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SPECTRAL PICTURES OF OPERATORS IN THE COWEN-DOUGLAS CLASS $\mathscr{B}_n(\Omega)$ AND ITS CLOSURE

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1. INTRODUCTION

In their article [4], M. J. Cowen and R. G. Douglas introduced and studied the following important class of (bounded linear) operators on a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$. Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all operators acting on $\mathscr{H}$, and let $\sigma(T)$ denote the spectrum of $T \in \mathscr{L}(\mathscr{H})$.

DEFINITION. For Ω a bounded connected open subset of $\mathbf{C}$ and n a positive integer, let $\mathscr{B}_n(\Omega)$ denote the operators T in $\mathscr{L}(\mathscr{H})$ which satisfy:

(a) $\Omega \subset \sigma(T)$;

(b) $\operatorname{ran}(T-\lambda) = \mathscr{H}$ for all λ in Ω;

(c) $\bigvee \{\ker(T-\lambda) : \lambda \in \Omega\} = \mathscr{H}$; and

(d) $\operatorname{nul}(T-\lambda) := \dim\ker(T-\lambda) = n$ for all λ in Ω.

Recall that $T \in \mathscr{L}(\mathscr{H})$ is a semi-Fredholm operator if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T$ or $\operatorname{nul} T^*$ is finite (T^* = the adjoint of T); in this case, the index of T is defined by $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$ [9]. It is completely apparent that $\mathscr{B}_n(\Omega)$ is a particular class of operators T with the property that $T-\lambda$ is semi-Fredholm and $\operatorname{ind}(T-\lambda) = n$ for all $\lambda \in \Omega$. Furthermore, $\mathscr{B}_n(\Omega)$ is invariant under similarities; that is, if $T \in \mathscr{B}_n(\Omega)$, then the whole *similarity orbit* of T,

$$\mathscr{S}(T) = \{WTW^{-1} : W \in \mathscr{L}(\mathscr{H}) \text{ is invertible}\},$$

is included in $\mathscr{B}_n(\Omega)$.

What else can be said about the different parts of the spectrum of an operator T in $\mathscr{B}_n(\Omega)$? What is the (norm) closure, $\mathscr{B}_n(\Omega)^-$, of $\mathscr{B}_n(\Omega)$ in $\mathscr{L}(\mathscr{H})$?

Since the class $\mathscr{B}_n(\Omega)$ is similarity-invariant, we can apply the machinery (for approximation of operators) developed in the monograph [3], [7]. The following three results provide complete answers to the above questions. We shall need some notation. In what follows, $\rho^{\text{s-F}}(T) \equiv \{\lambda \in \mathbf{C} : T-\lambda \text{ is semi-Fredholm}\}$

denotes the semi-Fredholm domain of $T \in \mathscr{L}(\mathscr{H})$, $\sigma_{\mathrm{lre}}(T) \equiv \mathbf{C} \setminus \rho_{\mathrm{s\text{-}F}}(T)$, and $\sigma_{\mathrm{W}}(T) \equiv \{\lambda \in \sigma(T) : T - \lambda$ is not semi-Fredholm of index $0\}$ is the Weyl spectrum of T; $\sigma_0(T)$ denotes the set of normal eigenvalues of T, i.e., those isolated points λ of $\sigma(T)$ such that the corresponding Riesz spectral subspace is finite dimensional. Let $\mathscr{K}(\mathscr{H})$ denote the ideal of all compact operators; then the essential spectrum of T, $\sigma_{\mathrm{e}}(T)$, is the spectrum of $\tilde{T} \equiv T + \mathscr{K}(\mathscr{H})$, the canonical projection of T in the quotient Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. Finally, $m_{\mathrm{e}}(T) = \min\{\lambda \in \sigma_{\mathrm{e}}([T^*T]^{1/2})\}$ denotes the essential minimum modulus of T, and we define $\Delta_\gamma(T) = \{\lambda \in \mathbf{C} : m_{\mathrm{e}}(T - \lambda) \leqslant \gamma\}$ $(\gamma \geqslant 0)$.

THEOREM 1.1. *If* $T \in \mathscr{B}_n(\Omega)$, *then*

(i) $\sigma(T)$ *coincides with* $\sigma_{\mathrm{W}}(T)$, *and is a connected set including* Ω;

(ii) $\mathrm{ind}(T - \lambda) \geqslant 0$ *for all* $\lambda \in \rho_{\mathrm{s\text{-}F}}(T)$;

(iii) $\mathrm{ind}(T - \lambda) = n$ *for all* $\lambda \in \Omega$; *and*

(iv) $\mathrm{nul}(T - \lambda)^* = 0$ *for all complex* λ.

Nothing else can be said, in general. Indeed, we have the following "converse":

THEOREM 1.2. *Assume that* $R \in \mathscr{L}(\mathscr{H})$ *satisfies*

(i′) $\sigma(R)$ *and* $\sigma_{\mathrm{W}}(R)$ *are connected sets including* Ω;

(ii) $\mathrm{ind}(R - \lambda) \geqslant 0$ *for all* $\lambda \in \rho_{\mathrm{s\text{-}F}}(R)$; *and*

(iii) $\rho_{\mathrm{s\text{-}F}}(R) \supset \Omega$ *and* $\mathrm{ind}(R - \lambda) = n$ *for all* $\lambda \in \Omega$.

Then, given $\varepsilon > 0$ *there exists exists* $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, *with* $\|K_\varepsilon\| < \varepsilon$, *such that* $R - K_\varepsilon \in \mathscr{B}_n(\Omega)$.

For example, if Ω is the open unit disk, $\{\Omega_k\}_{k=1}^\infty$ is a denumerable family of pairwise disjoint open disks, not included in Ω, but tangent to Ω, and $\{m_k\}_{k=1}^\infty$ $(1 \leqslant m_k \leqslant \infty)$ is a sequence of indices, then (from Theorem 1.2 and a perturbation argument) there exists R in $\mathscr{B}_n(\Omega)$ satisfying $\sigma(R) = \left[\Omega \cup \left(\bigcup_{k=1}^\infty \Omega_k\right)\right]^-$, $\mathrm{ind}(R - \lambda) = n$ for all $\lambda \in \Omega$, and $\mathrm{ind}(R - \lambda) = m_k$ for all $\lambda \in \Omega_k$ $(k = 1, 2, \ldots)$. The existence of this kind of example is not immediate from the definition of the class $\mathscr{B}_n(\Omega)$!

On the other hand, it is not difficult to show that if $T \in \mathscr{B}_n(\Omega)$ and $\varepsilon > 0$, then there exists $C_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|C_\varepsilon\| < \varepsilon$, such that $T - C_\varepsilon$ satisfies conditions (i)–(iv) of Theorem 1.1, but $T - C_\varepsilon \notin \mathscr{B}_n(\Omega)$. (Use Voiculescu's theorem to find C_ε as above, so that $T - C_\varepsilon$ is unitarily equivalent to $T \oplus L$, where L is the image of $\tilde{T}$ under a faithful unital $*$-representation of the C^*-algebra generated by $\tilde{1}$ and $\tilde{T}$; see [10] for details.) Thus, Theorems 1.1 and 1.2 provide the best possible results along these lines.

If, instead of "small compact perturbations" we allow "small (not necessarily compact) perturbations", then we obtain the (norm) closure of $\mathscr{B}_n(\Omega)$.

THEOREM 1.3. *$\mathscr{B}_n(\Omega)^-$ is the set of all those operators A in $\mathscr{L}(\mathscr{H})$ which satisfy:*

(i′) *$\sigma(A)$ and $\sigma_W(A)$ are connected sets including Ω;*

(ii) $\operatorname{ind}(A-\lambda)\geqslant 0$ *for all $\lambda\in\rho_{s\text{-}F}(A)$; and*

(iii′) *Ω does not include any component of $\sigma_{lre}(A)$, and* $\operatorname{ind}(A-\lambda)=n$ *for all $\lambda\in\Omega\cap\rho_{s\text{-}F}(A)$.*

Furthermore, for each S in $\mathscr{L}(\mathscr{H})$.

$$\operatorname{dist}[S,\mathscr{B}_n(\Omega)] = \varkappa(S) := \max\{\alpha(S),\alpha'(S),\beta(S),\delta(S)\},$$

where

(1) $\alpha(S)=\min\{\gamma\geqslant 0 : \operatorname{ind}(S-\lambda)\geqslant 0$ *for all* $\lambda\in\rho_{s\text{-}F}(S)\setminus\Delta_\gamma(S)\}$;

(2) $\alpha'(S)=\min\{\gamma\geqslant 0 : \operatorname{ind}(S-\lambda)=n$ *for all* $\lambda\in[\Omega\cap\rho_{s\text{-}F}(S)]\setminus\Delta_\gamma(S)$, *and Ω does not include any component of $\Delta_\gamma(S)$*$\}$;

(3) $\beta(S)=\min\{\gamma\geqslant 0 : \sigma_W(S)\cup\Delta_\gamma(S)$ *is a connected set including* $\Omega\}$; *and*

(4) $\delta(S)=\inf\{\|B\| : B\in\mathscr{L}(\mathscr{H}),\ \sigma_0(S-B)=\emptyset\}$.

The proofs of the three theorems will be sketched in Section 2. Sections 3 and 4 will be devoted to a similar analysis of closely related classes of operators.

2. PROOFS OF THE MAIN RESULTS

1. Suppose that $T\in\mathscr{B}_n(\Omega)$, and let ω be any point of Ω; (b) and (d) imply that $(T-\omega)$ is a Fredholm operator of index n and trivial cokernel. Therefore, $(T-\omega)^k$ is Fredholm and satisfies

$$\operatorname{nul}(T-\omega)^k = \operatorname{ind}(T-\omega)^k = kn$$

or all $k=1,2,\ldots$. Moreover, since $(T-\omega)$ is Fredholm with trivial cokernel and Ω is connected, it follows from (c) and [7, Lemma 3.15, p. 48] that $\mathscr{H} = \bigvee\{\ker(T-\omega)^k\}_{k=1}^{\infty}$ (see also [1], [4, Section 1]). It follows that T admits an upper triangular matrix (with respect to a suitable orthonormal basis of $\mathscr{H}$) with diagonal entries equal to ω. We infer from [6] (or [7, Corollary 3.40]) that (ii) $\operatorname{ind}(T-\lambda)\geqslant 0$ for all $\lambda\in\rho_{s\text{-}F}(T)$, $\sigma(T)=\sigma_W(T)$ is a connected set containing Ω, and $\operatorname{nul}(T-\lambda)^*=0$ for all $\lambda\in\mathbf{C}\setminus\{\omega\}$.

Combining these observations with (a) and (b) ($\operatorname{ran}(T-\omega)=\mathscr{H}$), we conclude that T satisfies (i) — (iv) of Theorem 1.1.

2. Now assume that $R\in\mathscr{L}(\mathscr{H})$ satisfies (i′), (ii) and (iii) of Theorem 1.2, and let $\varepsilon>0$ be given. Let $\omega\in\Omega$. By [8, Theorem 1.2] (or [3, Theorem 13.36]) applied to $R-\omega$, there exists $K_\varepsilon\in\mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\|<\varepsilon$, such that $T=R-K_\varepsilon$ admits an upper triangular matrix $(t_{ij})_{i,j=1}^{\infty}$ (with respect to a suitable orthonormal

basis of $\mathscr{H}$) such that $t_{ii} = \omega$ for all $i = 1, 2, \ldots$, and $t_{ij} = 0$ for $0 < j - i < n$; moreover, K_ε can be chosen so that $t_{i,i+n} \neq 0$ for all $i = 1, 2, \ldots$. The stability properties of the semi-Fredholm index imply that T satisfies (ii) and (iii) of Theorem 1.1, so that $\sigma(T) \supset \Omega$. On the other hand, the results of [6], [7, Corollary 3.40, p. 64] indicate that $\sigma(T) = \sigma_W(T)$, and this set is connected and contains ω; furthermore, $\text{nul}(T - \lambda)^* = 0$ for all $\lambda \neq \omega$. Since $t_{i,i+n} \neq 0$ for all $i = 1, 2, \ldots$, a simple computation shows that $\text{ran}(T - \omega)$ is dense in $\mathscr{H}$ (and therefore $\text{ran}(T - \omega) = \mathscr{H}$), that is, $\text{nul}(T - \omega)^* = 0$. It readily follows that T also satisfies (i) and (iv). Thus

(a) $\Omega \subset \{\lambda \in \rho_{s\text{-}F}(T) : \text{ind}(T - \lambda) = n\} \subset \sigma(T)$;

(b) $\text{ran}(T - \lambda) = [\ker(T - \lambda)^*]^\perp = \mathscr{H}$ for all $\lambda \in \Omega$;

(c) $\bigvee \{\ker(T - \lambda) : \lambda \in \Omega\} = \bigvee \{\ker(T - \omega)^k\}_{k=1}^\infty = \mathscr{H}$ because $\text{nul}(T - \lambda) \equiv n$ near ω (see [1], [7, Chapter 3]); and

(d) $\text{nul}(T - \lambda) = \text{ind}(T - \lambda) + \text{nul}(T - \lambda)^* = n$ for all $\lambda \in \Omega$.

Hence, $T \in \mathscr{B}_n(\Omega)$.

3. Let $A \in \mathscr{L}(\mathscr{H})$ and assume that A satisfies (i′), (ii) and (iii′) of Theorem 1.3.

For each bounded open subset Φ of $\mathbf{C}$ such that $\Phi = \text{interior}(\Phi^-)$ we choose a cosubnormal operator $C(\Phi)$ such that $\sigma(C(\Phi)) = \Phi^-$, $\sigma_e(C(\Phi)) = \partial\Phi$ (= the boundary of Φ) and $\text{ind}(C(\Phi) - \lambda) = \text{nul}(C(\Phi) - \lambda) = 1$ for all $\lambda \in \Phi$. (The particular form of $C(\Phi)$ is irrelevant here; a concrete example can be found, e.g., in [7, pp. 89—90].) $C(\Phi)^{(\alpha)}$ will denote, as usual, the direct sum of α $(0 \leqslant \alpha \leqslant \infty)$ copies of $C(\Phi)$ acting in the usual fashion on the orthogonal direct sum of α copies of the underlying space. Finally, let Q be any quasinilpotent operator such that $Q^k \notin \mathscr{K}(\mathscr{H})$ for any $k = 1, 2, \ldots$.

We define $B \in \mathscr{L}(\mathscr{H})$ as any operator unitarily equivalent to

$$N \oplus C(\Phi_n)^{(n)} \oplus [\oplus \{C(\Phi_\alpha)^{(\alpha)} : 1 \leqslant \alpha \leqslant \infty,\ \alpha \neq n\}] \oplus$$
$$\oplus [\oplus \{\mu + Q : \mu \text{ is an isolated point of } \sigma_e(A)\}],$$

where

$$N \text{ is a normal operator such that } \sigma(N) = \sigma_e(N) = \sigma_{\text{lre}}(A) \setminus \Omega,$$

$$\Phi_n = \text{interior}([\Omega \cup \{\lambda \in \rho_{s\text{-}F}(A) : \text{ind}(A - \lambda) = n\}]^-),$$

and

$$\Phi_\alpha = \text{interior}(\{\lambda \in \rho_{s\text{-}F}(A) : \text{ind}(A - \lambda) = \alpha\}^-) \quad (1 \leqslant \alpha \leqslant \infty,\ \alpha \neq n).$$

Then $\sigma(B) = \sigma_W(A)$, $\rho_{s\text{-}F}(B) = \rho_{s\text{-}F}(A) \cup \Omega$, $\text{ind}(B - \lambda) = \text{ind}(A - \lambda)$ for all $\lambda \in \rho_{s\text{-}F}(A)$, and $\text{ind}(B - \lambda) = n$ for all $\lambda \in \Omega$. Since B satisfies the hypotheses of Theorem 1.2, we can find a compact operator K such that $T = B - K \in \mathscr{B}_n(\Omega)$, and $\sigma(T) = \sigma_W(T) = \sigma_W(A)$. Clearly, $\rho_{s\text{-}F}(T) = \rho_{s\text{-}F}(B)$ and $\text{ind}(T - \lambda) = \text{ind}(B - \lambda)$ for all $\lambda \in \rho_{s\text{-}F}(B)$. Since $\sigma(A)$ is connected, we deduce from [3, Theorem 9.1] that

$$A \in \mathscr{S}(T)^- \subset \mathscr{B}_n(\Omega)^-.$$

Finally, suppose that $S \in \mathscr{L}(\mathscr{H})$, and let $\alpha(S)$, $\alpha'(S)$, $\beta(S)$ and $\delta(S)$ be defined as in Theorem 1.3. It is *easy* to check that if $\|S - M\| < \alpha(S)$, then $\operatorname{ind}(M - \lambda) < 0$ for some $\lambda \in \rho_{s\text{-}F}(M)$; if $\|S - M\| < \alpha'(S)$, then either $M - \lambda$ is not semi-Fredholm for some $\lambda \in \Omega$, or $M - \lambda$ is semi-Fredholm with $\operatorname{ind}(M - \lambda) \neq n$ for some $\lambda \in \Omega$, or Ω includes a component of $\sigma_{\text{lre}}(M)$. If $\|S - M\| < \beta(S)$, then $\sigma_W(M)$ cannot be a connected set including Ω; and, if $\|S - M\| < \delta(S)$, then $\sigma_0(M) \neq \emptyset$. In either case, M cannot belong to $\mathscr{B}_n(\Omega)$, whence we deduce that

$$\operatorname{dist}[S, \mathscr{B}_n(\Omega)] \geqslant \varkappa(S).$$

On the other hand, given $\varepsilon > 0$ we can use the general construction of [3, Chapter 12] in order to find an operator A_ε satisfying conditions (i'), (ii) and (iii') such that $\|S - A_\varepsilon\| < \varkappa(S) + \varepsilon$. Since $A_\varepsilon \in \mathscr{B}_n(\Omega)^-$, we conclude that

$$\operatorname{dist}[S, \mathscr{B}_n(\Omega)] \leqslant \varkappa(S).$$

The proofs of Theorems 1.1, 1.2, and 1.3 are now complete. ▨

From the spectral characterization of $\mathscr{B}_n(\Omega)^-$ (Theorem 1.3), and [3, Chapter 12], it is not difficult to deduce the following corollaries, whose proofs are left to the reader.

COROLLARY 2.1. *$\mathscr{B}_n(\Omega)^- + \mathscr{K}(\mathscr{H}) = \{A + K : A \in \mathscr{B}_n(\Omega)^-, \ K \in \mathscr{K}(\mathscr{H})\}$ is a closed subset of $\mathscr{L}(\mathscr{H})$. This set coincides with the set $\{A \in \mathscr{L}(\mathscr{H}) : A$ satisfies* (ii) *and* (iii') *of Theorem* 1.3, *and* (i'') *$\sigma_W(A)$ is a connected set including $\Omega\}$. Moreover, for each S in $\mathscr{L}(\mathscr{H})$,*

$$\operatorname{dist}[S, \mathscr{B}_n(\Omega) + \mathscr{K}(\mathscr{H})] = \varkappa_e(S) := \max\{\alpha(S), \alpha'(S), \beta(S)\},$$

where $\alpha(S)$, $\alpha'(S)$, and $\beta(S)$ are defined as in Theorem 1.3.
Furthermore, there exists R in $\mathscr{B}_n(\Omega)^- + \mathscr{K}(\mathscr{H})$ such that $\|S - R\| = \varkappa_e(R)$.

COROLLARY 2.2. *For each n $(1 \leqslant n < \infty)$,*

$$[\cup \mathscr{B}_n(\Omega)]^- = \{A \in \mathscr{L}(\mathscr{H}) : \operatorname{ind}(A - \lambda) \geqslant 0 \text{ for all } \lambda \in \rho_{s\text{-}F}(A),\ \sigma(A) \text{ and } \sigma_W(A) \text{ are connected sets}\}.$$

(The union is taken over all possible Ω's.)
For each S in $\mathscr{L}(\mathscr{H})$,

$$\operatorname{dist}[S, \cup \mathscr{B}_n(\Omega)] = \max\{\alpha(S), \beta'(S), \delta(S)\},$$

where $\alpha(S)$ *and* $\delta(S)$ *are defined as in Theorem* 1.3, *and*

$$\beta'(S) = \min\{\gamma \geqslant 0 : \sigma_{\mathrm{W}}(S) \cup \Delta_\gamma(S) \textit{ is connected}\}.$$

The set $[\cup \mathscr{B}_n(\Omega)]^- + \mathscr{K}(\mathscr{H})$ *is closed in* $\mathscr{L}(\mathscr{H})$. *It coincides with the set* $\{A \in \mathscr{L}(\mathscr{H}) : \mathrm{ind}(A - \lambda) \geqslant 0$ *for all* $\lambda \in \rho_{\text{s-F}}(A)$, *and* $\sigma_{\mathrm{W}}(A)$ *is a connected set*$\}$; *furthermore, for each* S *in* $\mathscr{L}(\mathscr{H})$, *there exists* R *in* $[\cup \mathscr{B}_n(\Omega)]^- + \mathscr{K}(\mathscr{H})$ *such that*

$$\|S - R\| = \mathrm{dist}[S, \cup B_n(\Omega) + \mathscr{K}(\mathscr{H})] = \max\{\alpha(S), \beta'(S)\}.$$

3. THE CASE WHEN $\sigma(T) = \Omega^-$

Let $\mathscr{B}_n'(\Omega) = \{T \in \mathscr{B}_n(\Omega) : \sigma(T) = \Omega^-\}$. The analogues of Theorems 1.1 and 1.2 can be easily obtained by replacing the condition "the Weyl spectrum includes Ω" by "the Weyl spectrum is equal to Ω^-". The proofs are identical.

The analogue of Theorem 1.3 is the following result, whose proof follows by the same arguments.

THEOREM 3.1. $\mathscr{B}_n'(\Omega)^-$ *is the set of all those operators* A *in* $\mathscr{L}(\mathscr{H})$ *which satisfy:*

(i') $\sigma(A)$ *and* $\sigma_{\mathrm{W}}(A)$ *are connected sets including* Ω;

(ii') $\mathrm{ind}(A - \lambda) = 0$ *for all* $\lambda \in \rho_{\text{s-F}}(A)\setminus\Omega$; *and*

(iii'') $\partial(\Omega^-) \subset \sigma_{\mathrm{lre}}(A)$, Ω *does not include any component of* $\sigma_{\mathrm{lre}}(A)$, *and* $\mathrm{ind}(A - \lambda) = n$ *for all* $\lambda \in \Omega \cap \rho_{\text{s-F}}(A)$.

Furthermore, for each S *in* $\mathscr{L}(\mathscr{H})$,

$$\mathrm{dist}[S, \mathscr{B}_n'(\Omega)] = \varkappa'(S) := \max\{\alpha(S), \alpha''(S), \beta(S), \delta(S)\},$$

where

(1) $\alpha(S) = \min\{\gamma \geqslant 0 : \mathrm{ind}(S - \lambda) = 0$ *for all* $\lambda \in \rho_{\text{s-F}}(S)\setminus[\Delta_\gamma(S) \cup \Omega]\}$;

(2) $\alpha''(S) = \min\{\gamma \geqslant 0 : \mathrm{ind}(S - \lambda) = n$ *for all* $\lambda \in [\Omega \cap \rho_{\text{s-F}}(S)]\setminus\Delta_\gamma(S)$, Ω *does not include any component of* $\Delta_\gamma(S)$, *and* $\partial(\Omega^-) \subset \Delta_\gamma(S)\}$;

(3) $\beta(S) = \min\{\gamma \geqslant 0 : \sigma_{\mathrm{W}}(S) \cup \Delta_\gamma(S)$ *is a connected set including* $\Omega\}$; *and*

(4) $\delta(S) = \inf\{\|B\| : B \in \mathscr{L}(\mathscr{H}),\ \sigma_0(S - B) = \emptyset\}$.

Moreover, $\mathscr{B}_n'(\Omega)^- + \mathscr{K}(\mathscr{H})$ *is a closed subset of* $\mathscr{L}(\mathscr{H})$, *and for each* S *in* $\mathscr{L}(\mathscr{H})$, *there exists* R *in* $\mathscr{B}_n'(\Omega)^- + \mathscr{K}(\mathscr{H})$ *such that*

$$\|S - R\| = \mathrm{dist}[S, \mathscr{B}_n'(\Omega) + \mathscr{K}(\mathscr{H})] = \varkappa_e'(S) := \max\{\alpha(S), \alpha''(S), \beta(S)\}.$$

REMARK. If $\Omega = \mathrm{interior}(\Omega^-)$, then it readily follows from [3, Theorem 9.1] that $\mathscr{B}_n'(\Omega)^- = \mathscr{S}(T)^-$ for each T in $\mathscr{B}_n'(\Omega)$.

4. THE CLASSES $\mathscr{B}_{m,n}(\Omega_1, \Omega_2)$

During the Annual Meeting of the American Mathematical Society (New Orleans, Louisiana, January, 1986), M. J. Cowen announced several results about a class closely related to $\mathscr{B}_n(\Omega)$.

DEFINITION. For Ω_1, Ω_2 bounded connected open subsets of $\mathbf{C}$ and positive integers m, n, let $\mathscr{B}_{m,n}(\Omega_1, \Omega_2)$ denote the operators T in $\mathscr{L}(\mathscr{H})$ which satisfy:

(a) $\Omega_1 \cup \Omega_2 \subset \sigma(T)$;

(b) $\mathrm{ran}(T - \omega_1)$ and $\mathrm{ran}(T - \omega_2)^*$ are closed for all $\omega_1 \in \Omega_1$ and, respectively, for all $\omega_2 \in \Omega_2$;

(c) $\mathrm{nul}(T - \omega_1) = m$ and $\mathrm{nul}(T - \omega_2)^* = n$ for all $\omega_1 \in \Omega_1$ and, respectively, for all $\omega_2 \in \Omega_2$; and

(d) $\bigvee (\{\ker(T - \lambda) : \lambda \in \Omega_1\}; \{\ker(T - \lambda)^* : \lambda \in \Omega_2\}) = \mathscr{H}$.

Let $T \in \mathscr{B}_{m,n}(\Omega_1, \Omega_2)$. By using the results of [1], [7, Chapter 3], we see that $\mathscr{H} = \mathscr{H}_{\mathrm{r}}(T) \oplus \mathscr{H}_{\mathrm{l}}(T)$ (orthogonal direct sum), where

$$\mathscr{H}_{\mathrm{r}}(T) = \bigvee \{\ker(T - \lambda) : \lambda \in \Omega_1\} = \bigvee \{\ker(T - \omega_1)^k\}_{k=1}^{\infty}$$

(for each ω_1 in Ω_1) is invariant under T, and

$$\mathscr{H}_{\mathrm{l}}(T) = \bigvee \{\ker(T - \lambda)^* : \lambda \in \Omega_2\} = \bigvee \{\ker[(T - \omega_2)^*]^k\}_{k=1}^{\infty}$$

(for each ω_2 in Ω_2) is invariant under T^*, so that

$$T = \begin{pmatrix} T_{\mathrm{r}} & T_{12} \\ 0 & T_{\mathrm{l}} \end{pmatrix} \begin{matrix} \mathscr{H}_{\mathrm{r}}(T) \\ \mathscr{H}_{\mathrm{l}}(T) \end{matrix};$$

furthermore, if $\Omega_1 \cap \Omega_2 \neq \emptyset$, then $\mathscr{B}_{m,n}(\Omega_1, \Omega_2) = \mathscr{B}_{m,n}(\Omega, \Omega)$, where $\Omega = \Omega_1 \cup \Omega_2$ (see [2]).

Thus, we have two essentially different cases:

$$\text{(I)} \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \text{(II)} \quad \Omega_1 = \Omega_2.$$

By applying to T_{r} and T_{l} the same arguments as in Section 2, we obtain the analogues of the main results for the classes $\mathscr{B}_{m,n}(\Omega_1, \Omega_2)$. The proofs are left to the reader.

In what follows, $\sigma_{\mathrm{p}}(T)$ denotes the point spectrum of T, and $\Gamma^* = \{\bar{\lambda} : \lambda \in \Gamma\}$ ($\Gamma \subset \mathbf{C}$).

THEOREM 4.1. *If $T \in \mathscr{B}_{m,n}(\Omega_1, \Omega_2)$, then*

(i) *$\sigma(T)$ is the union of two closed connected sets, $\sigma_1(T)$ and $\sigma_2(T)$, where $\sigma_1(T)$ ($\sigma_2(T)$) is the union of $\sigma_{\mathrm{p}}(T)$ ($\sigma_{\mathrm{p}}(T^*)^*$, resp.) and the components of $\sigma_{\mathrm{lre}}(T)$ that intersect $\sigma_{\mathrm{p}}(T)^-$ ($[\sigma_{\mathrm{p}}(T^*)^*]^-$, resp.). Moreover, $\sigma_j(T)$ includes $\Omega_j (j = 1, 2)$;*

(ii) $\operatorname{ind}(T-\lambda) > 0$ *and* $\operatorname{nul}(T-\lambda)^* = 0$ *for all* $\lambda \in \rho_{s\text{-}F}(T) \cap [\sigma_1(T) \setminus \sigma_2(T)]$; $\operatorname{ind}(T-\lambda) < 0$ *and* $\operatorname{nul}(T-\lambda) = 0$ *for all* $\lambda \in \rho_{s\text{-}F}(T) \cap [\sigma_2(T) \setminus \sigma_1(T)]$; *and*

(iii) *If* $\Omega_1 \cap \Omega_2 = \emptyset$, *then* $\operatorname{ind}(T-\lambda) = m$ *for all* $\lambda \in \Omega_1$, *and* $\operatorname{ind}(T-\lambda) = -n$ *for all* $\lambda \in \Omega_2$. *If* $\Omega_1 = \Omega_2$, *then* $\operatorname{ind}(T-\lambda) = m-n$, $\operatorname{nul}(T-\lambda) = m$, *and* $\operatorname{nul}(T-\lambda)^* = n$ *for all* $\lambda \in \Omega_1$.

Observe that $\operatorname{ind}(T-\lambda) = 0$ for all λ in a component Φ of $\rho_{s\text{-}F}(T) \cap \sigma_1(T) \cap \sigma_2(T)$ such that $0 < \operatorname{nul}(T-\lambda) = \operatorname{nul}(T-\lambda)^* < \infty$ $(\lambda \in \Phi)$. Clearly, Φ can be simply, or multiply, connected. We can use the results of [1], [7, Chapter 3] in order to find a small compact perturbation C_ε such that $\sigma(T-C_\varepsilon) = \sigma(T) \setminus \Phi$. If Φ is simply connected, then both $\sigma(T)$ and $\sigma(T-C_\varepsilon)$ will be connected sets. (Observe that $\Phi \subset \sigma_1(T) \cap \sigma_2(T) \neq \emptyset$, so $\sigma(T)$ is necessarily connected in this case.) But if Φ is not simply connected, then $\sigma(T-C_\varepsilon)$ will be a disconnected set. This phenomenon explains the necessity of the strange condition (i′) in Theorem 4.2 below.

The "converse" of Theorem 4.1 has the following form.

THEOREM 4.2. *Assume that* $R \in \mathcal{L}(\mathcal{H})$ *satisfies:*

(i′) *Let* $\sigma_1(R)$ $(\sigma_2(R))$ *be the union of* $\sigma_p(R)$ $(\sigma_p(R^*)^*$, *resp.) and all the components of* $\sigma_{lre}(R)$ *that intersect* $\sigma_p(R)^-$ $([\sigma_p(R^*)^*]^-$, *resp.); then* $\sigma_j(R) \supset \Omega_j$ $(j = 1, 2)$. *Let* $\sigma_j'(R)$ *denote the component of* $\sigma_j(R)$ *including* Ω_j. *Then* $\sigma(R) = \sigma_1'(R) \cup \sigma_2'(R)$, *and the multiply connected components of* $\{\lambda \in \sigma(R) \cap \rho_{s\text{-}F}(R) : \operatorname{ind}(R-\lambda) = 0\}$ *are included in* $\sigma_1'(R) \cap \sigma_2'(R)$;

(ii′)
$$\operatorname{ind}(R-\lambda)\begin{cases} > 0 & \text{for all } \lambda \in \rho_{s\text{-}F}(R) \cap [\sigma_1'(R) \setminus \sigma_2'(R)], \\ < 0 & \text{for all } \lambda \in \rho_{s\text{-}F}(R) \cap [\sigma_2'(R) \setminus \sigma_1'(R)]; \end{cases}$$

(iii′) *If* $\Omega_1 \cap \Omega_2 = \emptyset$, *then* $\operatorname{ind}(R-\lambda) = m$ *for all* $\lambda \in \Omega_1$, *and* $\operatorname{ind}(R-\lambda) = -n$ *for all* $\lambda \in \Omega_2$. *If* $\Omega_1 = \Omega_2$, *then* $\operatorname{ind}(R-\lambda) = m-n$, $\operatorname{nul}(R-\lambda) \geqslant m$ *and* $\operatorname{nul}(R-\lambda)^* \geqslant n$ *for all* $\lambda \in \Omega_1$.

Then, given $\varepsilon > 0$ *there exists* $K_\varepsilon \in \mathcal{K}(\mathcal{H})$, *with* $\|K_\varepsilon\| < \varepsilon$, *such that* $R - K_\varepsilon \in \mathcal{B}_{m,n}(\Omega_1, \Omega_2)$.

THEOREM 4.3. $\mathcal{B}_{m,n}(\Omega_1, \Omega_2)^-$ *is the set of all those operators* A *on* $\mathcal{L}(\mathcal{H})$ *which satisfy:*

(i″) *Let* $\sigma_1''(A)$ $(\sigma_2''(A))$ *be the component of the union of* $[\Omega_1 \cup \sigma_p(A)]^-$ $([\Omega_2 \cup \sigma_p(A^*)^*]^-$, *resp.) and all the components of* $\sigma_{lre}(A)$ *that intersect* $[\Omega_1 \cup \sigma_p(A)]^-$ $([\Omega_2 \cup \sigma_p(A^*)^*]^-$, *resp.) that includes* Ω_1 $(\Omega_2$, *resp.); then* $\sigma(A) = \sigma_1''(A) \cup \sigma_2''(A)$ *and the multiply connected components of* $\{\lambda \in \sigma(A) \cap \rho_{s\text{-}F}(A) : \operatorname{ind}(A-\lambda) = 0\}$, *are included in* $\sigma_1''(A) \cap \sigma_2''(A)$;

$$\text{(ii'')} \qquad \operatorname{ind}(A-\lambda) \begin{cases} > 0 & \textit{for all } \lambda \in \rho_{s\text{-}F}(A) \cup [\sigma_1''(A) \setminus \sigma_2''(A)], \\ < 0 & \textit{for all } \lambda \in \rho_{s\text{-}F}(A) \cup [\sigma_2''(A) \setminus \sigma_1''(A)]; \end{cases}$$

(iii'') $\Omega_1 \cup \Omega_2$ *does not include any component of* $\sigma_{\mathrm{lre}}(A)$. *If* $\Omega_1 \cap \Omega_2 = \emptyset$, *then* $\operatorname{ind}(A-\lambda) = m$ *for all* $\lambda \in \Omega_1 \cap \rho_{s\text{-}F}(A)$, *and* $\operatorname{ind}(A-\lambda) = -n$ *for all* $\lambda \in \Omega_2 \cap \rho_{s\text{-}F}(A)$. *If* $\Omega_1 = \Omega_2$, *then* $\operatorname{ind}(A-\lambda) = m-n$, $\operatorname{nul}(A-\lambda) \geqslant m$ *and* $\operatorname{nul}(A-\lambda)^* \geqslant n$ *for all* $\lambda \in \Omega_1 \cap \rho_{s\text{-}F}(A)$.

Furthermore, for each S *in* $\mathscr{L}(\mathscr{H})$,

$$\operatorname{dist}[S, \mathscr{B}_{m,n}(\Omega_1, \Omega_2)] = \max\{\eta(S), \delta(S)\},$$

where

(1) $\eta(S)$ *is the minimum over all* $\gamma \geqslant 0$ *such that* $\sigma(S \oplus N_\gamma) = \sigma_1''(S \oplus N_\gamma) \cup \sigma_2''(S \oplus N_\gamma) \cup \sigma_0(S)$, *the multiply connected components of* $\operatorname{interior}\{\lambda \in \sigma(S \oplus N_\gamma) \cap \rho_{s\text{-}F}(S \oplus N_\gamma) : \operatorname{ind}(S \oplus N_\gamma - \lambda) = 0\}$ *are included in* $\sigma_1''(S \oplus N_\gamma) \cap \sigma_2''(S \oplus N_\gamma)$, *and* $S \oplus N_\gamma$ *satisfies* (ii'') *and* (iii''). *(Here* N_γ *denotes a normal operator such that* $\sigma(N_\gamma) = \sigma_e(N_\gamma) = \Delta_\gamma(S) \cup \Delta_\gamma(S^*)^*$.*) And*

$$(2) \qquad \delta(S) = \inf\{\|B\| : B \in \mathscr{L}(\mathscr{H}),\ \sigma_0(S-B) = \emptyset\}.$$

COROLLARY 4.4. $\mathscr{B}_{m,n}(\Omega_1, \Omega_2)^- + \mathscr{K}(\mathscr{H})$ *is a closed subset of* $\mathscr{L}(\mathscr{H})$, *which coincides with the set of all those operators* $A \in \mathscr{L}(\mathscr{H})$ *satisfying the conditions:*

(i''') *Let* $\sigma_1''(A)$ *and* $\sigma_2''(A)$ *be defined as in Theorem* 4.3 (i''); *then* $\sigma(A) = \sigma_1''(A) \cup \sigma_2''(A) \cup \sigma_0(A)$, *and the multiply connected components of* $\{\lambda \in \sigma(A) \cap \rho_{s\text{-}F}(A) : \operatorname{ind}(A-\lambda) = 0\}$ *are included in* $\sigma_1''(A) \cap \sigma_2''(A)$;

(ii'') *(Exactly as in Theorem* 4.3*); and*

(iii''') $\Omega_1 \cup \Omega_2$ *does not include any component of* $\sigma_{\mathrm{lre}}(A)$. *If* $\Omega_1 \cap \Omega_2 = \emptyset$, *then* $\operatorname{ind}(A-\lambda) = m$ *for all* $\lambda \in \Omega_1 \cap \rho_{s\text{-}F}(A)$, *and* $\operatorname{ind}(A-\lambda) = -n$ *for all* $\lambda \in \Omega_2 \cap \rho_{s\text{-}F}(A)$. *If* $\Omega_1 = \Omega_2$, *then* $\operatorname{ind}(A-\lambda) = m-n$ *for all* $\lambda \in \Omega_1 \cap \rho_{s\text{-}F}(A)$.

Furthermore, for each S *in* $\mathscr{L}(\mathscr{H})$ *there exists* R *in* $\mathscr{B}_{m,n}(\Omega_1, \Omega_2)^- + \mathscr{K}(\mathscr{H})$ *such that*

$$\|S-R\| = \operatorname{dist}[S, \mathscr{B}_{m,n}(\Omega_1, \Omega_2) + \mathscr{K}(\mathscr{H})] = \eta(S).$$

By analogy with Corollary 2.2, we can similarly find spectral characterizations of the sets $[\cup \mathscr{B}_{m,n}(\Omega_1, \Omega_2)]^-$ (m, n fixed, $1 \leqslant m, n < \infty$; the union runs over all possible pairs (Ω_1, Ω_2) and $[\cup \mathscr{B}_{m,n}(\Omega_1, \Omega_2)]^- + \mathscr{K}(\mathscr{H})$ (which is closed in $\mathscr{L}(\mathscr{H})$), and find concrete formulas for the distance from a given operator S to each of these two sets.

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