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A NOTE ON THE CLASSES $(\mathrm{BCP})_\theta$

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$.

We are concerned with some classes of operators appearing in the theory of dual algebras, namely $(\mathrm{BCP})_\theta$ operators.

As it was shown in [3] (see also [2]), $(\mathrm{BCP})_\theta$ operators are contained in the class $\mathbf{A}_{\aleph_0}$ (see below the terminology). Both of these proofs use the Sz.-Nagy—Foiaş functional model. Our aim is to give a more direct proof of the above result, using systematically the minimal coisometric extension of a given contraction. The main tools in proving that $(\mathrm{BCP})_\theta$ are included in $\mathbf{A}_{\aleph_0}$ are Lemma 3 below combined with a certain criterion for membership in $\mathbf{A}_{\aleph_0}$ (cf. [2, Theorem 3.7]).

We also show (see Theorem 4) that several classes of contractions whose essential resolvent grows rapidly near the unit circle are included in $\mathbf{A}_{\aleph_0}$.

2. NOTATIONS AND TERMINOLOGY

Recall that $\mathscr{L}(\mathscr{H})$ is the dual space of $\mathscr{C}_1(\mathscr{H})$ — the trace-class operators on $\mathscr{H}$ — the duality map being

$$\langle T, L\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{L}(\mathscr{H}), \ L \in \mathscr{C}_1(\mathscr{H}).$$

A weak* closed subalgebra of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ is called a *dual algebra.* (The paper [2] is the basic reference for the notations and the terminology.) If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra, then $\mathscr{A}$ is identified with the dual space of the Banach space $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$ where ${}^{\perp}\mathscr{A}$ denotes the preannihilator of $\mathscr{A}$ in $\mathscr{C}_1(\mathscr{H})$; this duality is implemented by

$$\langle T, [L]\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{A}, \ [L] \in Q_{\mathscr{A}}.$$

If x and y are vectors from $\mathscr{H}$ then $x \otimes y$ denotes the rank-one operator, i.e.

$$(x \otimes y)(z) = (z, y)x \quad \forall z \in \mathscr{H}.$$

If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and n is a cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$, then $\mathscr{A}$ is said to have property (A_n) if for every system $\{[L_{ij}]\,;\ 0 \leqslant i, j < n\}$ of elements from $Q_{\mathscr{A}}$, there exist vectors $\{x_i, y_j\,;\ 0 \leqslant i, j < n\}$ in $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \quad 0 \leqslant i, j < n.$$

If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and θ is a nonnegative real number, then $\mathscr{X}_\theta(\mathscr{A})$ denotes the set of all $[L]$ in $Q_{\mathscr{A}}$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ satisfying the following conditions:

$$\limsup_i (\|[x_i \otimes y_i] - [L]\|) \leqslant \theta$$

$$\|x_i\| \leqslant 1, \quad \|y_i\| \leqslant 1, \quad i \in \mathbf{N}$$

$$\lim_{i\to\infty}(\|[x_i \otimes z]\| + \|[z \otimes y_i]\|) = 0 \quad \forall z \in \mathscr{H}.$$

Suppose now that $0 \leqslant \theta < \gamma$. Then a dual algebra is said to have property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathscr{X}_\theta(\mathscr{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at 0 in $Q_{\mathscr{A}}$.

As it was shown in [2, Theorem 3.7], if $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra which has property $X_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma$ then $\mathscr{A}$ has also property $(\mathbf{A}_{\aleph_0})$.

Let $\mathbf{D} := \{z \in \mathbf{C}\,;\ |z| < 1\}$ and let $\mathbf{T} = \partial\mathbf{D}$. Then H^∞ denotes, as usual, the Banach algebra of all bounded analytic functions on $\mathbf{D}$. A subset $S \subset \mathbf{D}$ is said to be dominating (for $\mathbf{T}$) if almost every point of $\mathbf{T}$ is a nontangential limit of a sequence of points from S.

If $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous (AC) contraction, then $\mathscr{A}_T$ denotes the dual algebra generated by T in $\mathscr{L}(\mathscr{H})$ and Q_T denotes the predual $Q_{\mathscr{A}_T}$.

The class $\mathbf{A} = \mathbf{A}(\mathscr{H})$ is defined to be the set of all AC contractions $T \in \mathscr{L}(\mathscr{H})$ for which the Sz.-Nagy—Foiaş functional calculus

$$\Phi_T : H^\infty \to \mathscr{A}_T$$

is an isometry.

For such T, one knows (cf. [2, Theorem 4.1]) that Φ_T is a weak* homeomorphism between H^∞ and $\mathscr{A}_T$ and for each $\lambda \in \mathbf{D}$, there exists a unique element $[C_\lambda]$ in Q_T such that

$$\langle \Phi_T(f), [C_\lambda]\rangle = f(\lambda) \quad \forall f \in H^\infty.$$

For any cardinal number n satisfying $1 \leqslant n \leqslant \aleph_0$ the class $\mathbf{A}_n$ consists of all those $T \in \mathbf{A}$ for which the dual algebra $\mathscr{A}_T$ has property (A_n). The basic reference for the theory of dual algebras is [2]. If T is any contraction in $\mathscr{L}(\mathscr{H})$ and $\mu \in \mathbf{D}$, then T_μ denotes the Möbius transform

$$T_\mu = (T - \mu I)(I - \bar{\mu} T)^{-1}.$$

For each θ satisfying $0 \leqslant \theta < 1$, the class $(\mathrm{BCP})_\theta$ consists of all completely nonunitary contractions T in $\mathscr{L}(\mathscr{H})$ for which the set

$$\{\mu \in D \ ; \ \inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leqslant \theta \quad \text{or} \quad \inf \sigma_e((T_\mu T_\mu^*)^{1/2}) \leqslant \theta\}$$

is dominating for $\mathbf{T}$.

Let us define

$$L_\theta(T) = \{\mu \in \mathbf{D} \ ; \ \inf \sigma_e((T_\mu^* T_\mu)^{1/2}) \leqslant \theta\}$$

and

$$R_\theta(T) = \{\mu \in \mathbf{D} \ ; \ \inf \sigma_e((T_\mu T_\mu^*)^{1/2}) \leqslant \theta\}.$$

Thus $T \in (\mathrm{BCP})_\theta$ if and only if the set $L_\theta(T) \cup R_\theta(T)$ is dominating for $\mathbf{T}$.

Let us denote, for an arbitrary contraction T in $\mathscr{L}(\mathscr{H})$, and for each $0 < \theta < 1$,

$$\tilde{\zeta}_\theta(T) = (\sigma_e(T) \cap \mathbf{D}) \cup \left\{\lambda \in \mathbf{D} \setminus \sigma_e(T) \ ; \ \theta \|(\pi(T) - \lambda)^{-1}\| > \frac{1}{1 - |\lambda|}\right\}$$

where $\pi(T)$ denotes the projection onto the Calkin algebra. These sets were introduced by Apostol in [1] and are closely related to the classes $(\mathrm{BCP})_\theta$. Indeed, we have

$$\tilde{\zeta}_\theta(T) \subset L_\theta(T) \cup R_\theta(T) \subset \tilde{\zeta}_{\frac{2\theta}{1-\theta}}(T)$$

for every contraction T in $\mathscr{L}(\mathscr{H})$ and for every θ, satisfying $0 < \theta < 1$ (cf. [2, Proposition 8.1]).

3. THE MAIN THEOREM

In this section we present our proof of the following result:

THEOREM 1. (cf. [2, Theorem 5.2]). *For every $\theta, 0 \leqslant \theta < 1$,* $(\mathrm{BCP})_\theta \subset \mathbf{A}_{\aleph_0}$.

Before proving this theorem, we need two lemmas. Recall that if T is a given contraction in $\mathscr{L}(\mathscr{H})$, then $D_T = (I - T^* T)^{1/2}$ is the defect operator; it is easy to see that $T D_T = D_{T^*} T$.

LEMMA 2. *Let T be in $\mathbf{A}(\mathscr{H})$ and (x_n) be a sequence converging weakly to 0 in $\mathscr{H}$. Then for any $x \in \mathscr{H}$,*

$$\|[D_T x_n \otimes x]\| \to 0 \quad \text{as } n \to \infty.$$

Proof. Let $\varepsilon > 0$ and $x \in \mathscr{H}$. Since $\sum_{j=0}^{\infty} \|D_{T^*} T^{*j} x\|^2 < \infty$ we may choose $N > 1$ such that

$$\sum_{j \geqslant N} \|D_{T^*} T^{*j} x\|^2 < \frac{\varepsilon^2}{4}.$$

Since $x_n \to 0$ weakly, there exists $n_0 > 1$ such that

$$|(T^j D_T x_n, x)| < \frac{\varepsilon}{2(N+1)}, \quad 0 \leqslant j \leqslant N,\ n > n_0$$

Since $TD_T = D_{T^*} T$, we obtain

$$(T^j D_T x_n, x) = (T D_T x_n, T^{*j-1} x) =$$

$$= (D_{T^*} T x_n, T^{*j-1} x) = (T x_n, D_{T^*} T^{*j-1} x), \quad j \geqslant 1.$$

Take $f \in H^\infty$, $\|f\|_\infty = 1$ and write

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in \mathbf{D}.$$

If $0 < r < 1$ and $f_r(z) = f(rz)$, $z \in D$, then we have

$$f_r(T) = \sum_{j=0}^{\infty} a_j r^j T^j$$

therefore

$$|(f_r(T) D_T x_n, x)| \leqslant \sum_{j=0}^{N} |a_j| \, |(T^j D_T x_n, x)| + \sum_{j>N} |a_j| \, |(r^j T^j D_T x_n, x)| \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \Big(\sum_{j>N} |a_j|^2\Big)^{1/2} \Big(\sum_{j \geqslant N} \|D_{T^*} T^{*j} x\|^2\Big)^{1/2} < \varepsilon, \quad \text{for } n > n_0.$$

Since $f_r(T) \xrightarrow[so]{} f(T)$ as $r \to 1$, we obtain

$$|(f(T) D_T x_n, x)| \leqslant \varepsilon \quad \text{for } n > n_0 \quad \text{and} \quad f \in H^\infty,\ \|f\|_\infty = 1$$

hence $\|[D_T x_n \otimes x]\| \leqslant \varepsilon$, $n > n_0$ and the proof is complete.

The following lemma is the main tool in proving that for each $T \in (\mathrm{BCP})_\theta$, $\mathcal{A}_T$ has property $X_{\theta+\varepsilon, 1}$, where $\theta + \varepsilon < 1$.

LEMMA 3. *Suppose* $T \in \mathbf{A}(\mathcal{H})$, $0 < \theta < 1$ *and* $0 < \varepsilon < 1 - \theta$. *If* $\mu \in \mathbf{D}$ *satisfies* $\inf(\sigma_e(T_\mu^* T_\mu)^{1/2}) \leqslant \theta$, *then there exists a sequence* (y_n) *in the unit ball of* $\mathcal{H}$ *such that*

a) $\|[C_\mu] - [y_n \otimes y_n]\| \leqslant \theta + \varepsilon, \quad n \in \mathbf{N}$ *and*

b) $\lim_{n\to\infty} (\|[y_n \otimes z]\| + \|[z \otimes y_n]\|) = 0 \quad \forall z \in \mathcal{H}$.

Proof. Since $T \in \mathbf{A}$, it is easy to see that $T_\mu \in \mathbf{A}$ and that for any x and y in $\mathcal{H}$ we have

$$\|[C_\mu] - [x \otimes y]\|_{Q_T} = \|[C_0] - [x \otimes y]\|_{Q_{T_\mu}}$$

and

$$\|[x \otimes y]\|_{Q_T} = \|[x \otimes y]\|_{Q_{T_\mu}}.$$

These comments show that we may assume that $\mu = 0$. Since $\inf \sigma_e(T^*T)^{1/2} \leqslant \theta$, it follows from elementary spectral theory that there exists an orthonormal sequence $\{x_n\}$ in $\mathcal{H}$ such that

$$\|Tx_n\| \leqslant \theta + \varepsilon \quad \forall n \in \mathbf{N}.$$

Let $V = S \oplus R \in \mathcal{L}(\mathcal{K})$ be the minimal isometric dilation of T^*, where $S \in \mathcal{L}(\mathcal{P})$ is a unilateral shift and $R \in \mathcal{L}(\mathcal{R})$ is an absolutely continuous unitary operator. Then it is well known that

$$V^*\mathcal{H} \subset \mathcal{H}$$

and

$$Th = V^*h \quad \forall h \in \mathcal{H}$$

(cf. [7, Chapters 3 and 4]).

Since $T \in \mathbf{A}(\mathcal{H})$ and T is a part of V^*, it follows that $V^* \in \mathbf{A}(\mathcal{K})$.

Let us consider the orthogonal projection P of $\mathcal{K}$ onto $\operatorname{Ker} V^*$. Then we have

$$\|Px_n\|^2 = \|(I - VV^*)x_n\|^2 = \|x_n\|^2 - \|V^*x_n\|^2 =$$

$$= 1 - \|Tx_n\|^2 \geqslant 1 - (\theta + \varepsilon)^2.$$

Let $z_n = \dfrac{Px_n}{\|Px_n\|}$ and let $y_n = P_{\mathcal{H}} z_n$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. We claim that the sequence $\{y_n\}$ satisfies the conditions a) and b).

We have

$$\|y_n\|^2 = \frac{\|P_{\mathscr{H}} P x_n\|^2}{\|P x_n\|^2} \geqslant \frac{(P_{\mathscr{H}} P x_n, x_n)^2}{\|P x_n\|^2} = \|P x_n\|^2 \geqslant 1 - (\theta + \varepsilon)^2$$

hence

$$\|y_n - z_n\|^2 = \|P_{\mathscr{H}} z_n - z_n\|^2 = \|z_n\|^2 - \|y_n\|^2 \leqslant (\theta + \varepsilon)^2.$$

Since $z_n \in \operatorname{Ker} V^*$ and $\|z_n\| = 1$, it is easy to see that

$$(f(V^*)z_n, z_n) = f(0) \quad \text{for all } f \in H^\infty,$$

or equivalently

$$[C_0]_{Q_{V^*}} = [z_n \otimes z_n]_{Q_{V^*}}.$$

Let us remark that there exists a natural isometry between Q_T and Q_{V^*}

$$j : Q_T \to Q_{V^*}$$

defined by

$$\langle f(T), [L]_T \rangle = \langle f(V^*), j([L]_T) \rangle$$

and it is easy to see that

$$j([C_0]_{Q_T}) = [C_0]_{Q_{V^*}}$$

and

$$j([x \otimes y]_{Q_T}) = [x \otimes y]_{Q_{V^*}} \quad \forall x, y \in T.$$

Since $V^* \mathscr{H} \subset \mathscr{H}$, we also have

$$[x \otimes z]_{Q_{V^*}} = [x \otimes P_{\mathscr{H}} z]_{Q_{V^*}} \quad \forall x \in \mathscr{H},\ z \in \mathscr{K}.$$

Therefore we obtain

$$\|[C_0] - [y_n \otimes y_n]\|_{Q_T} = \|[C_0] - [y_n \otimes y_n]\|_{Q_{V^*}} =$$

$$= \|[C_0] - [P_{\mathscr{H}} z_n \otimes z_n]\|_{Q_{V^*}} \leqslant \|[C_0] - [z_n \otimes z_n]\|_{Q_{V^*}} +$$

$$+ \|z_n - P_{\mathscr{H}} z_n\| \leqslant \theta + \varepsilon \quad \forall n \in \mathbf{N}.$$

It follows that the sequence $\{y_n\}_{n=1}^\infty$ satisfies a).

Let $x \in \mathscr{H}$ and write $x = x^1 \oplus x^2$, where $x^1 \in \mathscr{P}$, $x^2 \in \mathscr{R}$. Since $z_n \in \operatorname{Ker} V^* = \operatorname{Ker} S^*$

$$\|[x \otimes y_n]\|_{Q_T} = \|[x \otimes y_n]\|_{Q_{V^*}} =$$

$$= \|[x \otimes z_n]\|_{Q_{V^*}} = \sup_{\substack{f \in H^\infty \\ \|f\|_\infty \le 1}} |(f(V^*)x, z_n)| =$$

$$= \sup_{\substack{f \in H^\infty \\ \|f\|_\infty \le 1}} |(f(S^*)x^1, z_n)| = \|[x^1 \otimes z_n]\|_{Q_{S^*}}.$$

Since $z_n \underset{w}{\to} 0$ and $S^* \in \mathbf{A} \cap C_{0\cdot}$, this last term tends to 0 (cf. [2, Proposition 6.5]). Now, let us show that $\|[y_n \otimes x]\| \to \infty$ as $n \to \infty$. Let $f \in H^\infty$ and let us denote $x_n' = \dfrac{x_n}{\|Px_n\|}$. Then

$$(f(T)y_n, x) = (y_n, \tilde{f}(T^*)x) = (P_{\mathscr{H}} z_n, \tilde{f}(T^*)x) =$$

$$= (z_n, \tilde{f}(T^*)x) = ((I - VV^*)x_n', \tilde{f}(T^*)x) =$$

$$= ((I - T^*T)x_n', \tilde{f}(T^*)x) = (f(T)D_T^2 x_n', x),$$

where $\tilde{f}(z) = \overline{f(\bar{z})}$, $z \in \mathbf{D}$. Therefore

$$\|[y_n \otimes x]\| = \|[D_T^2 x_n' \otimes x]\|$$

and by virtue of Lemma 2, this last term tends to 0. The proof is complete.

Proof of the theorem. Fix an arbitrary T in $(\mathrm{BCP})_\theta$ and let $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. By virtue of [2, Theorem 3.7], it suffices to show that $T \in \mathbf{A}$ and $\mathscr{A}_T$ has property $X_{\theta+\varepsilon, 1}$. For each $\mu \in L_\theta(T) \cup R_\theta(T)$ we can construct as in the proof of Lemma 3 a vector y_μ in the unit ball of $\mathscr{H}$ such that

$$|f(\mu) - (f(T)y_\mu, y_\mu)| \leqslant (\theta + \varepsilon)\|f\|_\infty, \quad f \in H^\infty.$$

Since the set $L_\theta(T) \cup R_\theta(T)$ is dominating for $\mathbf{T}$, it follows that

$$\|f(T)\| \geqslant \sup_{\mu \in L_\theta \cup R_\theta} |(f(T)y_\mu, y_\mu)| \geqslant$$

$$\geqslant \sup_{\mu \in L_\theta \cup R_\theta} |f(\mu)| - (\theta + \varepsilon)\|f\|_\infty = (1 - \theta - \varepsilon)\|f\|_\infty.$$

If we replace f by f^n, we obtain

$$\|f(T)\|^n \geqslant (1 - \theta - \varepsilon)\|f\|_\infty^n.$$

Taking n-roots, we obtain, for $n \to \infty$

$$\|f(T)\| \geqslant \|f\|_\infty$$

hence $T \in \mathbf{A}$.

It follows from Lemma 3 that for any $\mu \in L_\theta(T)$, $[C_\mu]_T \in \mathscr{X}_{\theta+\varepsilon}(\mathscr{A}_T)$. Since $R_\theta(T) = \overline{L_\theta(T^*)}$ we also have $[C_\mu]_T \in \mathscr{X}_{\theta+\varepsilon}(\mathscr{A}_T)$ for all μ in $R_\theta(T)$. Since the set $L_\theta(T) \cup R_\theta(T)$ is dominating for $\mathbf{T}$, it follows from [2, Proposition 1.21] that $\overline{\mathrm{aco}}\{[C_\mu]\,;\ \mu \in L_\theta(T) \cup R_\theta(T)\}$ equals the unit ball in Q_T, therefore $\mathscr{A}_T$ has property $X_{\theta+\varepsilon,1}$. The proof is finished.

Lemma 3 together with a recent criterion for membership in $\mathbf{A}(\mathscr{H})$ (cf. [6, Theorem 2.1]) enable us to give a sufficient condition for membership in $\mathbf{A}_{\aleph_0}$, which improves a similar one from [4].

If $T \in \mathscr{L}(\mathscr{H})$, then we denote by $\mathscr{F}(T)$ the union of those holes H in $\sigma_e(T)$ such that $H \subset \sigma(T)$.

THEOREM 4. *Suppose $T \in C_{00}$ and there exists θ satisfying $0 < \theta < 1$ such that the set*

$$\Lambda(T) = (\sigma_e(T) \cap \mathbf{D}) \cup \mathscr{F}(T) \cup \left\{\lambda \in \mathbf{D} \setminus \sigma_e(T),\ \theta\|(\pi(T) - \lambda)^{-1}\| > \frac{1}{1 - |\lambda|}\right\}$$

is dominating for $\mathbf{T}$.

Then $T \in \mathbf{A}_{\aleph_0}$.

Proof. Let $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. Since $\Lambda(T)$ is dominating, it follows from [6, Theorem 2.1] that $T \in \mathbf{A}(\mathscr{H})$. Thus, by virtue of [2, Theorem 3.7] it suffices to show that for each $\mu \in \Lambda(T)$, there exists a sequence of vectors $\{x_n\}$ in the unit ball of $\mathscr{H}$ such that

$$\text{1)} \qquad \|[C_\mu] - [x_n \otimes x_n]\| \leqslant \theta + \varepsilon \quad \forall n \in \mathbf{N}$$

and

$$\text{2)} \qquad \lim_{n\to\infty} (\|[x_n \otimes z]\| + \|[z \otimes x_n]\|) = 0 \quad \forall z \in \mathscr{H}.$$

Suppose first that $\mu \in \tilde{\zeta}_\theta(T)$. Since $\tilde{\zeta}_\theta(T) \subset L_\theta(T) \cup R_\theta(T)$ (cf. [2, Proposition 8.1]), an application of Lemma 3 yields the existence of a sequence $\{x_n\}$ in the unit ball of $\mathscr{H}$ satisfying 1) and 2). If $\mu \in \mathscr{F}(T)$, then elementary properties of

Fredholm index combined with [5, Lemma 2.2 and Lemma 2.3] yield the existence of an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that

$$[C_\mu] = [x_n \otimes x_n], \quad n \geqslant 1.$$

Since $T \in \mathbf{A} \cap C_{00}$, the sequence $\{x_n\}$ also satisfies (2) (cf. [2, Proposition 6.5]), therefore $T \in \mathbf{A}_{\aleph_0}$.

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