geqslant \eta/4 - 2(\|x\| + 1)\gamma.\end{aligned} \tag{2.3}$$

For $t \in K_1$, $\varphi(t) = 1$ and hence by functional calculus

$$\begin{aligned}(x'^*x')(t) &= (x^*x)(t) + \gamma^2u^*u + \gamma(u^*x + x^*u) \geqslant \\ &\geqslant (x^*x)(t) + \gamma^2\chi_{[0,\eta/2]}(x^*x) - 2\gamma\|v^*x\| \geqslant \\ &\geqslant \gamma^2 - 2\gamma\|v^*x\| \quad \text{for } \gamma^2 \leqslant \eta/2.\end{aligned} \tag{2.4}$$

By (2.2) we have

$$\|v^*x\| = \|v^*h(xx^*)x\| \leqslant \|h(xx^*)x\| \leqslant 2\delta^{1/2}. \tag{2.5}$$

Let us choose γ and δ such that $0 < \gamma < \varepsilon$, $2\gamma(\|x\| + 1) < \eta/8$, $\delta^{1/2} < \gamma$ and such that $\|x' - x\| < 2\gamma$ implies that $\pi(x_1'), \ldots, \pi(x_s')$ still generate A as a left ideal. Then (2.3), (2.4) and (2.5) show that there exists $\lambda > 0$ such that $(x'^*x')(t) \geqslant \lambda$ for $t \in T$. Let φ be a pure state, π_φ the GNS representation associated with φ. If $\pi_\varphi \in \hat{I}$ then $\varphi(x'^*x') \geqslant \lambda > 0$; if $\pi_\varphi(I) = \{0\}$ then $\varphi(x'^*x') = \varphi'(\pi(x'^*x')) > 0$ since $x_1', \ldots, x_s'$ generate A/I as a left ideal (φ' is the induced state on A/I).We may conclude then that there exists $\lambda' > 0$ such that $x_1'^*x_1' + \ldots + x_s'^*x_s' \geqslant \lambda'$ and hence that $x_1', \ldots, x_s'$ generate A as a left ideal.

The following lemma is an unpublished result of G. Nagy.

LEMMA 3. *Let* $0 \to I \to A \to B \to 0$ *be an exact sequence of* C^**-algebras, such that* $\mathrm{sr}(I) = \mathrm{sr}(B) = 1$. *Then* $\mathrm{sr}(A) = 1$ *if and only if the index morphism* $\delta : \mathrm{K}_1(B) \to \mathrm{K}_0(I)$ *is zero.*

Proof. Suppose first that $\mathrm{sr}(A) = 1$. Choose u a unitary in $M_n(\tilde{B})$ and $v \in M_n(\tilde{A})$ a lifting of u. Choose also $w \in M_n(\tilde{A})$ an invertible element close enough to v such that $\pi(w)$ represents the same class as u does in $\mathrm{K}_1(B)$. Obviously $\delta([\pi(w)]) = 0$.

Conversely, we know that $\mathrm{sr}(A) = 1$ if and only if $\mathrm{sr}(A \otimes \mathcal{K}) = 1$ ([11], Theorem 3.6). Let $u \in \widetilde{\mathcal{K} \otimes A}$, $\varepsilon > 0$. There exists an invertible element $v \in \widetilde{\mathcal{K} \otimes B}$ such that $\|\pi(u) - v\| < \varepsilon$. Since $\delta([v]) = 0$, there exists an invertible element $w \in \widetilde{\mathcal{K} \otimes A}$ such that $\pi(w) = v$. Let $w_0 \in \widetilde{\mathcal{K} \otimes A}$ be such that $\pi(w_0) = v = \pi(w)$ and $\|u - w_0\| < \varepsilon$, then $w^{-1}w_0 \in 1 + \mathcal{K} \otimes I$. Choose an invertible element $x \in 1 + \mathcal{K} \otimes I$ such that $\|x - w^{-1}w_0\| < \|w\|^{-1}(\varepsilon - \|u - w_0\|)$ then wx is invertible and

$$\|wx - u\| \leqslant \|wx - w_0\| + \|w_0 - u\| < \|w\|\|w\|^{-1}(\varepsilon - \|w_0 - u\|) + \|w_0 - u\| = \varepsilon.$$

We shall study next the opposite case, namely for I a two-sided ideal of continuous trace such that the associated field of elementary C^*-algebras $J = ((I_t)_{t\in T}, \Gamma)$ ($T = \hat{I}$ — the spectrum of I, $I_t = I/\ker t$), be locally trivial with I_t a finite dimensional simple C^*-algebra. Let $T_n \subset T$ be the set of those $t \in T$ such that $I_t = M_n(\mathbf{C})$. By the assumption of locally triviality each T_n is open. Since $T = \bigcup_{n=1} T_n$, T_n is also closed. Let T_n correspond to the ideal $I_n \subset I$, $T_n = \hat{I}_n$, then I is the c_0-direct sum of the C^*-algebras I_n.

We notice that for a separable C^*-algebra I of continuous trace the spectrum $T = \hat{I}$ (which is a locally compact Hausdorff space [1]) is a separable σ-compact metric space.

We shall use the following technical result due to A. J.-L. Sheu ([14], Proposition 3.15):

LEMMA 4. *Let $\{J_\lambda\}_{\lambda\in\Lambda}$ be a net of closed ideals (ordered by inclusion) of a unital C^*-algebra A with J = the closure of the union of J_λ's. If K_λ are closed ideals of A such that $J_\lambda \cdot K_\lambda = 0$ for all $\lambda \in \Lambda$ then* $\mathrm{sr}(A) = \max\{\mathrm{sr}(A/J), \mathrm{sr}(A/K_\lambda) \mid \lambda \in \Lambda\}$.

LEMMA 5. a) *Let A be a C^*-algebra, $I \subset A$ a closed two-sided ideal as above then*

$$\mathrm{sr}(A) = \max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\}. \tag{5.1}$$

b) *If I is separable then*

$$\mathrm{sr}(I) = \sup_{n\geq 1} \{\{(\dim(T_n) - 1)/2n\}' + 1\}. \tag{5.2}$$

(Here $\{x\}'$ denotes the least integer m, $m \geq x$.)

Proof. a) Let $\Lambda = \{U \subset T \mid U$ open and relatively compact in $T\}$, J_U the ideal of A corresponding to U, K_U the ideal of A corresponding to $\hat{A}\setminus\dot{U}$.

We want to show that A/K_U identifies naturally with a quotient of I. This will folllow if we show that $K_U + I = A$ or equivalently that $\hat{K}_U \cup \hat{I} = (\hat{A}\setminus\dot{U}) \cup T = \hat{A}$.

We have to prove that $\dot{U} \subset T$.

In the following exact sequence

$$0 \to (I + K_U)/K_U \to A/K_U \to A/(I + K_U) \to 0$$

A/K_U has the spectrum $\bar{U}$ and $(I + K_U)/K_U$ has the spectrum $\bar{U} \cap T$. Using the compactness of $\dot{U} \cap T$ and the local triviality of J we obtain that $(I + K_U)/K_U$ has a unit. This shows that $\bar{U} \cap T$ is closed in $\bar{U}$ and hence closed. Since $U \subset \dot{U} \cap T$ it follows that $\dot{U} \subset \overline{\dot{U} \cap T} = \dot{U} \cap T$ and hence $\dot{U} \subset T$. It follows that we have the isomorphisms $A/K_U \cong (I + K_U)/K_U \cong I/I \cap K_U$. Theorem 4.3 of [11] shows that $\mathrm{sr}(A/K_U) \leq \mathrm{sr}(I)$.

We shall use Lemma 4 : $I = \bigcup_{\lambda\in\Lambda} J_\lambda$ and hence $\mathrm{sr}(A) = \max\{\mathrm{sr}(A/I), \mathrm{sr}(A/K_U) \mid U \in \Lambda\} \leq \max\{\mathrm{sr}(A/I), \mathrm{sr}(I)\}$.

b) Suppose first that $I = I_n$ and T_n is compact. Cover T_n by a finite number of open sets $V_1, \ldots, V_m$ such that $T \mid \bar{V}_k$ is trivial for any $k \in \{1, \ldots, m\}$. If J_k is the ideal of I corresponding to $T\setminus V_k$ the last statement is equivalent to the fact that $I/J_k = M_n(C(\bar{V}_k))$. By [8], Corollary 2.7

$$\mathrm{sr}(I) = \mathrm{sr}(I/J_1 \cap \ldots \cap J_m) = \max\{\mathrm{sr}(I/J_k) \mid 1 \leq k \leq m\}.$$

By [11], Theorem 6.1 $\mathrm{sr}(I/J_k) = \{(\dim(V_k) - 1)/2n\}' + 1$. By the sum theorem [6], $\dim(T) = \max\{\dim(V_k) \mid 1 \leqslant k \leqslant m\}$ and hence $\mathrm{sr}(I) = \{(\dim(T) - 1)/2n\}' - 1$.

The general statement can be obtained as follows: $\mathrm{sr}(I) = \max\{\mathrm{sr}(I_n) \mid n \in \mathbf{N}\}$ ([11], Theorem 5.2). All we have to prove is that $\mathrm{sr}(I_n) = \{(\dim(T_n) - 1)/2n\} + 1$. We shall use Lemma 4 in the following setting: let $L_1, L_2, \ldots, L_m, \ldots$ be compact subsets of T_n such that $T_n = \bigcup_{m=1} L_m$, $L_m \subset \overset{\circ}{L}_{m+1}$, $\Lambda = \mathbf{N}$, J_m the ideal corresponding to L_m in I_n, K_m the ideal corresponding to $(T_n \cup \{\infty\})\setminus L_m$ in $\tilde{I}_n$. $\bigcup_{m\geqslant 1} J_m = I_n$ and $\tilde{I}_n/I_n \cong \mathbf{C}$, hence

$$\mathrm{sr}(I_n) = \mathrm{sr}(\tilde{I}_n) = \max_{m\geqslant 1}\{\mathrm{sr}(\tilde{I}_n/K_U)\} =$$

$$= \sup_{m\geqslant 1}\{\{(\dim(L_m) - 1)/2n\}' + 1\} = \{(\dim(T_n) - 1)/2n\}' + 1$$

since $\dim(T_n) = \sup_{m\geqslant 1} \dim(L_m)$ by the sum theorem [6].

Definition 6. Let A be a separable C^*-algebra with a composition series $\{0\} = I_0 \subset I_1 \subset \ldots \subset I_{n+1} = A$ such that each of the subquotients I_{k+1}/I_k for $0 \leqslant k \leqslant n$ is of continuous trace and it satisfies either:

1°. I_{k+1}/I_k has only finite dimensional irreducible representations and the corresponding field of elementary C^*-algebras is locally trivial; or

2°. I_{k+1}/I_k has only infinite dimensional irreducible representations and the spectrum $(I_{k+1}/I_k)^\wedge$ has a finite dimension.

Then we say that *A satisfies condition* A.

Theorem 7. a) *Let I be a separable C^*-algebra of continuous trace such that the corresponding field of elementary C^*-algebras is locally trivial, $I = c_0\text{-}\bigoplus_{k\in \mathbf{N}\cup\{\infty\}} I_k$ with I_k homogeneous of degree k. Then*

$$\mathrm{sr}(I) = \sup\{s_\infty\} \cup \{\{(\dim(I_k) - 1)/2k\}' + 1 \mid k \in \mathbf{N}\}.$$

Here $s_\infty = 1$ if $\dim(\hat{I}_\infty) \leqslant 1$ and $s_\infty = 2$ else.

b) *Let A satisfy condition* A. *If $\mathrm{sr}(I_{k+1}/I_k) = 1$ for $0 \leqslant k \leqslant n$ and at least one of the index homomorphisms $\delta : \mathrm{K}_1(I_{k+1}/I_k) \to \mathrm{K}_0(I_k)$ for $1 \leqslant k \leqslant n$ is not zero then* $\mathrm{sr}(A) = 2$, *else*

$$\mathrm{sr}(A) = \max_{0\leqslant k\leqslant n}\{\mathrm{sr}(I_{k+1}/I_k)\}.$$

Proof. a) follows from Lemma 5 b) (for I_∞ we use an identical device and [11], Theorem 3.6).

b) follows by induction on n using Lemma 2, Lemma 3 and Lemma 5.

THEOREM 8. *Let A and B satisfy condition* A *with the spectrum of the subquotients* CW-*complexes then*

$$\mathrm{sr}(A \otimes B) \leqslant \mathrm{sr}(A) + \mathrm{sr}(B).$$

Proof. Let $\{0\} = I_0 \subset I_1 \ldots \subset I_{n+1} = A$ and $0 = J_0 \subset J_1 \ldots \subset J_{m+1} = B$ be composition series as in Definition 4; then $A \otimes B$ has a composition series with quotients isomorphic to $(I_{k+1}/I_k) \otimes (J_{e+1}/J_e)$.

If $\mathrm{sr}(A \otimes B) \in \{1, 2\}$ then (5.2) is obvious. Let $\mathrm{sr}(A \otimes B) \geqslant 3$; then $\mathrm{sr}(A \otimes B) = \mathrm{sr}((I_{k+1}/I_k) \otimes (J_{e+1}/J_e))$ for some k and e. It is obvious that I_{k+1}/I_k and J_{e+1}/J_e satisfy 1° of Definition 4. Let $I_{k+1}/I_k = c_0$-direct sum of the ideals L_j, $j \in \mathbf{N}$, $J_{e+1}/J_e = c_0$-direct sum of the ideals K_i, $i \in \mathbf{N}$ with L_j and K_j homogeneous of degree i. Since

$$\mathrm{sr}(L_j \otimes K_i) = \{(\dim(\hat{L}_j \times \hat{K}_i) - 1)/2ij\}' + 1 \leqslant$$

$$\leqslant (\{(\dim(\hat{L}_j) - 1)/2j\}' + 1) + (\{(\dim(\hat{K}_i) - 1)/2i\}' + 1) = \mathrm{sr}(L_j) + \mathrm{sr}(K_i)$$

we obtain

$$\mathrm{sr}(A \otimes B) = \mathrm{sr}((I_{k+1}/I_k) \otimes (J_{e+1}/J_e)) = \sup_{i,j} \mathrm{sr}(K_i \otimes L_j) =$$

$$= \sup_{i,j} (\mathrm{sr}(K_i) + \mathrm{sr}(L_j)) = \sup_i \mathrm{sr}(K_i) + \sup_j \mathrm{sr}(L_j) \leqslant$$

$$\leqslant \mathrm{sr}(I_{k+1}/I_k) + \mathrm{sr}(J_{e+1}/J_e) \leqslant \mathrm{sr}(A) + \mathrm{sr}(B).$$

REMARK. Theorem 8 answers Question 7.3 of [11] in a particular case.

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