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JORDAN OPERATORS ON HILBERT SPACES

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1. INTRODUCTION

A natural way to generalize the bounded selfadjoint operators on Hilbert spaces would be to replace the condition $T^* = T$ by the condition T^* be nilpotent equivalent to T or even quasi-nilpotent equivalent. Such a generalization was proposed in 1970–1972 by J. W. Helton, having in mind the multiplication with the variable on a weighted Sobolev space on $\mathbf{R}$ ([9]). An advanced version of the spectral theorem would be that any operator T with the property that T^* is quasi-nilpotent equivalent to T has a Jordan decomposition $T = A + Q$ where A is a selfadjoint operator and Q is a quasinilpotent operator commuting with A. J. W. Helton has proved that version for the order 3 of nilpotency: if T^* is nilpotent equivalent to T of order 3 then $T = A + Q$ where A is selfadjoint and Q is a nilpotent operator of order 2 commuting with A. Since the proof in ([9]) has an algebraic-spatial character, being specific to the order 3 of nilpotency, Helton himself has suggested that the geometric notion of orthogonally decomposable operator would help to generalize his result.

The purpose of this paper is to present the generalizations of Helton's result to an arbitrary order of nilpotency as well as for some cases of quasi-nilpotency. Moreover, we show how to formulate and solve the corresponding problems for unitary and normal operators.

2. PRELIMINARIES

Let H be a complex Hilbert space and $L(H)$ denote the algebra of all bounded linear operators on H. The commutator $C(T, S)$ of two operators $T, S \in L(H)$ is the operator defined on $L(H)$ by $C(T, S) = l(T) - r(S)$, where $l(T)$ is the left multiplication by T and $r(S)$ is the right multiplication by S on $L(H)$. Following ([5]) we say that T and S are quasi-nilpotent equivalent if

$$\lim_n \|C^n(T, S)(I)\|^{1/n} = 0 = \lim_n \|C^n(S, T)(I)\|^{1/n}.$$

An equivalent condition is that the operator entire functions $\exp(-zS)\exp(zT)$ and $\exp(-zT)\exp(zS)$ both have order one and minimal type (see [9], p. 328). With that remark in mind and inspired by a paper of A. Atzmon ([3]) we will distinguish several levels of quasi-nilpotent equivalence by the growth of those entire functions. In the same time we will need to "desymmetrize" the quasi-nilpotent equivalence. Thus we will use the following relations.

$T\mathrm{q}\,S$ if $\lim_n \|C^n(T, S)(I)\|^{1/n} = 0$;

$T\mathrm{n}_k S$ if $C^k(T, S)(I) = 0$, $k \in \mathbf{N}$;

$T\mathrm{q}_\alpha S$, $0 < \alpha < 1$, if $T\mathrm{q}\,S$ and $(\exists)\ c > 0$ such that

$$\|\exp(-itT)\exp(itS)\| = O(\exp c|t|^\alpha) \quad \text{for } |t| \to \infty;$$

$T\hat{\mathrm{q}}\,S$ if $T\mathrm{q}\,S$ and $\displaystyle\int_{-\infty}^{\infty} \log\|\exp(-itT)\exp(itS)\|/(1+t^2)\mathrm{d}t$ is convergent.

If r denotes the one of the relations above then by sr we will denote the corresponding symmetric relation. Thus T and S are quasi-nilpotent equivalent iff $T\mathrm{sq}\,S$.

An operator $T \in L(H)$ will be called *orthogonally decomposable* if T is decomposable in the sense of ([5]) and moreover any two invariant subspaces with disjoint spectra are orthogonal (see [9], p. 330).

An operator $T \in L(H)$ will be called *Jordan operator* if it has a decomposition $T = N + Q$ where N is normal and Q is quasi-nilpotent and $NQ = QN$.

3. RESULTS

THEOREM 1. *An operator $T \in L(H)$ has a Jordan decomposition $T = A + Q$, where A is a selfadjoint operator and Q is a quasi-nilpotent operator commuting with A if and only if T is decomposable and $T^*\mathrm{sq}\,T$.*

Of course it would be interesting to know if the condition $T^*\mathrm{sq}\,T$ does not imply the decomposability of T. A result in that direction is the following.

THEOREM 1′. *If $T^*\mathrm{r}\,T$ where r is one of the relations n_{2k-1}, q_α or $\hat{\mathrm{q}}$ then T is a $\mathscr{A}$-selfadjoint operator in the sense of* ([5]).

The correspondence between the "level" of quasi-nilpotent equivalence of T^* and T and the order of quasi-nilpotency of Q is the following.

THEOREM 1″. *An operator $T \in L(H)$ has a Jordan decomposition $T = A + Q$, where A is a selfadjoint operator and Q is an operator commuting with A and such* *that* $Q^k = 0$, $Q \,\mathrm{sq}_\alpha 0$ *or* $Q \,\mathrm{s\hat{q}}\, 0$ *if and only if respectively* $T^* \mathrm{sn}_{2k-1} T$, $T^* \mathrm{sq}_\alpha T$ *or* $T^* \,\mathrm{s\hat{q}}\, T$.

For Jordan operators with unitary scalar part we have the following results.

THEOREM 2. *An operator* $T \in L(H)$ *has a Jordan decomposition* $T = U + Q$ *where* U *is a unitary operator and* Q *is a quasi-nilpotent operator commuting with* U *if and only if* T *is invertible, decomposable and* $T^* \mathrm{sq}\, T^{-1}$.

THEOREM 2′. *If* T *is invertible and* $T^* \mathrm{sr}\, T^{-1}$ *where* r *is one of the relations* n_{2k-1} *or* q_α, *then* T *is a* $\mathcal{U}$*-unitary operator in the sense of* ([5]).

THEOREM 2″. *An operator* $T \in L(H)$ *has a Jordan decomposition* $T = U + Q$, *where* U *is unitary and* Q *is an operator commuting with* T *and such that* $Q^k = 0$ *or* $Q \,\mathrm{sq}_\alpha 0$ *if and only if* T *is invertible and* $T^* \mathrm{sn}_{2k-1} T^{-1}$ *respectively* $T^* \mathrm{sq}_\alpha T^{-1}$.

The conditions $T^* \mathrm{sq}\, T$ and $T^* \mathrm{sq}\, T^{-1}$ cannot be extended directly to cover the corresponding generalizations of the normal operators. A substitute would consist in replacing the equality $T^*T = TT^*$ by the condition

$$(*) \qquad \lim_n \|C^n(T, T)(T^*)\|^{1/n} = 0$$

which, as can be readily seen, is already symmetric.

THEOREM 3. *An operator* $T \in L(H)$ *has a Jordan decomposition* $T = N + Q$ *where* N *is a normal operator and* Q *is a quasi-nilpotent operator commuting with* N *if and only if* T *is decomposable and satisfies condition* $(*)$.

The following corollary of Theorem 3 was conjectured in ([9], p. 330).

COROLLARY. *If* T *has a* $C^n(\mathbf{R}^2)$*-functional calculus and both* T *and* T^* *are orthogonally separated, then* $T = N + Q$ *where* N *is normal,* Q *is nilpotent and* $NQ = QN$.

It seems difficult to prove a Theorem 3′ similar to Theorems 1′ and 2′ even in the case when T has real or unitary spectrum and the condition $(*)$ holds in the most particular way: $(\exists)\ k > 2$ such that $C^k(T, T)(T^*) = 0$.

Another interesting way to develop the potential ideas of ([9]) would be to characterize the sub-Jordan operators by the corresponding asymmetric relations as that was done for the order 2 in ([9]). The complete positivity technique used in ([1]) seems to be useful in this context too.

4. AUXILIARY RESULTS

LEMMA 1. *If* $T^* \mathrm{q}\, T$ *then* $\mathrm{sp}(T) \subset \{z \in \mathbf{C} \mid \mathrm{Im}\, z \leqslant 0\}$.

Proof. From $T^* \text{q} T$ and $\|\exp(-\text{i}sT^*)\exp(\text{i}sT)\| = \|\exp(\text{i}sT)\|^2$, $s \in \mathbf{R}$, we get for any $\varepsilon > 0$ an estimation of the form

$$\|\exp(-\text{i}sT)\| \leqslant M_\varepsilon \exp(\varepsilon s), \quad s > 0$$

which guarantees that the Laplace integral $\int_0^\infty \exp(zs)\exp(-\text{i}sT)\text{d}s$ is absolutely convergent in the operator norm for $\text{Re}\, z < 0$. It is a standard fact that this integral represents the resolvent $\text{R}(z; \text{i}T)$. Consequently $\text{sp}(\text{i}T) \subset \{z \in \mathbf{C} \mid \text{Re}\, z \geqslant 0\}$ whence we get $\text{sp}(T) \subset \{z \in \mathbf{C} \mid \text{Im}\, z \leqslant 0\}$, as desired.

COROLLARY. *If* T^* sq T *then* $\text{sp}(T) \subset \mathbf{R}$.

LEMMA 2. *If* T *is invertible and* $T^* \text{q} T^{-1}$ *then* $\text{sp}(T) \subset \{z \in \mathbf{C} \mid |z| \leqslant 1\}$.

Proof. We have

$$C^n(T^*, T^{-1})(I) = (T^*)^n - \binom{n}{1}(T^*)^{n-1}T^{-1} + \cdots + (-1)^n (T^{-1})^n$$

whence we get

$$C^n(T^*, T^{-1})(I)T^n = (T^*)^n T^n - \binom{n}{1}(T^*)^{n-1}T^{n-1} + \cdots + (-1)^n I.$$

From here one can prove, by induction on n, the relation

$$(T^*)^n T^n = C^n(T^*, T^{-1})(I)T^n + \binom{n}{1} C^{n-1}(T^*, T^{-1})(I)T^{n-1} + \cdots + I$$

which gives us the estimation

$$\|(T^*)^n T^n\| \leqslant M_\varepsilon(\varepsilon\|T\| + 1)^n, \quad n \in \mathbf{N}$$

where ε is an arbitrary positive number and M_ε is a positive constant depending on ε. Taking into account that $\|(T^*)^n T^n\| = \|T^n\|^2$, we get from the estimation above that the series $\sum_{n=0}^{\infty} T^n / z^{n+1}$ whose sum is the resolvent $\text{R}(z; T)$ is absolutely convergent in the operator norm for $|z| > 1$. Consequently we have $\text{sp}(T) \subset \{z \in \mathbf{C} \mid |z| \leqslant 1\}$ as desired.

COROLLARY. *If* T *is invertible and* $T^* \text{sq}\, T^{-1}$ *then* $\text{sp}(T) \subset \{z \in \mathbf{C} \mid |z| = 1\}$.

The following lemma is inspired from ([3], Proposition 3).

LEMMA 3. *$T\mathrm{q}_\alpha S$ if and only if $\|C^n(T, S)(I)\|^{1/n} = O(n^{-(1-\alpha)/\alpha})$ for $n \to \infty$.*

Proof. If $T\mathrm{q}_\alpha S$ then, by applying the analog for vector valued functions of ([4], p. 97, Theorem 6.69) we get $\|\exp(-zT)\exp(zS)\| = O(\exp c|z|^\alpha)$ for $|z| \to \infty$. Using now the relation between the order of an entire function and the growth of its Taylor coefficients we obtain the estimation

$$\|C^n(T, S)(I)/n!\| \leqslant C^n \cdot n^{-n/\alpha}, \quad n = 1,2, \ldots, \; C = \text{constant} > 0,$$

whence, by applying Stirling's formula, we obtain the estimation

$$\|C^n(T, S)(I)\| \leqslant C_1^n \cdot n^{-n/\beta}, \text{ where } \beta = \alpha/(1-\alpha)$$

and consequently

$$\|C^n(T, S)(I)\|^{1/n} = O(n^{-(1-\alpha)/\alpha}) \quad \text{for} \quad n \to \infty.$$

Conversely, if

$$\|C^n(T, S)(I)\|^{1/n} = O(n^{-(1-\alpha)/\alpha}) \quad \text{for} \quad n \to \infty,$$

then by applying again Stirling's formula and the relation order-coefficients (this time in the other direction), we get

$$\|\exp(-zT)\exp(zS)\| = O(\exp c|z|^\alpha) \quad \text{for} \quad |z| \to \infty$$

and consequently $T\mathrm{q}_\alpha S$. The proof is concluded.

One of the main tools used in the proof of the announced results will be a Jordan decomposition theorem for orthogonally decomposable operators. The first variant of this theorem ([8], Theorem 6) said that an operator $T \in L(H)$ is orthogonally decomposable if and only if it is a Dunford spectral operator with normal scalar part. Subsequently, M. Putinar (private communication) discovered by two examples that the theorem cannot be true in the form stated above. So I revised the proof, found the guilty lemma (it was Lemma 1 in [8]) and added a new assumption to get the correct variant. In fact there are two correct variants. Recall that an operator $T \in L(H)$ is called orthogonally separated if any two invariant subpaces with disjoint spectra are orthogonal.

THEOREM J1. *If $T \in L(H)$ is decomposable and both T and T^* are orthogonally separated then T is a Jordan operator with normal scalar part.*

The additional assumption is that T^* is orthogonally separated. Under this assumption the proof in [8] remains valid except we do not need Lemma 1 this time.

THEOREM J2. *If $T \in L(H)$ is orthogonally decomposable and the maximal spectral spaces of T are invariant for T^* then T is a Jordan operator with normal scalar part.*

The additional assumption is that the maximal spectral spaces of T are invariant for T^*. Under that assumption the proof of Lemma 1 goes as follows. Let $F = \sigma_T(x)$ and denote by T_F the restriction of T to $H_T(F)$. Then $(T_F)^*$ is the restriction of T^* to $H_T(F)$ and it is easy to see that the function $x^*(z) = \mathrm{R}(z; T_F^*)x$ which is defined and analytic on $(F^c)^*$ (the conjugate set of F^c) satisfies $(z - T^*)x^*(z) \equiv \equiv x$, $z \in (F^c)^*$. Consequently, $\sigma_{T^*}(x) \subset \sigma_T(x)^*$ and since T and T^* may be interchanged, we get finally the equality $\sigma_{T^*}(x) = \sigma_T(x)^*$.

REMARK. A. A. Jafarian informed me that a result similar to Theorem J2 may by found in his paper ([11]).

Finally we will need to know the behaviour of the quasinilpotent equivalence with respect to the functional analytic calculus.

PROPOSITION 4. *Let X be a complex Banach space and A, $B \in L(X)$. If $A \,\mathrm{r}\, B$ where* r *is one of the relations* n_k, q_α *or* q *then* $f(A)\,\mathrm{r}\,f(B)$ *for any function f analytic in an open neighbourhood* of $\mathrm{sp}(A) \cup \mathrm{sp}(B)$.

COROLLARY. *If* $A \,\mathrm{sr}\, B$ *where* r *is one of the relations* n_k, q_α *or* $\hat{\mathrm{q}}$ *then* $f(A)\,\mathrm{sr}\,f(B)$ *for any function f analytic in an open neighbourhood of* $\mathrm{sp}(A) \cup \mathrm{sp}(B)$.

The proof of Proposition 4 will be based on the following.

LEMMA 5. *Let X be a complex Banach space and A, $B \in L(X)$ be two commuting operators. If f is an analytic function in an open neighbourhood of* $\mathrm{sp}(A) \cup \cup \mathrm{sp}(B)$ *then there exists an operator $D \in L(X)$ commuting with A and B such that*

$$f(A) - f(B) = (A - B)D.$$

Proof of Proposition 4. We apply Lemma 5 for the commuting operators $\tilde{A} = l(A)$ and $\tilde{B} = r(B)$ on $L(X)$. Since $\mathrm{sp}(\tilde{A}) = \mathrm{sp}(A)$, $\mathrm{sp}(\tilde{B}) = \mathrm{sp}(B)$ and $f(\tilde{A}) = \widetilde{f(A)} = l(f(A))$, $f(\tilde{B}) = \widetilde{f(B)} = r(f(B))$ then there exists an operator $\mathscr{D}$ on $L(X)$ such that

$$C(f(A), f(B)) = l(f(A)) - r(f(B)) = f(\tilde{A}) - f(\tilde{B}) =$$

$$= (\tilde{A} - \tilde{B})\mathscr{D} = C(A, B)\mathscr{D} = \mathscr{D}\, C(A, B).$$

From here we get the following inequality

$$\|C^n(f(A), f(B))\,(I)\|^{1/n} \leqslant \|\mathscr{D}\| \cdot \|C^n(A, B)\,(I)\|^{1/n}$$

whence we obtain directly the assertion if r is n_k or q. For $\mathrm{r} = \mathrm{q}_\alpha$ we can apply Lemma 3.

Proof of Lemma 5. Let Γ be a finite system of C^1-Jordan curves contained in the neighbourhood V and containing inside the union $\mathrm{sp}(A) \cup \mathrm{sp}(B)$. Since $AB = BA$, we may write

$$\mathrm{R}(z;A) - \mathrm{R}(z;B) = (A-B)\,\mathrm{R}(z;A)\,\mathrm{R}(z;B), \qquad z \in \Gamma$$

and consequently

$$f(A) - f(B) = (2\pi\mathrm{i})^{-1}\int_{\Gamma} f(z)[\mathrm{R}(z;A) - \mathrm{R}(z;B)]\,\mathrm{d}z = (A-B)D$$

where

$$D = (2\pi\mathrm{i})^{-1}\int_{\Gamma} f(z)\,\mathrm{R}(z;A)\,\mathrm{R}(z;B)\,\mathrm{d}z$$

and it is clear that D commutes with A and B.

5. PROOFS

Proof of Theorem 1. If $T = A + Q$, where A is selfadjoint and Q is quasi-nilpotent commuting with A, then by ([5], Theorem 2.1) T is decomposable. Moreover, $\exp(-zT^*)\exp(zT) = \exp(-zA)\exp(-zQ^*)\exp(zA)\exp(zQ) = \exp(-zQ^*)\exp(zQ)$ and analogously $\exp(-zT)\exp(zT^*) = \exp(-zQ)\exp(zQ^*)$. Since both Q and Q^* are quasi-nilpotent, both $\exp(-zQ^*)\exp(zQ)$ and $\exp(-zQ)\exp(zQ^*)$ are entire functions of order one and minimal type so that $T^* \,\mathrm{sq}\, T$ and the "if" part of the theorem is proved. To prove the "only if" part, let us remark first that, by Lemma 1, we have $\mathrm{sp}(T) \subset \mathbf{R}$ and consequently the "support" of the spectral spaces $H(F, T)$ is contained in $\mathbf{R}$. Since T is decomposable then, by ([7]), T^* is also decomposable and $H(F, T^*) = H(F^c, T)^{\perp}$ for any closed subset $F \subset \mathbf{R}$ (F^c denotes the relative complement of F in $\mathbf{R}$). Taking into account that $T^* \,\mathrm{sq}\, T$, by ([5], Theorem 2.1) we obtain $H(F, T^*) = H(F, T)$ for any closed subset $F \subset \mathbf{R}$ and consequently both T and T^* are orthogonally separated. To conclude the proof it remains only to apply Theorem J1 and to remark that the scalar part of T must be selfadjoint as being normal with real spectrum.

Proof of Theorem 1'. If $T^* \,\mathrm{n}_{2k-1}\, T$ then $\exp(-\mathrm{i}tT^*)\exp(\mathrm{i}tT)$ is a polynomial of degree at most $2k-2$ and consequently $\|\exp(-\mathrm{i}tT^*)\exp(\mathrm{i}tT)\| = \mathrm{O}(|t|^{2k-2})$ for $|t| \to \infty$. Since $\|\exp(\mathrm{i}tT)\|^2 = \|\exp(-\mathrm{i}tT^*)\exp(\mathrm{i}tT)\|$, we get $\|\exp(\mathrm{i}tT)\| = \mathrm{O}(|t|^{k-1})$ for $|t| \to \infty$. By applying the theorem of Kantorovitz ([5]) it follows that T has a C^{k+1}-functional calculus on the real axis. If $T^* \,\hat{\mathrm{q}}\, T$ then

$$\int_{-\infty}^{\infty} \log\|\exp(\mathrm{i}tT)\|/(1+t^2)\mathrm{d}t = (1/2)\int_{-\infty}^{\infty} \log\|\exp(-\mathrm{i}tT^*)\exp(\mathrm{i}tT)\|/(1+t^2)\mathrm{d}t < \infty$$

and consequently by ([12]) T is $\mathscr{A}$-selfadjoint in the sense of ([5]). In particular that is the case if $T^* \mathrm{q}_\alpha T$.

Proof of Theorem 1''. Combining Theorem 1 with Theorem 1' and taking into account that any $\mathscr{A}$-selfadjoint operator is decomposable, it will be enough to prove that for an operator $T = A + Q$, where A is selfadjoint and Q is quasinilpotent commuting with A, the conditions $Q^k = 0$, $Q\,\mathrm{sq}_\alpha 0$ and $Q\,\mathrm{s}\hat{\mathrm{q}}\,0$ are respectively equivalent to $T^* \mathrm{sn}_{2k-1} T$, $T^* \mathrm{sq}_\alpha T$ and $T^* \mathrm{s}\hat{\mathrm{q}}\, T$. It is easy to see that the conditions $Q^k = 0$, $Q\,\mathrm{sq}_\alpha 0$ and $Q\,\mathrm{s}\hat{\mathrm{q}}\,0$ imply respectively $T^* \mathrm{sn}_{2k-1} T$, $T^* \mathrm{sq}_\alpha T$ and $T^* \mathrm{s}\hat{\mathrm{q}}\, T$. To prove the converse statements, let us assume first that $T^* \mathrm{sn}_{2k-1} T$. Then $\exp(-zT^*)\exp(zT) = \exp(-zQ^*)\exp(zQ)$ is a polynomial of degree at most $2k-2$, whence $\|\exp(-itQ^*)\exp(itQ)\| = \mathrm{O}(|t|^{2k-2})$ for $|t| \to \infty$ and therefore $\|\exp(itQ)\| = \mathrm{O}(|t|^{k-1})$ for $|t| \to \infty$. Since Q is quasinilpotent, the entire function $\exp(zQ)$ is of order one and minimal type so that by ([4], Theorem 6.69) we get $\|\exp(zQ)\| = \mathrm{O}(|z|^{k-1})$ for $|z| \to \infty$, whence $Q^k = 0$. Let us assume now that $T^* \mathrm{sq}_\alpha T$. Then $\|\exp(-itQ^*)\exp(itQ)\| = \mathrm{O}(\exp c|t|^\alpha)$ for $|t| \to \infty$ whence, by applying again ([4], Theorem 6.69) we get $\|\exp(tQ)\| = \mathrm{O}(\exp c|t|)^\alpha$, as desired. Finally, let $T^* \mathrm{s}\hat{\mathrm{q}}\, T$. Then

$$\int_{-\infty}^{\infty} \log\|\exp(itQ)\| / (1 + t^2)\mathrm{d}t = (1/2)\int_{-\infty}^{\infty} \log\|\exp(-itQ^*)\exp(itQ)\| / (1 + t^2)\mathrm{d}t =$$

$$= (1/2)\int_{-\infty}^{\infty} \log\|\exp(-itT^*)\exp(itT)\| / (1 + t^2)\mathrm{d}t < \infty$$

and thus $Q\,\mathrm{s}\hat{\mathrm{q}}\,0$.

Proof of Theorem 2. We can follow the same general lines as in the proof of Theorem 1, so that we will insist only to show that T and T^* are orthogonally separated. The duality formula for spectral spaces changes now in $H(F^*, T^*) = H(F^c, T)^\perp$ for any closed subset F of the unit circle, where F^* denotes the conjugate of F, and F^c the relative complement in the unit circle. Applying ([5], Theorem 2.1 and Theorem 1.6) we get $H(F^*, T^*) = H(F^*, T^{-1}) = H(F, T)$ since for a subset F of the unit circle we have $F^* = F^{-1}$. Finally, we have $H(F, T) = H(F^c, T)^\perp$ which shows that T is orthogonally separated; T^* is also orthogonally separated since $H(F, T^*) = H(F^*, T)$ by the relations above. The proof is completed.

Proof of Theorem 2'. Let us assume first that T is invertible and $T^* \mathrm{sn}_{2k-1} T^{-1}$. As we have seen in the proof of Lemma 2, we have

$$(T^*)^n T^n = C^n(T^*, T^{-1})(I)T^n + \binom{n}{1} C^{n-1}(T^*, T^{-1})(I)T^{n-1} + \cdots + I =$$

$$= \binom{n}{2k-2} C^{2k-2}(T^*, T^{-1})(I)T^{2k-2} + \cdots + I, \quad \text{for } n \geqslant 2k-1,$$

whence we get the estimation

$$\|T^n\| = O(n^{k-1}) \quad \text{for } n \to \infty.$$

Expressing in the same way $(T^{-1})^n(T^{-1})^{*n}$, we get analogously $\|(T^{-1})^n\| = O(n^{k-1})$ for $n \to \infty$. Therefore we have obtained the estimation $\|T^n\| = O(|n|^{k-1})$ for $|n| \to \infty$, so that by ([5]) T is C^k-unitary.

Let us assume now that T is invertible and $T^* \mathrm{sq}_\alpha T^{-1}$. By applying Lemma 3 we get $\|C^n(T^*, T^{-1})(I)\|^{1/n} = O(n^{-(1-\alpha)/\alpha})$ and $\|C^n(T^{-1}, T^*)(I)\|^{1/n} = O(n^{-(1-\alpha)/\alpha})$ for $n \to \infty$. By using these estimations and the relations above, we get the following recurrent inequalities for $a_n = \|T^n\|$ and $b_n = \|(T^{-1})^n\|$:

$$a_n^2 \leqslant a_0^2 + \sum_{k=0}^{n} C_1^k \binom{n}{k} k^{-k/\beta} a_k$$

$$b_n^2 \leqslant b_0^2 + \sum_{k=0}^{n} C_1^k \binom{n}{k} k^{-k/\beta} b_k$$

where $C_1 \geqslant 1$ is a constant. From here one can prove, by induction on $n \in \mathbf{N}$, that

$$a_n \leqslant a_0 + (1/2) \sum_{k=0}^{n} C_1^k \binom{n}{k} k^{-k/\beta} \quad \text{and} \quad b_n \leqslant b_0 + (1/2) \sum_{k=0}^{n} C_1^k \binom{n}{k}.$$

Now we have to estimate the sum $\sum_{k=0}^{n} C_1^k \binom{n}{k} k^{-k/\beta}$. We will do this separately for $\sum_{k<n^\alpha}$ and $\sum_{k>n^\alpha}$. We have

$$\sum_{k<n^\alpha} C_1^k \binom{n}{k} k^{-k/\beta} < C_1^{n^\alpha} (n^\alpha + 1) n^{n^\alpha}.$$

As for the second sum, we can write

$$\sum_{k>n^\alpha} C_1^k \binom{n}{k} k^{-k/\beta} < \sum_{k>n^\alpha} (C_1 n/k^{1/\beta})^k/k! <$$

$$< \sum_{k>n^\alpha} (C_1 n^\alpha)^k/k! < \exp(C_1 n^\alpha).$$

From the estimations above we get $\|T^n\| = O(\exp c|n|^{\alpha'})$ for $|n| \to \infty$, where $\alpha' > \alpha$, whence by ([5]) we deduce that T is a $\mathcal{U}$-unitary operator.

Proof of Theorem 2''. Combining Theorem 2 with Theorem 2' and taking into account that any $\mathcal{U}$-unitary operator is decomposable it will be sufficient to prove that for an operator $T = U + Q$, where U is unitary and Q is quasi-nilpotent commuting with U, the conditions $Q^k = 0$ and $Q \, \mathrm{sq}_\alpha 0$ are respectively equivalent to $T^* \, \mathrm{sn}_{2k-1} T^{-1}$ and $T^* \, \mathrm{sq}_\alpha T^{-1}$. If $Q^k = 0$ then we have $T^* = U^{-1} + Q^*$ and $T^{-1} = U^{-1} + Q_1$, where $Q_1^k = 0$ and Q_1 commutes with U^{-1}, whence

$$C^{2k-1}(T^*, T^{-1})(I) = \sum_{p=0}^{2k-1} \binom{2k-1}{p} C^{2k-1-p}(T^*, U^{-1})(I) C^p(U^{-1}, T^{-1})(I) = 0$$

and analogously

$$C^{2k-1}(T^{-1}, T^*)(I) = 0.$$

Let us assume now that $T^* \, \mathrm{sn}_{2k-1} T^{-1}$. As we have seen in the proof of Theorem 2, we have the estimation $\|T^n\| = O(|n|^{k-1})$ for $|n| \to \infty$ and consequently $(TU^{-1})^n = O(|n|^{k-1})$. Since $TU^{-1} = I + QU^{-1}$, we can apply the theorem of Polya-Hille to the resolvent of TU^{-1} ([10], Theorem 3.13.5) concluding thus that $(QU^{-1})^k = 0$ hence $Q^k = 0$. It remains to prove that the relation $Q \, \mathrm{sq}_\alpha \, 0$ is equivalent to $T^* \, \mathrm{sq}_\alpha \, T^{-1}$. Assume first that $Q \, \mathrm{sq}_\alpha 0$. Then, by Lemma 3, $\|Q^n\|^{1/n} = O(n^{-1/\beta})$ for $n \to \infty$, whence

$$\|C^n(T^*, U^{-1})(I)\|^{1/n} = \|(T^* - U^{-1})^n\|^{1/n} = \|(Q^*)^n\|^{1/n} = O(n^{-1/\beta})$$

and

$$\|C^n(U^{-1}, T^{-1})(I)\|^{1/n} = \|(U^{-1} - T^{-1})^n\|^{1/n} = \|Q_1^n\|^{1/n} = O(n^{-1/\beta}) \quad \text{for } n \to \infty.$$

Using these estimations and the relation

$$C^n(T^*, T^{-1})(I) = \sum_{k=0}^{n} \binom{n}{k} C^{n-k}(T^*, U^{-1})(I) \, C^k(U^{-1}, T^{-1})(I)$$

we get

$$\|C^n(T^*, T^{-1})(I)\| \leqslant M^n \sum_{k=0}^{n} \binom{n}{k} (n-k)^{-(n-k)/\beta} k^{-k/\beta}.$$

Using the elementary inequality $(n-k)^{n-k} k^k \geqslant (n/2)^n$ we get finally the estimation $\|C^n(T, T^{-1})(I)\|^{1/n} = O(n^{-1/\beta})$ for $n \to \infty$. Analogously one can prove the estimation $\|C^n(T^{-1}, T^*)(I)\|^{1/n} = O(n^{-1/\beta})$ for $n \to \infty$. Finally, let us show that if T has a representation as above $T = U + Q$ and $T^* \, \mathrm{sq}_\alpha T^{-1}$ then $Q \, \mathrm{sq}_\alpha 0$. We may write $T = U(I + U^{-1}Q)$ and since $U^{-1}Q$ is quasinilpotent, the operator $I + U^{-1}Q$ has a logarithm $\mathrm{i}Q_1$ given by the usual power series expansion; moreover Q_1 is a quasinilpotent operator commuting with U. Thus $T = U\exp(\mathrm{i}Q_1)$, $\mathrm{i}Q_1 = \ln(I +$

$+ U^{-1}Q)$, whence $T^* = U^{-1}\exp(-\mathrm{i}Q_1^*)$ and $T^{-1} = U^{-1}\exp(-\mathrm{i}Q_1)$. By applying Lemma 3 we get from $T^* \mathrm{sq}_\alpha T^{-1}$, $T^*U \mathrm{sq}_\alpha T^{-1}U$ whence $\exp(-\mathrm{i}Q_1^*) \mathrm{sq}_\alpha \exp(-\mathrm{i}Q_1)$ so that, by the corollary of Proposition 4, $Q_1^* \mathrm{sq}_\alpha Q_1$ which is equivalent to $Q_1 \mathrm{sq}_\alpha 0$. By applying the same corollary again we get $\exp(\mathrm{i}Q_1) \mathrm{sq}_\alpha I$ that is $I + U^{-1}Q \mathrm{sq}_\alpha I$, whence $U^{-1}Q \mathrm{sq}_\alpha 0$ and consequently $Q \mathrm{sq}_\alpha 0$. The proof is completed.

REMARK. We could use the Cayley transform to reduce Theorems 1 and 1′ to Theorems 2 and 2′, but not conversely.

Proof of Theorem 3. The result is implicitely contained in ([2]). Since the arguments used there are quite complicated, we include here a simple proof using Theorem J2. The operator T being already supposed to be decomposable it remains only to show that T is orthogonally separated and the spectral spaces of T are invariant for T^*. Applying ([5], Theorem 3.3) we deduce that $H(F, T)$ is invariant for T^*. Using the duality formula $H(F, T) = H((F^*)^c, T^*)^\perp$ and the decomposability of T^* ([7]) we get $\mathrm{sp}(T^*, H(F, T)) = \mathrm{sp}(T^*, H((F^*)^c, T^*)) \subset F$ and consequently, by symmetry, $H(F, T) = H(F^*, T^*) = H(F^c, T)^\perp$ whence it follows that T is orthogonally separated. The proof is thus finished.

Proof of the Corollary. Applying Theorem 3 we obtain the decomposition $T = N + Q$ where N is normal and Q is quasinilpotent commuting with N. Since Q is the difference of two commuting scalar generalized operators, it must be also scalar generalized ([5]) and therefore, again by ([5]), it must be nilpotent.

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