J. OPERATOR THEORY
18(1987), 327–337

© Copyright by INCREST, 1987

K-THEORY FUNDAMENTAL CLASSES AND SMOOTH STRUCTURES

MIKIYA MASUDA

INTRODUCTION

Let M be a closed topological manifold which admits a smooth structure. Choosing a smooth structure on M, the norm closure $\bar{\mathscr{P}}_0(M)$ of the algebra of 0^{th}-order pseudo-differential operators on M determines a C^*-algebra extension

$$0 \to \mathscr{K}(L^2(M)) \to \bar{\mathscr{P}}_0(M) \xrightarrow{\bar{\sigma}} C(SM) \to 0.$$

Here $\mathscr{K}(L^2(M))$ is the algebra of compact operators on square integrable functions $L^2(M)$ over M, $\bar{\sigma}$ is the extended symbol map, and $C(SM)$ is the algebra of continuous complex valued functions over SM the total space of the tangent sphere bundle of M. By BDF theory [5], [8] this extention defines an element P_M of Ext(SM) and Ext(SM) is isomorphic to K homology $K_1(SM)$. Apparently the definition of P_M depends on the chosen smooth structure on M. I. M. Singer ([17]) raised a question of whether there is a path, in the space of extensions, between the pseudo-differential operator extensions associated with different smooth structures on M, to be brief, of whether P_M is independent of the smooth structure. The question is related to the famous Novikov theorem on the topological invariance of rational Pontryagin classes. In fact, if one can find such a path directly, then the Novikov theorem follows. This was one of the motivations to study the above extension.

Recently J. Kaminker, however, gave a negative answer to the question. First of all in [9] he observed that P_M is a fundamental class of Ext(SM) (note that there are many fundamental classes in Ext(SM)) and defined an invariant θ which detects the variation of P_M when the smooth structure on M is replaced by another one. Next in [11] he analyzed θ in homotopy theoretic terms under the assumption that M is 2-connected (Theorem 2.9 of [11]) and exhibited some (highly connected manifold) M with non-trivial θ (§ 3 of [11]).

In this article we show that the above 2-connectedness assumption is unnecessary and then θ is non-trivial for some standard (high dimensional) lens spaces, while the manifolds discovered by Kaminker are rather unfamiliar to us. In addition, as pointed out in § 5 of [11], our analysis of θ is related to the problem: To what extent do elliptic pseudo-differential operators depend on smooth structures? (cf. Remark 1.5 (c) of [7]). We observe it for the Dirac operator.

The organization of this article is as follows. In Section 1 we review the work of Kaminker. Section 2 is devoted to removal of 2-connectedness assumption in Theorem 1.3. In Section 3 the non-triviality of θ is verified for some standard lens spaces in two ways. In Section 4 we interpret Theorem 2.1 from the point of view of operator K theory. As a consequence the dependence on smooth structures of the Dirac operator is made clear.

Finally the author would like to express his gratitude to H. Moriyoshi for useful conversations.

1. KAMINKER'S WORK

For details on the content of this section we refer the reader to [9], [10], and [11].

Let M be a closed topological manifold (of dimension $\geqslant 5$). We choose and fix a smooth structure on M for a while. The resulting smooth manifold will also be denoted by M. Let $\mathrm{D}M$ be the total space of the tangent disk bundle of M. The (stable) normal bundle of $\mathrm{D}M$, written $\mathrm{ND}M$, is isomorphic to the pullback bundle $\pi^*(\mathrm{N}M \oplus \mathrm{N}M)$ of $\mathrm{N}M \oplus \mathrm{N}M$ by the projection map $\pi: \mathrm{D}M \to M$ and $\mathrm{N}M \oplus \mathrm{N}M$ is naturally isomorphic to $\mathrm{N}M \otimes \mathbf{C}$; so $\mathrm{ND}M$ inherits the complex structure from $\mathrm{N}M \otimes \mathbf{C}$. This determines a unique Spin^c structure on $\mathrm{ND}M$ and then the Thom class $U_{\mathrm{ND}M} \in \mathrm{K}^0(\mathrm{ND}M)$. As is well known, the one point compactification $(\mathrm{ND}M)^+$ of $\mathrm{ND}M$ is S-dual to the quotient space $\mathrm{D}M/\mathrm{S}M$, i.e. there is a duality map

$$d: S^p \to (\mathrm{D}M/\mathrm{S}M) \wedge (\mathrm{ND}M)^+ .$$

A K *theory fundamental class* $[\mathrm{T}M] \in \mathrm{K}_0(\mathrm{T}M) = \mathrm{K}_0(\mathrm{D}M, \mathrm{S}M)$ is then defined as the S-dual element of $U_{\mathrm{ND}M}$, i.e.

$$[\mathrm{T}M] = U_{\mathrm{ND}M} \backslash d_*(s)$$

where $s \in \mathrm{K}_0(S^p)$ is the Bott generator and $\backslash$ is the slant product. A deformation retraction of $\mathrm{D}M$ into M induces that of $(\mathrm{ND}M)^+$ into $(\mathrm{N}M \otimes \mathbf{C})^+$. This allows us to make use of $U_{\mathrm{N}M \otimes \mathbf{C}}$ instead of $U_{\mathrm{ND}M}$.

The class $[\mathrm{T}M]$ is related to the question on pseudo-differential operators by the following result, which is a consequence of the Atiyah-Singer index theorem for family of operators together with the Novikov theorem.

PROPOSITION 1.1. (see § 4 of [9] or Proposition 1.2 of [11]). *If* $P_M \in \mathrm{Ext}(SM) = K_1(SM)$ *is the pseudo-differential operator extension, then the boundary operator* $\partial : K_0(TM) \to K_1(SM)$ *satisfies* $\partial([TM]) = P_M$.

In general, there may be many elements in the preimage $\partial^{-1}(P_M)$. However ∂ is sometimes injective, for instance, this is the case if M is of odd dimension (more generally if the Euler characteristics of M is zero). Thus we are led to study the difference of the K theory fundamental classes associated with different smooth structures on M, which is a purely topological subject.

At this point we should precisely make mention of smooth structures. By a smooth structure on M we mean a maximal set of compatible local coordinate systems covering M. We specify it by (M, Σ) for a moment. One says that two smooth structures (M, Σ_0) and (M, Σ_1) on M are concordant if there is a smooth structure $\tilde{\Sigma}$ on $M \times I$ ($I = [0, 1]$) such that $(M \times I, \tilde{\Sigma}) | M \times 0 = (M, \Sigma_0)$ and $(M \times I, \tilde{\Sigma}) | M \times 1 = (M, \Sigma_1)$. Then we denote by $\mathscr{S}(M)$ the set of concordance classes of smooth structures on M.

Let TOP/O be the fiber of a natural map $i : \mathrm{BO} \to \mathrm{BTOP}$, where BO (resp. BTOP) is the classifying space of stable vector (resp. Euclidean) bundles. The topological manifold M has a unique Euclidean tangent bundle tM of M. In fact the total space of tM is given by a suitable neighborhood of the diagonal set of $M \times M$ and the bundle projection map is given by the projection to the first factor. There is a map $\alpha_M : M \to \mathrm{BTOP}$ classifying tM stably. Each smooth structure on M determines a vector bundle structure on tM, in other words, a lifting of α_M into BO. Since the difference of two vector bundle structures on tM can be described by a map from M to the fiber TOP/O (up to homotopy), we have a correspondence between $\mathscr{S}(M)$ and the homotopy set $[M, \mathrm{TOP/O}]$ (this set admits an abelian group structure induced from the H-space structure of TOP/O) whenever we fix a smooth structure on M. As a matter of fact the correspondence is bijective (here the assumption $\dim M \geqslant 5$ is necessary). Henceforth we identify $\mathscr{S}(M)$ with $[M, \mathrm{TOP/O}]$ and denote by M_α the smooth manifold corresponding to an element α of $[M, \mathrm{TOP/O}]$. Obviously M_α agrees with M if α is the zero element.

Kaminker defined an invariant $\theta(\alpha) \in 1 + \tilde{K}^0(M)$ by

$$[TM]_\alpha = \theta(\alpha) \cap [TM] \quad \text{for } \alpha \in [M, \mathrm{TOP/O}], \tag{1.2}$$

which turns out to be a homomorphism from $[M, \mathrm{TOP/O}]$ to the 2-subgroup of the multiplicative group of units $1 + \tilde{K}^0(M)$ (Theorems 6.9 and 7.3 of [9]). He analyzed θ in homotopy theoretic terms as follows. If E is a vector bundle and $\varphi : E \to X \times \mathbf{R}^k$ is a topological trivialization, then E admits a Spin^c structure (induced from a Spin structure) and hence has a Thom class U_E. There is a unique unit $u \in 1 + \tilde{K}^0(X)$ such that

$$u \cup U_E = \varphi^*(U_{X \times \mathbf{R}^k}).$$

Defining $e_{\mathbf{C}}(E, \varphi) = u$ yields a map $e_{\mathbf{C}} : \mathrm{TOP/O} \to \mathrm{B}U^{\otimes}$ where $\mathrm{B}U^{\otimes}$ denotes the classifying space (of stable complex vector bundles) with the H-space structure induced from the tensor product. Note that $[X, \mathrm{B}U^{\otimes}] = 1 + \tilde{\mathrm{K}}^0(X)$.

THEOREM 1.3. (Theorem 2.9 of [11]). *If M is a closed 2-connected smooth manifold, then $\theta(\alpha) = e_{\mathbf{C}}(\alpha)^2$ for $\alpha \in [M, \mathrm{TOP/O}]$.*

The map $e_{\mathbf{C}}$ is well understood by efforts of Sullivan and others, and Kaminker used it to find M of odd dimension with non-trivial θ.

2. REMOVAL OF THE 2-CONNECTEDNESS ASSUMPTION

The purpose of this section is to remove the 2-connectedness assumption in Theorem 1.3, namely we verify

THEOREM 2.1. *If M is a closed smooth manifold (of dimension ≥ 5), then $\theta(\alpha) = e_{\mathbf{C}}(\alpha)^2$ for $\alpha \in [M, \mathrm{TOP/O}]$.*

Before beginning the proof we first express (1.2) in terms of Thom classes. Let $M_{(\alpha)}$ stand for either M or M_α. Since their underlying topological manifolds are the same, one can identify $\mathrm{T}M_\alpha$ with $\mathrm{T}M$ as Euclidean bundles. Adding $\mathrm{N}M \oplus \mathrm{N}M_\alpha$ to both sides, one can also identify $\mathrm{N}M$ with $\mathrm{N}M_\alpha$ as Euclidean bundles (in particular as topological spaces), where Whitney sum of $\mathrm{N}M_{(\alpha)}$ and some trivial bundle is again denoted by $\mathrm{N}M_{(\alpha)}$, $\mathrm{N}M_{(\alpha)}$ being the normal bundle of $M_{(\alpha)}$ in a stable sense.

Remember that we may replace $\mathrm{ND}M_{(\alpha)}$ by $\mathrm{N}M_{(\alpha)} \otimes \mathbf{C}$ in the definition of the K theory fundamental class $[\mathrm{T}M_{(\alpha)}]$, namely we may think of $[\mathrm{T}M_{(\alpha)}]$ as the S-dual element of the Thom class $U_{\mathrm{N}M_{(\alpha)} \otimes \mathbf{C}}$. Hence (1.2) is equivalent that

$$U_{\mathrm{N}M_\alpha \otimes \mathbf{C}} = \theta(\alpha) \cup U_{\mathrm{N}M \otimes \mathbf{C}}, \tag{2.2}$$

where $\mathrm{K}^0(\mathrm{N}M_\alpha \otimes \mathbf{C})$ and $\mathrm{K}^0(\mathrm{N}M \otimes \mathbf{C})$ are identified through the identity map from $\mathrm{N}M \otimes \mathbf{C}$ to $\mathrm{N}M_\alpha \otimes \mathbf{C}$. This simple observation plays a role in our argument.

LEMMA 2.3. (Naturality of θ). *Let $f : Y \to X$ be a continuous map between closed smooth manifolds. Then the following diagram commutes:*

$$\begin{array}{ccc} [X, \mathrm{TOP/O}] & \xrightarrow{\theta} & 1 + \tilde{\mathrm{K}}^0(X) \\ \downarrow f^* & & \downarrow f^* \\ [Y, \mathrm{TOP/O}] & \xrightarrow{\theta} & 1 + \tilde{\mathrm{K}}^0(Y). \end{array}$$

Proof. Let α be an element of $[X, \mathrm{TOP/O}]$. Then there exists a closed smooth manifold X_α such that the identity map from TX_α to TX represents α. As observed above this induces the identity map from NX to NX_α which also represents α. Quite similarly there exists a smooth manifold $Y_{f^*\alpha}$ such that the identity map from NY to $NY_{f^*\alpha}$ represents $f^*\alpha$. By the definition of $*$ there exist vector bundles E (over X) and F (over Y) such that

$$f^*(NX_{(\alpha)} \oplus E) = NY_{(f^*\alpha)} \oplus F. \tag{2.4}$$

Thus we have

$$U_{(NY_{f^*\alpha}\oplus F)\otimes C} = f^*U_{(NX_\alpha\oplus E)\otimes C} = \qquad \text{(by (2.4))}$$

$$= f^*(\theta(\alpha) \cup U_{(NX\oplus E)\otimes C)} = \qquad \text{(by (2.2))}$$

$$= f^*\theta(\alpha) \cup f^*U_{(NX\oplus E)\otimes C}\,,$$

and similarly

$$U_{(NY_{f^*\alpha}\oplus F)\otimes C} = \theta(f^*\alpha) \cup U_{(NY\oplus F)\otimes C} = \qquad \text{(by (2.2))}$$

$$= \theta(f^*\alpha) \cup f^*U_{(NX\oplus E)\otimes C} \qquad \text{(by (2.4))}.$$

Comparing these, the lemma follows. Q.E.D.

Proof of Theorem 2.1. We distinguish three cases.

Case 1. The case where M admits a Spin structure. The argument of [11] essentially works in this case. We shall give a simplified proof based on the idea of [11]. Of importance is the existence of the decomposition

$$U_{NM_{(\alpha)}\otimes C} = U_{NM_{(\alpha)}} U_{NM_{(\alpha)}}, \tag{2.5}$$

which can be justified below in the present case.

Remember that Spin^c structures on a vector bundle are classified by the first Chern classes of their associated complex line bundles (see Corollary 3.6 of [16] for example). If a Spin^c structure happens to be induced from a Spin structure, then the first Chern class vanishes. In particular, such a Spin^c structure is uniquely determined. Now $NM_{(\alpha)}$ and hence $NM_{(\alpha)} \oplus NM_{(\alpha)}$ admit Spin structures by the assumption. We equip them with the induced Spin^c structures whose first Chern classes vanish. On the other hand, the Spin^c structure on $NM \otimes C$ was one inherited from the complex structure. The first Chern class of such a Spin^c structure agrees with that of the complex vector bundle $NM \otimes C$. The latter is an order 2 element obtained as the Bockstein image of $w_1(NM_{(\alpha)})$ the first Stiefel-Whitney class of

$NM_{(\alpha)}$ (see Problem 15-D of [15]). However $w_1(NM_{(\alpha)})$ vanishes, $M_{(\alpha)}$ being orientable by the assumption. Thus (2.5) has been justified.

Now remember that the identity map from NM to NM_α represents $\alpha \in [M, \mathrm{TOP/O}]$. It follows from the definition of $e_{\mathbf{C}}$ that

$$U_{NM_\alpha} = e_{\mathbf{C}}(\alpha) \cup U_{NM}.$$

This together with (2.5) yields

$$U_{NM_\alpha \otimes \mathbf{C}} = U_{NM_\alpha} U_{NM_\alpha} = e_{\mathbf{C}}(\alpha)^2 \cup (U_{NM} U_{NM}) =$$

$$= e_{\mathbf{C}}(\alpha)^2 \cup U_{NM \otimes \mathbf{C}}.$$

Compairing this identity with (2.2), the theorem is verified in Case 1.

Case 2. The case where M is orientable. Let S be the total space of a sphere bundle of $TM \oplus (M \times \mathbf{R})$. Let $\rho : S \to M$ be the projection map and let $s : M \to S$ be a section. Since TS is isomorphic to $\rho^*(TM \oplus TM \oplus (M \times \mathbf{R}))$, the second Stiefel-Whitney class of TS is equal to $\rho^* w_1^2(TM)$ which vanishes as M is orientable. This shows that S admits a Spin structure.

Consider the commutative diagram (Lemma 2.3)

$$\begin{array}{ccc} [S, \mathrm{TOP/O}] & \xrightarrow{\theta} & 1 + \tilde{K}^0(S) \\ \downarrow{s^*} & & \downarrow{s^*} \\ [M, \mathrm{TOP/O}] & \xrightarrow{\theta} & 1 + \tilde{K}^0(M) \end{array}$$

where the vertical homomorphisms are surjective as s is a section, and the upper θ agrees with $e_{\mathbf{C}}^2$ by Case 1 as S admits a Spin structure. Clearly the functor $e_{\mathbf{C}}^2$ is natural with respect to continuous maps; so the commutativity of the above diagram implies that the lower θ also agrees with $e_{\mathbf{C}}^2$.

Case 3. The case where M is non-orientable. Let S be the same as in Case 2. Then S is orientable. Thus the same argument as in Case 2 verifies $\theta = e_{\mathbf{C}}^2$ also in this case.

These complete the proof of Theorem 2.1. Q.E.D.

3. EXAMPLES OF NON-TRIVIAL θ

Kaminker [11] exhibited closed highly connected manifolds with non-trivial θ using Theorem 1.3 and an affirmed Adams conjecture. In fact he first constructed a finite complex for which $e_{\mathbf{C}}^2$ is non-trivial. Next, by embedding it in a sphere and taking the double of a smooth regular neighborhood, he obtained a desired manifold. In this section we verify that some high dimensional standard lens spaces have non-trivial θ in two ways; one is based on Kaminker's method and the other uses a result of Cappell-Shaneson [6].

Let $L^n(q)$ be the standard lens space of dimension $2n+1$ with $\pi_1(L^n(q)) \cong \mathbf{Z}_q$. One can see that the Kaminker's argument at the above first step applies to a stunted lens complex $L^n(q)/L^m(q)$ $(n \leqslant 2m)$, where the condition $n \leqslant 2m$ is necessary to ensure that the obstruction map $\mathcal{O}$ in the proof of Proposition 3.1 of [11] becomes a homomorphism. It tells us that $e_{\mathbf{C}}^2$ is non-trivial for $L^n(q)/L^m(q)$ (for an appropriate m) provided q is even and n is sufficiently large. On the other hand, $e_{\mathbf{C}}^2$ is natural with respect to a continuous map and the projection map from $L^n(q)$ to $L^n(q)/L^m(q)$ induces a monomorphism on the torsion subgroups of their K cohomology. This observation together with Theorem 2.1 establishes

THEOREM 3.1. *Suppose that q is even. Then there is a natural number N_q such that θ is non-trivial for $L^n(q)$ provided $n \geqslant N_q$.*

REMARK 3.2. (1) As indicated before the image of θ is contained in the 2-subgroup of units $1 + \tilde{\mathrm{K}}^0(L^n(q))$. Hence, in case q is odd, θ must be trivial.

(2) It is not difficult to see that the above argument gives an estimate $N_q \leqslant 8$ for every q divisible by 8.

In the rest of this section we shall give an alternative proof to Theorem 3.1, which seems interesting by itself although we require that q is divisible by 8. We begin it in a rather more general setting.

Let T be a Lie group. Suppose we are given a T homeomorphism $\omega : U \to V$ between T representations and a principal smooth T bundle $p : P \to M$. Combining them, we get in a natural manner an Euclidean bundle map, written ω_p, between associated vector bundles $U_p = P \underset{T}{\times} U$ and $V_p = P \underset{T}{\times} V$ over M. This ω_p determines an element of $[M, \mathrm{TOP/O}]$.

LEMMA 3.3. *If $2(U_p - V_p)$ is non-zero in $\tilde{\mathrm{KO}}^0(M)_{(2)}$ and the complexification homomorphism $c : \tilde{\mathrm{KO}}^0(M)_{(2)} \to \tilde{\mathrm{K}}^0(M)_{(2)}$ is injective, then $\theta(\omega_p)$ is non-trivial.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{TOP/O} & & \\ & \swarrow^{i} & & \searrow^{e_{\mathbf{C}}} & \\ G/O_{(2)} & \xrightarrow{e=\beta_2} & BSO^{\otimes}_{(2)} & \xrightarrow{c} & BU^{\otimes}_{(2)} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1/\Psi^3} & & \downarrow{\scriptstyle 1/\Psi^3} \\ BSO^{\oplus}_{(2)} & \xrightarrow{\rho^3_A} & BSO^{\otimes}_{(2)} & \xrightarrow{c} & BU^{\otimes}_{(2)} \end{array}.$$

Here j is the natural inclusion map, the commutativity of the left square is well known, which can be proved in a similar fashion to Lemma 5.7 in p. 103 of [13], and the notations $1/\Psi^3$ and ρ_A^3 are the same as in there. The reader is referred to 9.16 and Remark 9.17 in p. 184 of [13] for the H-equivalence of ρ_A^3 between $\mathrm{BSO}_2^{\oplus}$ and $\mathrm{BSO}_2^{\otimes}$. The above diagram and the injectivity of c mean that if $j_* i_*(\omega_p) \in \widetilde{\mathrm{KO}}^0(M)_{(2)}$ is non-trivial, then so is $e_{\mathbf{C}}(\omega_p)$. On the other hand $j_* i_*(\omega_p) = U_p - V_p$ by the definition of the composition map ji. Thus the lemma follows because $\theta = e_{\mathbf{C}}^2$ by Theorem 2.1 and every map in the diagram is an H-map. Q.E.D.

Needless to say Lemma 3.3 is nonsense for the same U and V. It was a long-standing problem to find out a pair of different T representations admitting a T homeomorphism. However Cappell-Shaneson recently provided such examples. To state it let σ_k ($k \in \mathbf{Z}$) denote the 2-dimensional real representation of order $2m$ of the cyclic group $\mathbf{Z}_{2m}$ given by

$$\sigma_k(g) = \begin{pmatrix} \cos \pi k/m & \sin \pi k/m \\ -\sin \pi k/m & \cos \pi k/m \end{pmatrix}$$

for g a fixed generator of $\mathbf{Z}_{2m}$ and let δ_{-1} denote the unique non-trivial one dimensional real representation of $\mathbf{Z}_{2m}$.

PROPOSITION 3.4. (Proposition 7 of [6]). *If m is even, then $4\sigma_1 + \delta_{-1}$ is $\mathbf{Z}_{2m}$ homeomorphic to $4\sigma_{m+1} + \delta_{-1}$.*

Now we are in a position to prove Theorem 3.1.

Set $U = 4\sigma_1 + \delta_{-1}$ and $V = 4\sigma_{m+1} + \delta_{-1}$. As a principal T bundle we take a natural $\mathbf{Z}_{2m}$ bundle $p_n : S^{2n+1} \to L^n(2m)$. To check the conditions of Lemma 3.3 for a sufficiently large n we recall the Atiyah-Segal's correspondence ([2])

$$\varinjlim_n \alpha_n : \mathrm{RO}(\mathbf{Z}_{2m}) \to \mathrm{RO}(\mathbf{Z}_{2m})^\wedge \cong \varinjlim \mathrm{KO}^0(L^n(2m))$$

which is defined by $\alpha_n(W) = W_{p_n}$ for $W \in \mathrm{RO}(\mathbf{Z}_{2m})$. Note that $\alpha_n(U - V)$ is contained in $\widetilde{\mathrm{KO}}^0(L^n(2m))$ and $2(U - V) \in \mathrm{RO}(\mathbf{Z}_{2m})$ is non-trivial in case $2m$ is divisible by 8.

Case 1. If $2m$ is a power of 2, then $\varinjlim \alpha_n$ is monic (Proposition 6.11 of [1]) and the localization $\widetilde{\mathrm{KO}}^0(L^n(2m)) \to \widetilde{\mathrm{KO}}^0(L^n(2m))_{(2)}$ is an isomorphism since $\widetilde{\mathrm{KO}}^0(L^n(2m))$ is a 2-group (see p. 201, Proposition 4.6 of [14]). These mean that $\alpha_n(2(U - V)) = 2(U_{p_n} - V_{p_n})$ is non-trivial in $\widetilde{\mathrm{KO}}^0(L(2m))_{(2)}$ for a sufficient large n. Moreover one can neglect the injectivity condition of c, for c is monic if $n \equiv 3 \pmod 4$ (see [19], or p. 202 of [14]) and $\theta(\omega_{p_s}) = r^*\theta(\omega_{p_t})$ with respect to the inclusion map r from $L^s(2m)$ to $L^t(2m)$ $(s \leqslant t)$.

Case 2. Let $2m = 2^l s$ (s odd). Then the projection map $\pi: L^n(2^l) \to L^n(2m)$ induces a monomorphism $\pi^* : \widetilde{\mathrm{KO}}^0(L^n(2m))_{(2)} \to \widetilde{\mathrm{KO}}^0(L^n(2^l))_{(2)}$, for there is a transfer map $t : \widetilde{\mathrm{KO}}^0(L^n(2^l)) \to \widetilde{\mathrm{KO}}^0(L^n(2m))$ such that the composition $\pi^* t$ is the multiplication by s. Hence Case 2 reduces to Case 1. Q.E.D.

4. A REMARK ON THE DIRAC OPERATOR

Suppose M is a closed Spin manifold of even dimension $\geqslant 5$. Then there is the (half) Dirac operator D associated with M (see § 5 of [3]). According to Kasparov [12] (see also [4]), it determines an element $[D]$ of $\mathrm{K}_0(M)$ for which

$$(4.1) \qquad [D] = U_{\mathrm{T}M} \cap [\mathrm{T}M]$$

where $U_{\mathrm{T}M}$ is the Thom class of the Spin^c structure on $\mathrm{T}M$ induced from a Spin structure as before. Similarly, if we take M_α in place of M, then we obtain the Dirac operator D_α for which

$$(4.2) \qquad [D_\alpha] = U_{\mathrm{T}M_\alpha} \cap [\mathrm{T}M_\alpha] \in \mathrm{K}_0(M_\alpha) = \mathrm{K}_0(M).$$

With these understood

THEOREM 4.3. *The Dirac operators* $[D]$, $[D_\alpha] \in \mathrm{K}_0(M)$ *associated with* M, M_α *are related with*

$$[D_\alpha] = e_{\mathrm{C}}(\alpha) \cap [D].$$

In particular $[D_\alpha] = [D]$ *if and only if* $e_{\mathrm{C}}(\alpha) = 1$.

Proof. By the definition of e_{C} we have $U_{\mathrm{N}M_\alpha} = e_{\mathrm{C}}(\alpha) \cup U_{\mathrm{N}M}$; so $U_{\mathrm{T}M_\alpha} = e_{\mathrm{C}}(\alpha)^{-1} \cup U_{\mathrm{T}M}$. On the other hand $[\mathrm{T}M_\alpha] = e_{\mathrm{C}}(\alpha)^2 \cap [\mathrm{T}M]$ by Theorem 2.1 and (1.2). Putting these together and using (4.1), (4.2) turns into

$$[D_\alpha] = (e_{\mathrm{C}}(\alpha) \cup U_{\mathrm{T}M}) \cap [\mathrm{T}M] = e_{\mathrm{C}}(\alpha) \cap [D],$$

which proves the former assertion. Since the cup product $\cup\, U_{\mathrm{T}M}: \mathrm{K}^0(M) \to \mathrm{K}^0(\mathrm{T}M)$ and the cap product $\cap\, [\mathrm{T}M] : \mathrm{K}^0(\mathrm{T}M) \to \mathrm{K}_0(M)$ are both isomorphisms, the latter assertion is straightforward from the former. Q.E.D.

COROLLARY 4.4. *If* $\mathrm{K}^0(M)$ *has no 2-torsion, then* $[D_\alpha] = [D]$ *for any* $\alpha \in [M, \mathrm{TOP/O}]$.

Proof. The assumption implies that the multiplicative 2-subgroup of $1 + \tilde{\mathrm{K}}^0(M)$ is trivial. On the other hand, the image of e_{C} is shown to be contained in the

2-subgroup by Sullivan (see § 4 of [11]). Hence e_{C} must be trivial and the corollary follows from Theorem 4.3. Q.E.D.

REMARK 4.5. If M is a Spin^c manifold of even dimension $\geqslant 5$, then the above results hold for the (half). Dirac operator associated with the Spin^c structures having the same first Chern classes.

If M is of odd dimension, then the Dirac operator is still defined and determines an element of $K_1(M)$ ([4]); so the formula of Theorem 3.3 still makes sense. But the author does not know whether it is true or not.

In contrast with the Dirac operator, the result of Teleman [18] or Hilsum [20] implies that the signature operator is independent of smooth structures.

REFERENCES

1. ATIYAH, M. F., Characters and cohomology of finite groups, *Inst. Hautes Études Sci. Publ. Math.*, **9**(1961), 23–64.
2. ATIYAH, M. F.; SEGAL, G. B., Equivariant K-theory and completion, *J. Differential Geom.*, **3**(1969), 1–18.
3. ATIYAH, M. F.; SINGER, I.M., The index of elliptic operators. III, *Ann. of Math.*, **87**(1968), 546–604.
4. BAUM, P.; DOUGLAS, R. G., K-homology and index theory, *Proc. Sympos. Pure Math.*, **38**(1982), 117–173.
5. BROWN, L. G.; DOUGLAS, R. G.; FILLMORE, P., Extensions of C^*-algebras and K-homology, *Ann. of Math.*, **105**(1977), 265–324.
6. CAPPELL, S. E.; SHANESON, J. L., The topological rationality of linear representations, *Inst. Hautes Études Sci. Publ. Math.*, **56**(1982), 309–336.
7. CONNES, A.; SKANDALIS, G., The longitudinal index theorem for foliations, *Publ. Res. Inst. Math. Sci.*, **20**(1984), 1139–1183.
8. DOUGLAS, R. G., *C^*-algebra extensions and K-homology*, Ann. of Math. Studies, **95**, Princeton Univ. Press, 1980.
9. KAMINKER, J., Pseudo-differential operators and differential structures. I, *Contemporary Math.*, **10**(1982), 99–128.
10. KAMINKER, J., C^*-algebras and differential topology, *Bull. Amer. Math. Soc.*, **12**(1985), 125–127.
11. KAMINKER, J., Pseudo-differential operators and differential structures. II, *Amer. J. Math.*, **108**(1986), 703–718.
12. KASPAROV, G. G., Topological invariants of elliptic operators. I: K-homology, *Math. USSR-Izv.*, **9**(1975), 751–792.
13. MADSEN, I.; MILGRAM, R. J., *The classifying spaces for surgery and cobordism of manifods*, Ann. of Math. Studies, **92**, Princeton Univ. Press, 1979.
14. MAHAMMED, N.; PICCININI, R.; SUTER, U., *Some applications of topologcal K-theory*, North-Holland Math. Studies, **45**, 1980.
15. MILNOR, J. W.; STASHEFF, J. D., *Characteristic classes*, Ann. of Math. Studies, **76**, Princeton Univ. Press, 1974.

16. PETRIE, T., Smooth S^1 actions on homotopy complex projective spaces and related topics, *Bull. Amer. Math. Soc.*, **78**(1972), 105–153.
17. SINGER, I. M., Some remarks on operator theory and index theory, in *Lect. Notes in Math.*, **575**, Springer-Verlag, pp. 128–138.
18. TELEMAN, N., The index of signature operators on Lipschitz manifolds, *Inst. Haute Études Sci. Publ. Math.*, **58**(1983), 39–78.
19. YASUO, M., γ-dimension and products of lens spaces, *Mem. Fac. Sci. Kyushu Univ. Ser. A.*, **31**(1977), 113–126.
20. HILSUM, M., Signature operators on Lipschitz manifolds and unbounded Kasparov bimodules, *Lect. Notes in Math.*, Springer-Verlag, **1132**(1985), pp. 254–288.

MIKIYA MASUDA
Department of Mathematics,
Osaka City University,
Sugimoto, Sumiyoshi-ku,
Osaka 558,
Japan.

Received December 24, 1986; revised June 15, 1987.