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STRONGLY QUASIDIAGONAL C^*-ALGEBRAS

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(with an Appendix by Jonathan Rosenberg)

1. INTRODUCTION

In [13] P. R. Halmos introduced the notion of a quasidiagonal operator, which was further studied by R. Smucker [19] and more recently by D. A. Herrero [14, 15]. Unlike quasitriangularity [12], quasidiagonality is not preserved under similarity [15], [14]. Moreover, if T is a quasidiagonal operator, then so is every operator in the C^*-algebra generated by 1 and T [19].

Later F. J. Thayer [22] introduced the notion of a quasidiagonal C^*-algebra of operators and of a quasidiagonal representation of an arbitrary C^*-algebra. Quasidiagonality has appeared in the work of E. Effros and J. Rosenberg [8], M.-D. Choi [7], and in work related to the Brown-Douglas-Fillmore theory of N. Salinas [18], M. Pimsner and D. Voiculescu [17], and L. G. Brown [5].

The notion of quasidiagonality of C^*-algebras of operators is not a C^*-algebraic property since there are two $*$-isomorphic C^*-algebras of operators, one of which is quasidiagonal and the other is not (e.g., Example 7). However, in [8] Effros and Rosenberg defined an abstract C^*-algebra to be quasidiagonal if it is $*$-isomorphic to a quasidiagonal C^*-algebra of operators. (We use the term "weakly quasidiagonal" to describe this property.)

In this paper we define a more restricted version of quasidiagonality, called strong quasidiagonality, and we study and compare these two notions of quasidiagonality. Our main results concern tensor products (Theorem 18) and C^*-crossed products (Theorem 25). We also include some of the basic properties of strong quasidiagonality; many of these properties and related examples appear in [22], [8], [5].

The author wishes to express his heartfelt gratitude to the referee for many valuable suggestions for improving this paper, particularly in regard to Theorem 16, which in the original version has "GCR" instead of "nuclear". The author also wishes to express his thanks to L. G. Brown for several valuable discussions; the ideas in Theorem 20 (and the remarks that follow) that were not taught to me by him were certainly known to him when I discovered them for myself.

2. BASIC PROPERTIES

Throughout, let H denote a separable Hilbert space, let $B(H)$ denote the set of (bounded linear) operators on H, and let $\mathcal{K}(H)$ denote the set of compact operators on H. A C^*-subalgebra $\mathcal{A}$ of $B(H)$ is *quasidiagonal* [22] if there is an orthogonal sequence $\{P_n\}$ of finite-rank projections whose sum is 1 such that $A - \sum_n P_n A P_n \in \mathcal{K}(H)$ for every A in $\mathcal{A}$. A representation of an arbitrary C^*-algebra is *quasidiagonal* if its range is quasidiagonal. We shall refrain from using the terminology in [8] and shall use the term "quasidiagonal" only for operators, C^*-algebras of operators, or representations of C^*-algebras. Instead, we call a separable C^*-algebra *weakly quasidiagonal* if it is $*$-isomorphic to a quasidiagonal C^*-algebra of operators, i.e., if it has a faithful quasidiagonal representation. We call a C^*-algebra *strongly quasidiagonal* if it is separable and all of its separable representations (i.e., representations on separable Hilbert spaces) are quasidiagonal. In the original preprint form of this paper the term "completely quasidiagonal" was used instead of "strongly quasidiagonal".

It is clear that a C^*-algebra without identity is strongly (weakly) quasidiagonal if and only if the C^*-algebra obtained by adjoining an identity is. Thus we shall mainly restrict ourselves to unital separable C^*-algebras.

There are many equivalent formulations of quasidiagonality [22], [5]. We list a few of them here and provide a brief outline of the proofs.

LEMMA 1. *Suppose $\mathcal{A}$ is a separable C^*-subalgebra of $B(H)$, suppose $\mathcal{S} \subset \mathcal{A}$ and $\mathcal{S}$ generates $\mathcal{A}$ as a C^*-algebra, and suppose $M \subset H$ and the closed linear span of M is H. The following are equivalent:*

(1) *$\mathcal{A}$ is quasidiagonal;*

(2) *there is an increasing sequence $\{P_n\}$ of finite-rank projections with $P_n \to 1$ strongly such that $\|AP_n - P_nA\| \to 0$ for every A in $\mathcal{A}$;*

(3) *there is a quasidiagonal C^*-algebra $\mathcal{A}_1$ such that $\mathcal{A} \subset \mathcal{A}_1 + \mathcal{K}(H)$;*

(4) *for every $\varepsilon > 0$, every finite subsest $\mathcal{S}_0$ of $\mathcal{S}$, and every finite subset M_0 of M there is a finite-rank projection P in $B(H)$ such that $\|SP - PS\| < \varepsilon$ and $\|(1 - P)f\| < \varepsilon$ for every S in $\mathcal{S}_0$ and every f in M_0.*

Sketch of proof. The implications $(3) \Leftrightarrow (1) \Rightarrow (2) \Rightarrow (4)$ are obvious. The implication $(2) \Rightarrow (1)$ is contained in [22] and the key idea is due to P. R. Halmos [13]. We need only prove $(4) \Rightarrow (2)$. Since $\mathcal{A}$ and H are separable, we can assume that $\mathcal{S}$ and M are countable, i.e., $\mathcal{S} = \{S_1, S_2, \ldots\}$ and $M = \{f_1, f_2, \ldots\}$. Thus, by (4), we can choose a finite-rank projection P_n so that $\|P_nS_k - S_kP_n\| + \|(1 - P_n)f_k\| < 1/n$ for $1 \leqslant k \leqslant n$ for each positive integer n. It follows that $P_n \to 1$ strongly and that $\|P_nA - AP_n\| \to 0$ for every A in $\mathcal{A}$. The sequence $\{P_n\}$ may not be increasing. However, it is not difficult to construct a subsequence

$\{P_{n_k}\}$ and an increasing sequence $\{Q_k\}$ of finite-rank projections so that $\|Q_k - P_{n_k}\| \to 0$. Thus $Q_k \to 1$ strongly and $\|AQ_k - Q_kA\| \to 0$ for every A in $\mathscr{A}$. This proves (4) $\Rightarrow$ (2). ▨

COROLLARY 2. *Suppose $\mathscr{A}$ is a separable C^*-algebra and π, ρ are representations such that, for some unitary operator U, we have $U^*\pi(A)U - \rho(A)$ is compact for every A in $\mathscr{A}$. Then π is quasidiagonal if and only if ρ is.*

REMARK. The preceding corollary implies that the quasidiagonality of a C^*-subalgebra of $B(H)$ depends only on its image in the Calkin algebra (i.e., the quotient $B(H)/\mathscr{K}(H)$). It therefore makes sense to talk of a quasidiagonal subalgebra of or a quasidiagonal representation into the Calkin algebra.

Two separable representations π, ρ of a separable C^*-algebra $\mathscr{A}$ are *approximately equivalent* [23] provided there is a sequence $\{U_n\}$ of unitary operators such that $U_n^*\pi(A)U_n - \rho(A)$ is compact for all $n \geqslant 1$ and all A in $\mathscr{A}$ and $\|U_n^*\pi(A)U_n - \rho(A)\| \to 0$ for every A in $\mathscr{A}$. A reformulation [10] of a deep theorem of D. Voiculescu [23] states that if $\pi(1) = 1$ and $\rho(1) = 1$, then π and ρ are approximately equivalent precisely when $\operatorname{rank}\pi(A) = \operatorname{rank}\rho(A)$ for every A in $\mathscr{A}$.

COROLLARY 3. *If π and ρ are approximately equivalent representations of a separable C^*-algebra, then π is quasidiagonal precisely when ρ is.*

It follows from Voiculescu's theorem [23, Theorem 1.5] that two irreducible representations of a separable C^*-algebra are approximately equivalent if and only if they have the same kernel (see [10, Corollary 2.11]).

COROLLARY 4. *If π and ρ are irreducible representations of a separable C^*-algebra with the same kernel, then π is quasidiagonal precisely when ρ is.*

In [22] Thayer proved that the separable quasidiagonal representations of a separable C^*-algebra are closed under countable direct sums and certain kinds of direct integrals; the author [10] proved that they are closed under arbitrary (separable) direct integrals.

Recall that an ideal of a C^*-algebra is primitive if it is the kernel of an irreducible representation. The following proposition shows that to prove strong quasidiagonality it is necessary to check only the irreducible representations.

PROPOSITION 5. *A separable C^*-algebra $\mathscr{A}$ is strongly quasidiagonal if and only if, for each primitive ideal $\mathscr{J}$ of $\mathscr{A}$ there is an irreducible quasidiagonal representation of $\mathscr{A}$ whose kernel is $\mathscr{J}$.*

Proof. This follows from Corollary 4 and a result of Voiculescu [23, Corollary 1.6], which says that every separable representation of $\mathscr{A}$ is approximately equivalent to a direct sum of irreducible representations. ▨

For simple C^*-algebras the notions of strong and weak quasidiagonality coincide. This is a result of L. G. Brown, and it appears in [8].

COROLLARY 6. *If a simple separable C^*-algebra has a non-degenerate quasidiagonal representation, then it is strongly quasidiagonal.*

REMARK. Note that M.-D. Choi [7] provided an example of a simple separable irreducible C^*-algebra of operators that is not quasidiagonal (although every operator in it is quasitriangular).

The following example illustrates the difference between strong and weak quasidiagonality. It is no coincidence that the most popular example (counterexample) among quasidiagonal operators generates the most popular example (counterexample) concerning quasidiagonal C^*-algebras.

EXAMPLE 7. Let S be the unilateral shift operator of multiplicity 1, let $A = S \oplus S^*$, and let $B = S \oplus S \oplus S^*$. Since A is the sum of a diagonal operator and a compact operator [6], we conclude that A, and hence $C^*(A)$, is quasidiagonal. On the other hand, B is not quasidiagonal (e.g., its Fredholm index is -1) and thus $C^*(B)$, which is $*$-isomorphic to $C^*(A)$, is not quasidiagonal. Norberto Salinas has pointed out that while B is not quasidiagonal, $B^{(\infty)} = B \oplus B \oplus \ldots$ is quasidiagonal (since $B^{(\infty)}$ is unitarily equivalent to $A^{(\infty)}$). Moreover, L. G. Brown [5] has constructed a separable C^*-algebra and a non-quasidiagonal representation π such that $\pi \oplus \pi$ is quasidiagonal.

The next proposition follows immediately from analogous results in [22].

PROPOSITION 8. *Suppose $\mathscr{A}$ is a separable C^*-algebra.*

(1) *If $\mathscr{A}$ is the direct limit of separable strongly quasidiagonal C^*-algebras, then $\mathscr{A}$ is strongly quasidiagonal.*

(2) *If $\mathscr{A}$ is* CCR *or* AF, *then $\mathscr{A}$ is strongly quasidiagonal.*

(3) *If $\{\mathscr{A}_n\}$ is a sequence of separable strongly (weakly) quasidiagonal C^*-algebras, and $\mathscr{A} = \sum_n^{\oplus} \mathscr{A}_n = \{A_1 \oplus A_2 \oplus \ldots : A_n \in \mathscr{A}_n \text{ for all } n, \|A_n\| \to 0\}$, then $\mathscr{A}$ is strongly (weakly) quasidiagonal.*

The following example shows that part (3) of the preceding proposition cannot be extended to more general direct sums.

EXAMPLE 9. Let $\mathscr{A}_n$ be the C^*-algebra of all complex $n \times n$ matrices for $n = 1, 2, \ldots$, and let $\mathscr{A}_\infty = \sum_n^{\oplus} \mathscr{A}_n$. For each positive integer n, let T_n be the $n \times n$ nilpotent Jordan block, and let $T = T_1 \oplus T_2 \oplus \ldots$. Let $\mathscr{A}$ be the C^*-algebra generated by 1, T, and $\mathscr{A}_\infty$. The preceding proposition implies that $\mathscr{A}_\infty$ is strongly quasidiagonal; however, $\mathscr{A}$ is not. To see this note that T is a power partial isometry [11], $\mathscr{A}_\infty$ is an ideal in $\mathscr{A}$, and $\mathscr{A}/\mathscr{A}_\infty$ is $*$-isomorphic to $C^*(S \oplus S \oplus S^*)$, the non-quasidiagonal C^*-algebra in Example 7.

It is clear that every subalgebra of a weakly quasidiagonal C^*-algebra is weakly quasidiagonal; the same is not true of quotients.

PROPOSITION 10. *Every separable C^*-algebra is a $*$-homomorphic image of a weakly quasidiagonal C^*-algebra.*

Proof. This is an immediate consequence of Proposition 3 in [2]. We briefly sketch an alternative proof.

Suppose $\mathscr{A}$ is a C^*-subalgebra of $B(H)$, and let $\{P_n\}$ be an increasing sequence of projections converging strongly to 1 such that rank $P_n = n$ for each n. Let $H^{(\infty)} = H \oplus H \oplus \dots$, and define a self-adjoint linear isometry $\rho : \mathscr{A} \to B(H^{(\infty)})$ by $\rho(A) = (P_1 A P_1) \oplus (P_2 A P_2) \oplus \dots$. The map ρ^{-1} extends to a $*$-homomorphism $\pi : C^*(\rho(\mathscr{A})) \to \mathscr{A}$. Clearly, π is onto and $C^*(\rho(\mathscr{A}))$ is quasidiagonal. ▨

REMARK. Note that if S is the unilateral shift operator, then $C^*(S)$ is not weakly quasidiagonal, although the ideal $\mathscr{J}$ of compact operators and the quotient $C^*(S)/\mathscr{J}$ are both strongly quasidiagonal.

PROPOSITION 11. *Suppose $\mathscr{A}$ is a separable strongly quasidiagonal C^*-algebra and $\mathscr{J}$ is a closed ideal in $\mathscr{A}$. Then $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ are strongly quasidiagonal.*

Proof. It is clear from the definition of strong quasidiagonality that $\mathscr{A}/\mathscr{J}$ is strongly quasidiagonal. The strong quasidiagonality of $\mathscr{J}$ follows from Proposition 5 and the fact that every non-zero irreducible representation of $\mathscr{J}$ is the restriction to $\mathscr{J}$ of an irreducible representation of $\mathscr{A}$ [1, Theorem 1.3.4]. ▨

The following example shows that the converse of the preceding proposition is false even for weakly quasidiagonal C^*-algebras.

EXAMPLE 12. Let S be the unilateral shift operator, $\mathscr{A} = C^*(S \oplus S^*)$, and let $\mathscr{J} = \mathscr{A} \cap \mathscr{K}(H \oplus H)$. Then $\mathscr{J}$ and $\mathscr{A}/\mathscr{J}$ are both strongly quasidiagonal, and $\mathscr{A}$ is weakly quasidiagonal, but $\mathscr{A}$ is not strongly quasidiagonal.

Since subalgebras of weakly quasidiagonal C^*-algebras are weakly quasidiagonal, it is reasonable to ask if the same holds for strongly quasidiagonal C^*-algebras. The negative answer is provided by the following example, which also shows that the analogue of Corollary 2 for strong quasidiagonality is false.

EXAMPLE 13. Let S be the unilateral shift operator, and let $\mathscr{A} = \{A + K : A \in C^*(S \oplus S^*)$ and K is compact$\}$. Note that an irreducible representation of $\mathscr{A}$ that annihilates the compact operators is 1-dimensional, and an irreducible representation that does not annihilate the compact operators is unitarily equivalent to the identity representation. Thus, by Proposition 5, $\mathscr{A}$ is strongly quasidiagonal. However, $C^*(S \oplus S^*)$ is a hereditary subalgebra of $\mathscr{A}$ that is not strongly quasidiagonal.

3. TENSOR PRODUCTS

We now consider tensor products and quasidiagonality. Recall that if H_1 and H_2 are Hilbert spaces with orthonormal bases $\{e_i : i \in I\}$ and $\{e_j : j \in J\}$, respectively, then $\{e_i \otimes e_j : (i,j) \in I \times J\}$ is an orthonormal basis for $H_1 \otimes H_2$. Also if $A_i \in B(H_i)$ for $i = 1, 2$, then the operator $A_1 \otimes A_2$ is defined on $H_1 \otimes H_2$ by $(A_1 \otimes A_2)(e_i \otimes e_j) = (A_1 e_i) \otimes (A_2 e_j)$. Note that $\|A_1 \otimes A_2\| = \|A_1\| \, \|A_2\|$ and $\operatorname{rank}(A_1 \otimes A_2) = (\operatorname{rank} A_1)(\operatorname{rank} A_2)$. It follows that if A_1 and A_2 are compact, then $A_1 \otimes A_2$ is compact.

LEMMA 14. *If $\mathscr{A}_i$ is a separable quasidiagonal C^*-subalgebra of $B(H_i)$ for $i = 1, 2$, then $C^*(\{A_1 \otimes A_2 : A_i \in \mathscr{A}_i, \ i = 1, 2\})$ is quasidiagonal.*

Proof. Suppose $\{P_n\}$, $\{Q_n\}$ are increasing sequences of projections in $B(H_1)$, $B(H_2)$, respectively, that converge strongly to 1 and such that $\|P_n A_1 - A_1 P_n\| \to 0$, and $\|Q_n A_2 - A_2 Q_n\| \to 0$ for every A_i in $\mathscr{A}_i$, $i = 1, 2$. If $E_n = P_n \otimes Q_n$, then it follows from Lemma 1 and a computation similar to that in the proof of continuity of multiplication that the C^*-algebra in question is indeed quasidiagonal. ▨

Note that if $\mathscr{A}_1$ and $\mathscr{A}_2$ are C^*-algebras, then there is not necessarily a unique tensor product C^*-algebra, unless one of the algebras is nuclear [21]; however, $\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2$ always exists and is defined by taking faithful representations and using the construction in Lemma 14.

COROLLARY 15. *If $\mathscr{A}_1$ and $\mathscr{A}_2$ are weakly quasidiagonal C^*-algebras, then $\mathscr{A}_1 \otimes_{\min} \mathscr{A}_2$ is weakly quasidiagonal.*

THEOREM 16. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are separable unital strongly quasidiagonal C^*-algebras and $\mathscr{A}_1$ is nuclear. Then $\mathscr{A}_1 \otimes \mathscr{A}_2$ is strongly quasidiagonal.*

Proof. Suppose that $\mathscr{I}$ is a primitive ideal of $\mathscr{A}_1 \otimes \mathscr{A}_2$. It follows from Theorem 3.3 in [3] that there are irreducible representations $\pi_i : \mathscr{A}_i \to B(H_i)$ for $i = 1, 2$ such that if π is the representation from $\mathscr{A}_1 \otimes \mathscr{A}_2$ into $B(H_1 \otimes H_2)$ defined by $\pi(A_1 \otimes A_2) = \pi_1(A_1) \otimes \pi_2(A_2)$, then π is irreducible and $\mathscr{I} = \ker \pi$. It follows from Lemma 14 that π is quasidiagonal (since π_1 and π_2 are). Thus, by Proposition 5, $\mathscr{A}_1 \otimes \mathscr{A}_2$ is strongly quasidiagonal. ▨

COROLLARY 17. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are separable strongly quasidiagonal C^*-subalgebras of a C^*-algebra $\mathscr{A}$. If $\mathscr{A}_1$ is nuclear and each element of $\mathscr{A}_1$ commutes with each element of $\mathscr{A}_2$, then $C^*(\mathscr{A}_1 \cup \mathscr{A}_2)$ is strongly quasidiagonal.*

Proof. $C^*(\mathscr{A}_1 \cup \mathscr{A}_2)$ *is a $*$-homomorphic image of* $\mathscr{A}_1 \otimes \mathscr{A}_2$. ▨

The following example shows that the analogue of the preceding corollary is false for weak quasidiagonality.

EXAMPLE 18. Let S be the unilateral shift, let $\mathscr{A}_1 = C^*(S \oplus S^*)$, and et $\mathscr{A}_2 = C^*(1 \oplus 0)$. Then both $\mathscr{A}_1$ and $\mathscr{A}_2$ are nuclear (in fact, GCR), $\mathscr{A}_1$ is quasidiagonal, and $\mathscr{A}_2$ is strongly quasidiagonal, but, in spite of the fact that every operator in $\mathscr{A}_1$ commutes with every operator in $\mathscr{A}_2$, $C^*(\mathscr{A}_1 \cup \mathscr{A}_2)$ is not quasidiagonal (or even weakly quasidiagonal) since it contains $S \oplus 0$.

Two separable C^*-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ are *stably isomorphic* if $\mathscr{A}_1 \otimes \mathscr{K}(H)$ is isomorphic to $\mathscr{A}_2 \otimes \mathscr{K}(H)$, where H is separable and infinite-dimensional. It follows from Corollary 15 and Theorem 16 that both strong and weak quasidiagonality are preserved under stable isomorphism if the strong (weak) quasidiagonality of $\mathscr{A} \otimes \mathscr{K}(H)$ implies the strong (weak) quasidiagonality of $\mathscr{A}$. This is precisely the content of the following theorem. First we need a lemma.

LEMMA 19. *Suppose $\mathscr{A}$ is a quasidiagonal C^*-subalgebra of $B(H)$ and P is a projection in $\mathscr{A}$. Then $P\mathscr{A}P|\operatorname{ran} P$ is quasidiagonal.*

Proof. Suppose $\{P_n\}$ is an increasing sequence of finite-rank projections converging strongly to 1 such that $\|P_nA - AP_n\| \to 0$ for every A in $\mathscr{A}$. Then $\{PP_nP\}$ is a sequence of positive operators converging strongly to P. Since $\|P_nP - PP_n\| \to 0$, it follows that $\|(PP_nP)^2 - PP_nP\| \to 0$. Thus there is a sequence $\{Q_n\}$ of projections whose ranges are contained in $\operatorname{ran} P$ such that $\|PP_nP - Q_n\| \to 0$. Thus $\{Q_n|\operatorname{ran} P\}$ is the sequence of finite-rank projections needed to prove (using Lemma 1) that $P\mathscr{A}P|\operatorname{ran} P$ is quasidiagonal. ▨

THEOREM 20. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are separable unital and stably isomorphic. Then $\mathscr{A}_1$ is strongly (weakly) quasidiagonal if and only if $\mathscr{A}_2$ is.*

Proof. In view of the remarks preceding Lemma 19, it suffices to show that if $\mathscr{A} \otimes \mathscr{K}(H)$ is strongly (weakly) quasidiagonal, then so is $\mathscr{A}$. The implication for weak quasidiagonality follows from the fact that if P is a non-zero projection in $\mathscr{K}(H)$, then $\{A \otimes P : A \in \mathscr{A}\}$ is a C^*-subalgebra of $\mathscr{A} \otimes \mathscr{K}(H)$ that is $*$-isomorphic to $\mathscr{A}$.

Suppose that $\mathscr{A} \otimes \mathscr{K}(H)$ is strongly quasidiagonal, and let $\pi : \mathscr{A} \to B(H_\pi)$ be a separable representation of $\mathscr{A}$. Let Q be a projection in $\mathscr{K}(H)$ whose rank is 1, and let $P = \pi(Q) \otimes 1$. Then there is a representation $\rho : \mathscr{A} \otimes \mathscr{K}(H) \to B(H_\pi \otimes H)$ defined by $\rho(A \otimes K) = \pi(A) \otimes K$. Then, by Lemma 19, $P\rho(\mathscr{A} \otimes \mathscr{K}(H))P|\operatorname{ran} P$ is quasidiagonal, and is clearly unitarily equivalent to $\pi(\mathscr{A})$. Thus π is quasidiagonal, and therefore $\mathscr{A}$ is strongly quasidiagonal. ▨

REMARK. The preceding theorem reduces the problem of determining the quasidiagonality (weak or strong) to the case of singly generated C^*-algebras.

This follows from the theorem of C. Olsen and W. Zame [16] that says that $\mathscr{A}\otimes\mathscr{K}(H)$ is singly generated when $\mathscr{A}$ is separable and H is separable and infinite-dimensional. This in turn reduces the problem to that of determining which irreducible Hilbert space operators are quasidiagonal, i.e., $C^*(A)$ is strongly quasidiagonal if and only if $\pi(A)$ is quasidiagonal for every irreducible representation π.

4. CROSSED PRODUCTS

Let Aut $\mathscr{A}$ denote the group of $*$-automorphisms of the unital C^*-algebra $\mathscr{A}$. An automorphism α in Aut $\mathscr{A}$ induces a homeomorphism on the quasispectrum of $\mathscr{A}$ [24]; we call α freely acting if this homeomorphism has no periodic points. The automorphism α also induces a natural homomorphism, $n \to \alpha^n$, from the additive group, $(\mathbf{Z}, +)$, of integers into Aut $\mathscr{A}$. The crossed product C^*-algebra $\mathscr{A}\times_\alpha \mathbf{Z}$ can be defined as a C^*-algebra generated by $\mathscr{A}$ and a unitary element u such that:

(1) $u^*au = \alpha(a)$ for every a in $\mathscr{A}$, and

(2) if $\pi : \mathscr{A} \to B(H_\pi)$ is a representation and U is a unitary operator in $B(H_\pi)$ such that $\pi(\alpha(a)) = U^*\pi(a)U$ for every a in $\mathscr{A}$, then there is a representation $\rho : \mathscr{A}\times_\alpha \mathbf{Z} \to B(H_\pi)$ such that $\rho(u) = U$ and $\rho|\mathscr{A} = \pi$.

We wish to discuss a certain class of representations of $\mathscr{A}\times_\alpha \mathbf{Z}$. Suppose H is a Hilbert space. Let $\ell^2(\mathbf{Z}, H)$ denote the set of functions $f : \mathbf{Z} \to H$ such that $\sum_n \|f(n)\|^2 = \|f\|^2$ is finite, and let $\ell^\infty(\mathbf{Z}, B(H))$ denote the set of bounded functions from $\mathbf{Z}$ into $B(H)$. It is clear that $\ell^2(\mathbf{Z}, H)$ is a Hilbert space, and each φ in $\ell^\infty(\mathbf{Z}, B(H))$ can be identified with an operator (which we shall still call φ) on $\ell^2(\mathbf{Z}, H)$ defined by $(\varphi f)(n) = \varphi(n)f(n)$ for each f in $\ell^2(\mathbf{Z}, H)$ and each n in $\mathbf{Z}$. More generally, if $\mathscr{A}$ is a C^*-subalgebra of $B(H)$, then $\ell^\infty(\mathbf{Z}, \mathscr{A})$ denotes the set of bounded functions from $\mathbf{Z}$ into $\mathscr{A}$.

Suppose $\pi : \mathscr{A} \to B(H_\pi)$ is a representation of $\mathscr{A}$. We define a representation $\pi_\alpha : \mathscr{A} \to \ell^\infty(B(H_\pi))$ by $\pi_\alpha(a)(n) = \pi(\alpha^n(a))$. If W is the bilateral shift operator on $\ell^2(\mathbf{Z}, H_\pi)$, i.e., $(Wf)(n) = f(n-1)$, then $W^*\pi_\alpha(a)W = \pi_\alpha(\alpha(a))$ for every a in $\mathscr{A}$. It follows from (2) above that there is a representation $\operatorname{Ind}(\pi) : \mathscr{A}\times_\alpha \mathbf{Z} \to B(\ell^2(\mathbf{Z}, H_\pi))$ that sends u to W and whose restriction to $\mathscr{A}$ is π_α.

It is shown in [24] that if π is faithful, then so is $\operatorname{Ind}(\pi)$: this gives us a way to visualize $\mathscr{A}\times_\alpha \mathbf{Z}$. Moreover, it is shown in [24] that if α is freely acting, then $\{\ker(\operatorname{Ind}\pi) : \pi$ is an irreducible representation of $\mathscr{A}\}$ is the set of primitive ideals of $\mathscr{A}$, and that $\operatorname{Ind}\pi$ is irreducible whenever π is.

The following result was proved by M. Pimsner and D. Voiculescu [17, Lemma 3.6].

LEMMA 21. [17]. *Suppose $\mathscr{A}$ is weakly quasidiagonal, $\alpha \in \operatorname{Aut}\mathscr{A}$, and there is a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$, such that $\|a - \alpha^{n_k}(a)\| \to 0$ for every a in $\mathscr{A}$. Then $\mathscr{A} \times_\alpha \mathbf{Z}$ is weakly quasidiagonal.*

The main idea of the proof in [17] of the preceding lemma appears in the doctoral dissertation of R. Smucker [19], [20], who characterized the unilateral and bilateral weighted shift operators that are quasidiagonal. The proof in [17] actually proves the following more general result.

LEMMA 22. *Suppose H is a separable Hilbert space, $\mathscr{A}$ is a quasidiagonal C^*-subalgebra of $B(H)$, $\{\varphi_n\}$ is a sequence in $\ell^\infty(\mathbf{Z}, \mathscr{A})$, and W is the bilateral shift on $\ell^2(\mathbf{Z}, H)$. Suppose also that $\{m_k\}$, $\{n_k\}$ are sequences of integers with $m_k \to -\infty$ and $n_k \to +\infty$ such that*

$$\|\varphi_n(j + m_k) - \varphi_n(j + n_k)\| \to 0 \quad \text{as } k \to \infty, \text{ for } n, j = 1, 2, \ldots .$$

Then $C^*(W, \varphi_1, \varphi_2, \ldots)$ is quasidiagonal.

We now obtain our main result on the strong quasidiagonality of crossed products.

THEOREM 23. *Suppose $\mathscr{A}$ is strongly quasidiagonal and unital, $\alpha \in \operatorname{Aut}\mathscr{A}$ is freely acting. Suppose that for each irreducible representation π of $\mathscr{A}$, there are sequences $\{m_k\}$, $\{n_k\}$ of integers with $m_k \to -\infty$ and $n_k \to +\infty$ such that*

$$\|\pi(\alpha^{m_k+j}(a)) - \pi(\alpha^{n_k+j}(a))\| \to 0 \quad \text{for } a \in \mathscr{A},\ j = 1, 2, \ldots . \tag{1}$$

Then $\mathscr{A} \times_\alpha \mathbf{Z}$ is strongly quasidiagonal.

Proof. Suppose $\mathscr{J}$ is a primitive ideal of $\mathscr{A} \times_\alpha \mathbf{Z}$. Since α is freely acting, there is an irreducible representation π of $\mathscr{A}$ so that $\mathscr{J}$ is the kernel of $\operatorname{Ind}\pi$. Let $\{a_1, a_2, \ldots\}$ be a dense subset of $\mathscr{A}$. If we let $\varphi_n = \operatorname{Ind}\pi(a_n)$ for $n = 1, 2, \ldots$, and if we choose sequences $\{m_k\}$ and $\{n_k\}$ of integers with $m_k \to -\infty$ and $n_k \to +\infty$ so that (1) above holds, then it follows from Lemma 22 that $\operatorname{Ind}\pi$ is quasidiagonal. Thus, by Proposition 5, $\mathscr{A} \times_\alpha \mathbf{Z}$ is strongly quasidiagonal. ▨

The following example shows that the assumption that α is freely acting cannot be dropped from Theorem 23.

EXAMPLE 24. Let S be the unilateral shift on H, let $\mathscr{A} = C^*(S \oplus S^*) + \mathscr{K}(H \oplus H)$ let $V = 1 \oplus -1$, and define α in $\operatorname{Aut}\mathscr{A}$ by $\alpha(A) = V^*AV$. Since α^2 is the identity automorphism, it is clear that $\alpha^{-k} = \alpha^k$ for $k = 1, 2, \ldots$. However, if $\mathscr{A} \times_\alpha \mathbf{Z}$ were strongly quasidiagonal, it would follow that $C^*(\mathscr{A} \cup \{V\})$ is quasidiagonal, an impossibility, since the latter algebra contains the non-quasidiagonal operator $S \oplus 0 = (1/2)(V + V^2)(S \oplus S^*)$.

In the case when $\mathscr{A}$ is commutative, not only can the assumption that α be freely acting be dropped in Theorem 23, but the converse of the theorem is also true. Suppose $\mathscr{A}$ is commutative and separable, and let X be the maximal ideal space of $\mathscr{A}$. Then X is a compact metric space (assuming $\mathscr{A}$ is unital), and the Gelfand map $\Gamma : \mathscr{A} \to C(X)$ is an isomorphism. An automorphism α in Aut $\mathscr{A}$ corresponds to a homeomorphism $\tau : X \to X$ $(\Gamma(\alpha(a)) = \Gamma(a)\circ\tau$ for all a in $\mathscr{A}$). In this case the notation $C(X)\times_\tau \mathbf{Z}$ is used in place of $\mathscr{A}\times_\alpha \mathbf{Z}$. The reader should consult the paper [9] of P. Green, where a number of interesting examples are worked out and where the idea of the "only if" part of the proof of the following theorem appears [9, Lemma 5]. The following theorem appeared in the original preprint version of this paper and was discovered independently of the results of [17].

THEOREM 25. *Suppose X is a compact metric space and $\tau : X \to X$ is a homeomorphism. Then $C(X)\times_\tau \mathbf{Z}$ is strongly quasidiagonal if and only if, for every x in X, $\{\tau^n(x) : n \geqslant 0\}^- \cap \{\tau^m(x) : m < 0\}^- \neq \emptyset$.*

Proof. Suppose $\{\tau^n(x) : n \geqslant 0\}^- \cap \{\tau^m(x) : m < 0\}^- \neq \emptyset$ for every x in X. It follows from [24] that the kernel of an infinite-dimensional irreducible representation of $C(X)\times_\tau \mathbf{Z}$ is the kernel of $\mathrm{Ind}\,\pi$ for some irreducible representation π of $C(X)$. Hence there is an x in X such that $\pi(f) = f(x)$ for every f in $C(X)$. However, there is an y in X and sequences $\{m_k\}$ and $\{n_k\}$ of integers with $m_k \to -\infty$ and $n_k \to +\infty$ such that $\tau^{m_k}(x) \to y$ and $\tau^{n_k}(x) \to y$. It follows from Lemma 22 that $\mathrm{Ind}\,\pi$ is quasidiagonal. Hence $C(X)\times_\tau \mathbf{Z}$ is strongly quasidiagonal.

Conversely, suppose that $x \in X$ and $\{\tau^n(x) : n \geqslant 0\}^- \cap \{\tau^m(x) : m < 0\}^-$ is empty. Define π on $C(X)$ by $\pi(f) = f(x)$. If we choose a function f in $C(X)$ so that $f|\{\tau^n(x) : n \geqslant 0\} = 1$ and $f|\{\tau^m(x) : m < 0\} = 0$, then $\mathrm{Ind}\,\pi$ sends f to an operator that is unitarily equivalent to $0 \oplus S$, where S is the unilateral shift. Hence $\mathrm{Ind}\,\pi$ is not quasidiagonal, and it follows that $C(X)\times_\tau \mathbf{Z}$ is not strongly quasidiagonal. ∎

The proof of the preceding theorem also proves the following.

COROLLARY 26. *If τ is a homeomorphism on a compact metric space X, and if the collection of x's in X such that $\{\tau^n(x) : n \geqslant 0\}^- \cap \{\tau^m(x) : m < 0\}^- \neq \emptyset$ is dense in X, then $C(X)\times_\tau \mathbf{Z}$ is weakly quasidiagonal.*

COROLLARY 27. *Suppose τ is a homeomorphim on a compact metric space X, $\mathscr{B}$ is a strongly quasidiagonal unital C^*-algebra, and $\mathscr{A}$ is the C^*-algebra of all continuous functions from X into $\mathscr{B}$. Let α be the automorphism on $\mathscr{A}$ defined by $\alpha(f) = f\circ\tau$. Then $\mathscr{A}\times_\alpha \mathbf{Z}$ is strongly quasidiagonal if and only if $\{\tau^n(x) : n \geqslant 0\}^- \cap \{\tau^m(x) : m < 0\}^- \neq \emptyset$ for every x in X.*

Proof. The algebra $\mathscr{A}$ is isomorphic to $C(X)\otimes\mathscr{B}$, and since the automorphism α only acts on $C(X)$, $\mathscr{A}\times_\alpha \mathbf{Z}$ is isomorphic to $(C(X)\times_\tau \mathbf{Z})\otimes\mathscr{B}$. The corollary follows from Theorem 25 and Theorem 16 (since $C(X)$ is nuclear). ▨

It is interesting to compare Theorem 25 with known characterizations of other properties of $C(X)\times_\tau \mathbf{Z}$.

LEMMA 28. [24]. *Suppose τ is a homeomorphism on a compact Hausdorff space X. Then*

(1) *$C(X)\times_\tau \mathbf{Z}$ is* GCR *if and only if $\{\tau^n(x) : n \in \mathbf{Z}\}$ is discrete in the relative topology for each x in X, and*

(2) *$C(X)\times_\tau \mathbf{Z}$ is simple if and only if $\{\tau^n(x) : n \in \mathbf{Z}\}^- = X$ for every x in X.*

COROLLARY 29. *If $C(X)\times_\tau \mathbf{Z}$ is simple, then it is strongly quasidiagonal.*

We therefore see that M. Rieffel's irrational rotation algebras are strongly quasidiagonal.

EXAMPLE 30. If we let $X = [0, 1]$ and let $\tau(x) = x^2$, then $C(X)\times_\tau \mathbf{Z}$ is GCR but not strongly quasidiagonal. If we identify the points 0 and 1, then the associated crossed product is both GCR and strongly quasidiagonal. The irrational rotation algebras give examples that are strongly quasidiagonal but not GCR.

5. QUESTIONS AND COMMENTS

1. Call a locally compact group G strongly or weakly quasidiagonal if $C^*(G)$ is. Which groups are strongly (weakly) quasidiagonal? Note that Choi [7] has shown that the free group on two generators is not strongly quasidiagonal.

2. Is every strongly quasidiagonal C^*-algebra nuclear? Note that Example 30 shows that there is no relation between strong quasidiagonality and being GCR. Which GCR C^*-algebras are strongly quasidiagonal? Note that if π is an irreducible representation of a GCR C^*-algebra $\mathscr{A}$, then $\pi(\mathscr{A})$ contains all of the compact operators. Thus the finite-rank projections needed to show that $\pi(\mathscr{A})$ is quasidiagonal are in $\pi(\mathscr{A})$. This suggests that there may be some "internal" way of determining when a GCR C^*-algebra is strongly quasidiagonal.

3. What about the strong or weak quasidiagonality of $\mathscr{A}\otimes_{\max}\mathscr{B}$ when $\mathscr{A}$ and $\mathscr{B}$ are strongly or weakly quasidiagonal C^*-algebras that are not nuclear. The proof we gave of Theorem 16 applies to $\mathscr{A}\otimes_{\min}\mathscr{B}$ whenever Blackadar's characterization [3] of the primitive ideals holds; Blackadar [4] has shown that it holds in the case that one of the algebras is a C^*-subalgebra of a nuclear C^*-algebra.

4. Note that Lemma 19 implies an improved version of Corollary 2: if $\mathscr{A}$ and $\mathscr{B}$ are C^*-subalgebras of $B(H)$ whose images in the Calkin algebra $B(H)/\mathscr{K}(H)$ are unitarily equivalent (using a unitary element from the Calkin

algebra), and if $\mathscr{A}$ is quasidiagonal, then so is $\mathscr{B}$. Using the characterization of unitary elements in the Calkin algebra [6], the problem easily reduces to showing that if $\mathscr{A}$ is quasidiagonal and S is a unilateral shift of finite multiplicity, then $S^*\mathscr{A}S$ is quasidiagonal. Clearly, we can assume that $\mathscr{A}$ contains $\mathscr{K}(H)$. Then $P = SS^*$ is a projection in $\mathscr{A}$, and $S^*\mathscr{A}S$ is unitarily equivalent to $P\mathscr{A}P|\operatorname{ran} P$.

5. For some interesting relationships between quasidiagonality and the Brown-Douglas-Fillmore theory, see the paper of L. G. Brown [5].

6. The results on crossed products might be pushed in several directions. In which cases can the "freely acting" assumption in Theorem 23 be dropped? Is it always true that $\mathscr{A} \times_\alpha \mathbf{Z}$ is strongly quasidiagonal whenever it is simple and $\mathscr{A}$ is strongly quasidiagonal? Perhaps some of the results can be extended to the case when the group $\mathbf{Z}$ is replaced by the group $\mathbf{R}$ of real numbers or by $\mathbf{Z}\oplus\mathbf{Z}$.

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APPENDIX:

QUASIDIAGONALITY AND NUCLEARITY

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In § 5 of the above paper, Hadwin raises the questions of which groups have quasidiagonal C^*-algebras, and whether every strongly quasidiagonal C^*-algebra is nuclear. We partially answer the first question and present some evidence for a positive answer to the second question in the following two theorems. The first of the two generalizes Example B of [A2, § 3].

THEOREM A1. *Let G be a countable discrete group, and suppose the left regular representation λ of $C^*(G)$ on $L^2(G)$ is quasidiagonal. Then G is amenable.*

COROLLARY. *If G is not amenable $C^*(G)$ and $C_r^*(G)$ are not strongly quasidiagonal. If G is nonamenable and $C_r^*(G)$ has no faithful representation whose image contains a non-zero compact operator (which is almost always the case and, for instance, is obvious if $C_r^*(G)$ is simple, which already occurs in many cases—see* [A8], [A2], [A1], [A5], *and* [A6]*), then $C_r^*(G)$ is not weakly quasidiagonal.*

Proof of Corollary. This is immediate from the theorem, given Hadwin's Corollaries 3 and 6.

Note that by [A7, Theorem 4.2], $C^*(G)$ is nuclear if and only if $C_r^*(G)$ is nuclear, which occurs if and only if G is amenable. Thus Theorem A1 provides evidence that (at least under extra hypotheses) strongly quasidiagonal C^*-algebras must be nuclear. Some additional evidence is provided by the following.

THEOREM A2. *If a separable C^*-algebra $\mathscr{A}$ is strongly quasidiagonal, then every completely positive contraction from $\mathscr{A}$ to an injective von Neumann algebra $\mathscr{R}$ is nuclear (*i.e., *is a point-norm limit of composites of completely positive contractions $\mathscr{A} \to M_n \to \mathscr{R}$ factoring through finite-dimensional matrix rings).*

COROLLARY. *If $\mathscr{A}$ is strongly quasidiagonal, then $\mathscr{A}$ does not contain an operator system matricially order-isomorphic to the finite-dimensional operator system $N^d_{2n_0}$ of* [A3, Theorem 10.2].

Proof of Corollary. As in the proof of [A3, Theorem 10.2], there is a non-nuclear completely positive contraction $N^d_{2n_0} \to \mathscr{B}(H)$ (H separable), and if there were a matricial order embedding of $N^d_{2n_0}$ into $\mathscr{A}$, one could use injectivity of $\mathscr{B}(H)$ to extend this to a non-nuclear completely positive contraction from $\mathscr{A}$ to $\mathscr{B}(H)$, contradicting the theorem.

Note that had the conjecture of [A3, § 10] been correct, this corollary would show that a strongly quasidiagonal C^*-algebra must be nuclear. As we now know [A2] that there are subnuclear algebras which are not nuclear, things are more subtle, but it is plausible that one could show, say, that strongly quasidiagonal algebras are C^*-exact. It would also be interesting to determine if $\mathscr{R}$ in Theorem A2 could be replaced by the Calkin algebra, as this is what one would need to show that $\mathrm{Ext}(\mathscr{A})$ is a group.

Proof of Theorem A1. If λ is quasidiagonal, then given $x_1, \ldots, x_n \in G$ and $\varepsilon > 0$ (small), $N > 0$ (large), there must be a projection p in $\mathscr{B}(L^2(G))$ of finite rank $> N$ and with

$$\|\lambda(x_i)p\lambda(x_i)^{-1} - p\| < \varepsilon, \quad 1 \leqslant i \leqslant n.$$

View p as an element of the Hilbert space $H \otimes H$ (identified with the Hilbert-Schmidt operators on H). The action π of G on Hilbert-Schmidt operators via

$$\pi(x)a = \lambda(x)a\lambda(x)^{-1}$$

corresponds to translation of kernels, and thus π on $H \otimes H$ is an infinite multiple of λ (unless G is finite, in which case there is nothing to prove, anyway). Furthermore,

$$\|p\|_2 = \sqrt{\operatorname{rk} p} > \sqrt{N},$$

and

$$\|\pi(x_i)p - p\|_2 \leqslant \sqrt{\operatorname{rk}(\pi(x_i)p - p)}\|\lambda(x_i)p\lambda(x_i)^{-1} - p\|$$

so that if $\xi = p/\|p\|_2$,

$$\|\xi\|_2 = 2 \quad \text{and} \quad \|\pi(x_i)\xi - \xi\|_2 < \sqrt{2}\varepsilon, \quad 1 \leqslant i \leqslant n.$$

Thus the trivial representation of G is weakly contained in π, hence in λ, and G is amenable ([A4, Theorem 3.5.2]).

Proof of Theorem A2. Let $\mathscr{R}$ be nuclear and $\varphi : \mathscr{A} \to \mathscr{R}$ a completely positive contraction. Since we may embed $\mathscr{R}$ in $\mathscr{B}(H)$ for some H and use injectivity of $\mathscr{R}$ to push maps down from $\mathscr{B}(H)$ to $\mathscr{R}$, it is no loss of generality to assume $\mathscr{R}$ is a type I factor. Also, we may adjoin an identity to $\mathscr{A}$ if necessary and assume $\mathscr{A}$ and φ are unital. By Stinespring's Theorem, φ is the compression on H of a representation π of $\mathscr{A}$ on a larger Hilbert space $H \oplus K$. If we can show π is nuclear, then clearly so is φ. Finally, since $\mathscr{A}$ is separable and the statement only needs to be proved on finitely many elements of $H \oplus K$ at a time, we may cut down to a subrepresentation σ of π on a separable Hilbert space.

In other words, we are reduced to showing that if $\mathscr{A}$ is unital and σ is a representation of $\mathscr{A}$ on a separable Hilbert space H_1, then σ is nuclear. By strong quasidiagonality of $\mathscr{A}$, σ is quasidiagonal. If we now choose finite-rank projections p_n with $p_n \uparrow 1$ and with

$$\|\sigma(a)p_n - p_n\sigma(a)\| \to 0 \quad \text{for all } a \in \mathscr{A}$$

then

$$\begin{cases} a \mapsto p_n\sigma(a)p_n : \mathscr{A} \to \mathscr{B}(p_nH_1) \\ b \mapsto b + 1 - p_n : \mathscr{B}(p_nH_1) \to \mathscr{B}(H_1) \end{cases}$$

give approximate factorizations of σ as composites

$$\mathscr{A} \to M_{k_n} \to \mathscr{B}(H_1)$$

of unital completely positive maps, as required.

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