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INVARIANT SUBSPACES FOR ℓ^p-OPERATORS HAVING BISHOP'S PROPERTY (β) ON A LARGE PART OF THEIR SPECTRUM

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To the memory of our friend C. Apostol who made the first successful attempts to apply the S. Brown technique outside the Hilbert space framework.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In [4] it is proved that a hyponormal operator with thick spectrum has nontrivial invariant subspaces. One essential ingredient in the proof is a result of [8] to the effect that hyponormal operators are subscalar. In fact, as pointed out in [4], only the subdecomposability of hyponormal operators is needed to make a certain variant of the S. Brown machinery go through. In Sections 3 and 4 of this paper we fully explicit and also extend this aspect. Using the internal characterization of sub-residual-decomposable operators by means of Bishop's property (β) obtained in [1], we thus prove that operators $S \in \mathscr{L}(X)$, where X is topologically isomorphic to a quotient of two closed subspaces of ℓ^p, $1 < p < \infty$, and S or S^* has property (β) on a large part of its spectrum, have nontrivial invariant subspaces. Then, after developing a version of the S. Brown technique adapted to our context (inspired by that of [2] but simplified by the use of convexity properties) we obtain additional local structure theory results, the most important one having to do with the richness of the lattice of invariant subspaces in the case that S or S^* has property (β) on a thick part of its essential spectrum.

Before stating our main results we first recall some basic notations and facts from local spectral theory [6, 10]. Let X be a Banach space, $T \in \mathscr{L}(X)$, and let F be a closed subset of the spectrum $\sigma(T)$ of T. The operator T is said to be *F-decomposable* if for every finite open covering $U_0, U_1, \ldots, U_n$ of $\sigma(T)$ such that $F \subset U_0$ there are $X_j \in \operatorname{Lat} T$ $(0 \leqslant j \leqslant n)$ with $\sigma(T | X_j) \subset U_j$ and $X = X_0 + X_1 + \ldots + X_n$. It turns out that it is sufficient to require this property for $n = 1$. For general $n \in \mathbf{N}$ it can be derived from this special case (cf. [10], Theorem IV.4.26). Of course every $T \in \mathscr{L}(X)$ is $\sigma(T)$-decomposable. Thus, the definition is only interesting if F is a proper subset of $\sigma(T)$. If F is not specified, then we speak of *residually decomposable* operators. A $\emptyset$-decomposable operator will be called *decomposable in the sense of C. Foiaş.*

If $U \subset \mathbf{C}$ is open, we write $\mathcal{O}(U, X)$ for the space of all analytic X-valued functions on U. Endowed with the topology of uniform convergence on all compact subsets of U, $\mathcal{O}(U, X)$ is a Fréchet space. For $T \in \mathcal{L}(X)$ a continuous linear mapping $\alpha_T^U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ is defined by $(\alpha_T^U f)(z) := (z - T)f(z)$ for $z \in U$, $f \in \mathcal{O}(U, X)$. Let now G be an open set in $\mathbf{C}$. We say that an operator $T \in \mathcal{L}(X)$ has

— the *single valued extension property* (SVEP) *on* G if for all open subsets U of G the mapping $\alpha_T^U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ is injective;

— *Bishop's property* (β) *on* G (cf. [3]), if for all open subsets U of G the mapping $\alpha_T^U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ is injective and has closed range.

If $T \in \mathcal{L}(X)$ is F-decomposable, then T has property (β) and hence the SVEP on $\mathbf{C} \setminus F$ (see [10], Lemma IV.4.16). Also, if $T \in \mathcal{L}(X)$ has property (β) on G then $T \mid Y$ obviously has property (β) on G for all $Y \in \operatorname{Lat} T$. Conversely, it has recently be shown in [1] that, if $S \in \mathcal{L}(X)$ has property (β) on $\mathbf{C} \setminus F$ for some closed $F \subset \sigma(S)$, then there exists a Banach space Z and an F-decomposable operator $T \in \mathcal{L}(Z)$ such that S is similar to $T \mid Y$ for some $Y \in \operatorname{Lat} T$; the operator S will then be called F-subdecomposable (resp. subdecomposable if $F = \emptyset$). Moreover, if X is a (separable) Hilbert space then also Z can be chosen to be a (separable) Hilbert space. If X is topologically isomorphic to a quotient X_0/X_1 of two closed subspaces of $\ell^p = \ell^p(\mathbf{N})$ $(1 < p < \infty)$ then there are an open rectangle $R \supset \sigma(S)$, two closed subspaces $Z_1 \subset Z_0$ of ℓ^p, and an operator $T \in \mathcal{L}(Z_0/Z_1)$ which is $(F \cup \partial R)$-decomposable such that S is similar to $T \mid Y$ for some $Y \in \operatorname{Lat} T$ (see [1] for details).

Let Ω be an open set in $\mathbf{C}$. Recall that a subset $G \subset \Omega$ is said to be *dominating for* Ω, if for all $f \in H^\infty(\Omega)$ we have

$$\sup_{\lambda \in G} |f(\lambda)| = \sup_{\lambda \in \Omega} |f(\lambda)| =: \|f\|_{H^\infty(\Omega)}.$$

We can now state our first main result.

1.1. THEOREM. *Let X be a Banach space which is topologically isomorphic to a quotient of two closed subspaces of ℓ^p $(1 < p < \infty)$ and let S be an operator in $\mathcal{L}(X)$ such that S or S^* has property (β) on $\mathbf{C} \setminus F$ for some closed $F \subset \sigma(S)$. Suppose that there exists a sequence $(\Omega_n)_{n=1}^\infty$ of open sets in $\mathbf{C}$ such that*

$$(1) \qquad \begin{cases} \emptyset \neq \bar{\Omega}_n \subset \Omega_{n+1} \text{ for all } n \in \mathbf{N}, \\ \Omega_n \cap \sigma(S) \text{ is dominating in } \Omega_n \text{ for all } n \in \mathbf{N}, \\ \Omega_\infty \cap \sigma(S) \text{ is dominating in } \Omega_\infty := \bigcup_{n=1}^\infty \Omega_n, \text{ and } \bar{\Omega}_\infty \cap F = \emptyset. \end{cases}$$

Then S has a nontrivial invariant subspace.

In particular, if $\sigma(S) \setminus F$ has nonempty interior then there is always a sequence $(\Omega_n)_{n=1}^\infty$ of open sets in $\sigma(S) \setminus F$ satisfying (1) and S must thus have a nontrivial invariant subspace. In this case we will show that even $\operatorname{Lat} \mathcal{W}_S$ is nontrivial, where

$\mathscr{W}_S$ is the closure of $\{f(S) \mid f$ rational with poles off $\sigma(S)\}$ in the weak operator topology (WOT). This fact and Theorem 1.1 will be proved in Section 4. In the Hilbert space case we obtain:

1.2. Corollary. *Let $\mathscr{H}$ be a Hilbert space and let $S \in \mathscr{L}(\mathscr{H})$ be an operator such that S resp. S^* (here we mean the adjoint operator) has property (β) on $\mathbf{C}\setminus F$, resp. on $\mathbf{C}\setminus F^*$, where $F^* := \{\bar{z}\ ;\ z \in F\}$. Suppose that there exists a sequence $(\Omega_n)_{n=1}^{\infty}$ of open sets in $\mathbf{C}$ satisfying (1). Then S has a nontrivial invariant subspace.*

Proof. Of course, we may assume that $\mathscr{H}$ is separable. Because of (1) $\mathscr{H}$ must be infinite dimensional. Hence, $\mathscr{H}$ is isometrically isomorphic to $\ell^2 = \ell^2/\{0\}$ and we are in the situation of Theorem 1.1.

1.3. Corollary. *Let S be a subdecomposable operator on a Hilbert space $\mathscr{H}$ such that* $\operatorname{int}\sigma(S) \neq \emptyset$. *Then S has a nontrivial invariant subspace.*

Our next main result states that Lat S will be very rich if (1) is satisfied for the essential spectrum $\sigma_e(S)$ instead of $\sigma(S)$. More precisely,

1.4. Theorem. *Let X be as in Theorem* 1.1 *and let $S \in \mathscr{L}(X)$ be an operator such that S or S^* has property (β) on $\mathbf{C}\setminus F$ for some closed $F \subset \sigma(S)$. Suppose that there exists a sequence $(\Omega_n)_{n=1}^{\infty}$ of open sets in $\mathbf{C}$ such that* (1) *is fulfilled with $\sigma(S)$ replaced by $\sigma_e(S)$. Then there exists $\mathscr{M}, \mathscr{N} \in$* Lat S *with $\mathscr{N} \subset \mathscr{M}$ such that $\mathscr{M}/\mathscr{N}$ is infinite dimensional and such that* Lat S *contains a sublattice (order) isomorphic to* Lat $\mathscr{M}/\mathscr{N}$, *the lattice of all closed subspaces of $\mathscr{M}/\mathscr{N}$.*

Notice, that a sequence $(\Omega_n)_{n=1}^{\infty}$ satisfying (1) for $\sigma_e(S)$ will exist if $\sigma_e(S)\setminus F$ has nonempty interior. Theorem 1.4 will be proved in Section 5.

1.5. Corollary. *Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space and let $S \in \mathscr{L}(\mathscr{H})$ be as in Corollary 1.2. Suppose that there is some sequence $(\Omega_n)_{n=1}^{\infty}$ of open sets satisfying (1) with $\sigma(S)$ replaced by $\sigma_e(S)$. Then* Lat S *has a sublattice which is order isomorphic to* Lat $\mathscr{H}$.

Notice, that in this case $\mathscr{M}/\mathscr{N}$ is topologically isomorphic to $\mathscr{H}$.

1.6. Corollary. *Let S be a subdecomposable operator on a separable infinite dimensional Hilbert space $\mathscr{H}$ such that* $\operatorname{int}\sigma_e(S) \neq \emptyset$. *Then* Lat S *contains a sublattice order isomorphic to* Lat $\mathscr{H}$.

This result seems to be new even for the special cases of subnormal or hyponormal operators.

It should be noted that the present methods do not admit an immediate generalization to arbitrary Banach spaces. Indeed, the Rosenthal theorem used in Section 2 actually characterizes Banach spaces not containing ℓ^1 as a closed subspace. Also the version of Lemma 5.3 needed for the case $X = \ell^1$ (i.e. $r = \infty$ and weak null sequence replaced by weak* null sequence) does not hold. In this connection a closer look to the example found by Read [9] of an operator $S \in \mathscr{L}(\ell^1)$ without nontrivial invariant subspaces could be of some interest.

2. A CHARACTERIZATION OF THE LEFT ESSENTIAL SPECTRUM FOR OPERATORS ON SEPARABLE BANACH SPACES CONTAINING NO CLOSED SUBSPACE TOPOLOGICALLY ISOMORPHIC TO ℓ^1

Let X be a Banach space and $T \in \mathscr{L}(X)$. Recall that the left essential spectrum $\sigma_{le}(T)$ is the set of all $\lambda \in \mathbf{C}$ such that $\ker(\lambda - T)$ is infinite dimensional or such that $\operatorname{ran}(\lambda - T)$ is not closed. The following proposition is well known for Hilbert spaces. In fact, in this case, the sequence $(x_n)_{n=1}^{\infty}$ may be chosen orthonormal and one may take $x_n^* = x_n$ for all $n \in \mathbf{N}$.

2.1. PROPOSITION. *Let X be a separable Banach space such that no closed subspace of X is topologically isomorphic to ℓ^1 and let $T \in \mathscr{L}(X)$. Then a point $\lambda \in \mathbf{C}$ is in $\sigma_{le}(T)$ if and only if there are sequences $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in X^* such that*

(a) $\|x_n\| = 1$ *for all* $n \in \mathbf{N}$ *and* $x_n \to 0$ *for* $n \to \infty$ *in the weak topology* $\sigma(X, X^*)$.

(b) $\|(\lambda - T)x_n\| \to 0$ *for* $n \to \infty$.

(c) $\|x_n^*\| = 1$ *for all* $n \in \mathbf{N}$ *and* $x_n^* \to 0$ *for* $n \to \infty$ *in the weak* topology* $\sigma(X^*, X)$.

(d) *The limit* $\eta := \lim_{n\to\infty} \langle x_n^*, x_n\rangle$ *exists and* $\eta \geqslant 1/2$. *(Here, $\langle \cdot , \cdot \rangle$ denotes the duality $\langle X^*, X\rangle$.)*

Proof. (I) Fix an arbitrary $\lambda \in \sigma_{le}(T)$.

(α) First we construct a sequence $(x_n)_{n=1}^{\infty}$ in X satisfying (a) and (b). For this purpose we consider two cases:

Case 1: $\dim\ker(\lambda - T) = \infty$. Then, by the Riesz lemma there is a sequence $(u_n)_{n=1}^{\infty}$ in $\ker(\lambda - T)$ with $\|u_n\| = 1$ ($n \in \mathbf{N}$) and $\|u_n - u_m\| > 1/2$ for all $n, m \in \mathbf{N}$ with $n \neq m$. By the complex version of the Rosenthal ℓ^1 theorem (see [7], p. 201), the sequence $(u_n)_{n=1}^{\infty}$ has a weak Cauchy subsequence which will be again denoted by $(u_n)_{n=1}^{\infty}$. Therefore, $u_{n+1} - u_n \to 0$ weakly for $n \to \infty$. Since the sequence $(\|u_{n+1} - u_n\|^{-1})_{n=1}^{\infty}$ is bounded, the sequence $(x_n)_{n=1}^{\infty}$, defined by

$$x_n := \|u_{n+1} - u_n\|^{-1}(u_{n+1} - u_n),$$

tends weakly to 0. It follows that $(x_n)_{n=1}^{\infty}$ satisfies (a) and (b).

Case 2: $\dim\ker(\lambda - T) < \infty$. Then $\operatorname{ran}(\lambda - T)$ cannot be closed. Moreover, as a finite dimensional space, $\ker(\lambda - T)$ has a topological complement $Y \in \operatorname{Lat} X : X = Y \oplus \ker(\lambda - T)$. Since $(\lambda - T)(Y) = \operatorname{ran}(\lambda - T)$ is not closed, there is some sequence $(u_n)_{n=1}^{\infty}$ in Y such that $\|u_n\| = 1$ for all $n \in \mathbf{N}$ and

(2) $$\|(\lambda - T)u_n\| \to 0 \quad \text{for } n \to \infty.$$

As in the first case we may now assume (dropping to a subsequence) that $(u_n)_{n=1}^{\infty}$ is a weak Cauchy sequence. The sequence $(u_n)_{n=1}^{\infty}$ cannot have any norm conv ergen subsequence. Indeed, if $u_{k(n)} \to u$ in Y, then, because of (2), we would have $u \in Y \cap \cap \ker(\lambda - T) = \{0\}$ in contradiction to $1 = \lim_{n\to\infty} \|u_{k(n)}\| = \|u\|$. Since $(u_n)_{n=1}^{\infty}$ has no norm-convergent subsequence there are some $\varepsilon > 0$ and a subsequence $(u_{j(n)})_{n=1}^{\infty}$, such that

$$\inf_{n\in\mathbf{N}} \|u_{j(n)} - u_{j(n+1)}\| \geqslant \varepsilon.$$

Let $x_n := \|u_{j(n)} - u_{j(n+1)})\|^{-1} (u_{j(n)} - u_{j(n+1)})$. It is straightforward to check that the sequence $(x_n)_{n=1}^{\infty}$ satisfies (a) and (b).

(β) Because of (α) we now have a sequence $(x_n)_{n=1}^{\infty}$ with properties (a) and (b). Notice, that then also every subsequence of $(x_n)_{n=1}^{\infty}$ must have these properties. For the construction of $(x_n^*)_{n=1}^{\infty}$ we first remark that by the Hahn-Banach theorem we can find $v_n^* \in X^*$ such that

(3) $$\|v_n^*\| = 1 = \langle v_n^*, x_n\rangle \quad \text{for all } n \in \mathbf{N}.$$

The unit ball X_1^* of X^* is weak* compact and, due to the separability of X, metrizable in the weak* topology. Hence, by passing to subsequences (if necessary) we may assume that $v_n^* \to v^* \in X_1^*$ for $n \to \infty$ with respect to $\sigma(X^*, X)$. Then $w_n^* := v_n^* - v^* \to 0$ for $n \to \infty$ with respect to $\sigma(X^*, X)$ and $\|w_n^*\| \leqslant 2$ for all $n \in \mathbf{N}$. Moreover, because of (a) and (3), we have

$$\lim_{n\to\infty} \langle w_n^*, x_n\rangle = \lim_{n\to\infty} \langle v_n^*, x_n\rangle = 1,$$

and therefore

$$1 \leqslant \delta := \liminf_{n\to\infty} \|w_n^*\| \leqslant 2.$$

The sequence $(x_n^*)_{n=1}^{\infty}$ with $x_n^* := \|w_n^*\|^{-1} w_n$ now satisfies (c). Taking again subsequences, we may assume that $\|w_n^*\| \to \delta$ for $n \to \infty$. It follows that

$$\langle x_n^*, x_n\rangle = \|w_n^*\|^{-1}\langle w_n^*, w_n\rangle \to 1/\delta \geqslant 1/2$$

for $n \to \infty$. Hence also (d) is fulfilled and the construction is complete.

(II) Conversely, suppose now that for a given $\lambda \in \mathbf{C}$ there are sequences $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in X^* satisfying (a)—(d). If $\dim\ker(\lambda - T) = \infty$ then $\lambda \in \sigma_{le}(T)$ and we are done. Hence, suppose that $\dim\ker(\lambda - T) < \infty$ and assume that $\operatorname{ran}(\lambda - T)$ is closed. Then the mapping $S_\lambda : X/\ker(\lambda - T) \to X$ induced by $\lambda - T$ must be bounded below. Hence, because of $\|(\lambda - T)x_n\| \to 0$ there exist $y_n \in \ker(\lambda - T)$ with $\|x_n - y_n\| \to 0$ for $n \to \infty$. As $\ker(\lambda - T)$ is finite dimensional, the bounded sequence $(y_n)_{n=1}^{\infty}$ has a convergent subsequence $(y_{k(n)})_{n=1}^{\infty}$ with $y := \lim_{n\to\infty} y_{k(n)} \in \ker(\lambda - T)$. But then we also have $x_{k(n)} \to y$ in norm and because of (a) obtain $y = 0$. This gives the contradiction

$$1 = \lim_{n\to\infty} \|x_{k(n)}\| = \|y\| = 0.$$

Hence, $\operatorname{ran}(\lambda - T)$ cannot be closed and we obtain $\lambda \in \sigma_{le}(T)$.

2.2. REMARK. The proof in (II) shows that the existence of a sequence $(x_n)_{n=1}^{\infty}$ in X satisfying (a) and (b) already implies $\lambda \in \sigma_{le}(T)$ (without any restrictions for the Banach space X).

3. LOCAL APPROXIMATION

Throughout this section, $Z = Z_0/Z_1$ will be a quotient of two closed subspaces $Z_1 \subset Z_0$ of ℓ^p, $1 < p < \infty$, and $T \in \mathscr{L}(Z)$ will be a fixed F-decomposable operator, F a closed subset of $\sigma(T)$. We write

$$\mathscr{F}_F := \{A = \bar{A} \subset \mathbf{C}\,;\, A \cap F = \emptyset \ \text{ or } \ F \subset A\}$$

and we define for $A \in \mathscr{F}_F$

$$\mathscr{E}(A, T) := Z \cap \operatorname{ran} \alpha_T^{\mathbf{C}\setminus A}$$

where we identify Z with the set of all constant functions in $\mathscr{O}(\mathbf{C}\setminus A, Z)$. Then $\mathscr{E}(A, T) \in \operatorname{Lat} T$ (see [10], IV.4.19 and (4.2) in the proof of IV.4.21) and the mapping

$$\mathscr{E}(\cdot, T) : \mathscr{F}_F \to \operatorname{Lat} T,\ A \mapsto \mathscr{E}(A, T)$$

is a *spectral capacity* for T, i.e. it has the following properties:

(4) $\mathscr{E}(\emptyset, T) = \{0\}$, $\mathscr{E}(\mathbf{C}, T) = Z$.

(5) $\mathscr{E}\left(\bigcap_{n=1}^{\infty} A_n, T\right) = \bigcap_{n=1}^{\infty} \mathscr{E}(A_n, T)$ for every sequence $(A_n)_{n=1}^{\infty}$ in $\mathscr{F}_F$.

(6) For every finite open covering $\{U_0, U_1, \ldots, U_n\}$ of $\sigma(T)$ with $F \subset U_0$

and $U_j \cap F = \emptyset$ ($j = 1, \ldots, n$) we have

$$Z = \mathscr{E}(\overline{U}_0, T) + \mathscr{E}(\overline{U}_1, T) + \ldots + \mathscr{E}(\overline{U}_n, T).$$

(7) $\forall A \in \mathscr{F}_F : \sigma(T \mid \mathscr{E}(A, T)) \subset A \cap \sigma(T)$.

For the proof we refer to [10], IV. 4.19—4.26.

One knows from [10], IV.5.5, that the transposed operator $T^* \in \mathscr{L}(Z^*)$ is also F-decomposable. It turns out that the spectral capacity $\mathscr{E}(\cdot, T^*)\colon \mathscr{F}_F \to \operatorname{Lat} T^*$ of T^* is given by

$$\mathscr{E}(A, T^*) = \mathscr{E}(\sigma(T)\setminus A, T)^\perp \quad \text{for } A \in \mathscr{F}_F, \tag{8}$$

where $\mathscr{E}(\sigma(T)\setminus A, T) := \bigcup\{\mathscr{E}(B, T)\,;\, B \in \mathscr{F}_F,\ B \subset \sigma(T)\setminus A\}$. Indeed we have $\sigma(T^* \mid \mathscr{E}(\sigma(T)\setminus A, T)^\perp) \subset A$ by [10], IV.5.3, and hence $\mathscr{E}(\sigma(T)\setminus A, T)^\perp \subset \mathscr{E}(A, T^*)$. Conversely, if $x^* \in \mathscr{E}(A, T^*)$, then $x^* \equiv (\lambda - T^*)f^*(\lambda)$ on $\mathbf{C}\setminus A$ for some $f^* \in \mathscr{O}(\mathbf{C}\setminus A, Z^*)$. Fix an arbitrary $x \in \mathscr{E}(\sigma(T)\setminus A, T)$. Then $x \in \mathscr{E}(B, T)$ for some $B \in \mathscr{F}_F$ with $B \subset \sigma(T)\setminus A \subset \mathbf{C}\setminus A$. For $\lambda \in \mathbf{C}$ we define

$$\varphi(\lambda) := \begin{cases} \langle x^*, (\lambda - T \mid \mathscr{E}(B, T))^{-1}x\rangle & \text{for } \lambda \in \mathbf{C}\setminus B \\ \langle f^*(\lambda), x\rangle & \text{for } \lambda \in \mathbf{C}\setminus A. \end{cases}$$

φ is well defined since for $\lambda \in (\mathbf{C}\setminus B) \cap (\mathbf{C}\setminus A)$ we have

$$\langle x^*, (\lambda - T \mid \mathscr{E}(B, T))^{-1}x\rangle = \langle(\lambda - T^*)f^*(\lambda), (\lambda - T \mid \mathscr{E}(B, T))^{-1}x\rangle =$$

$$= \langle f^*(\lambda), (\lambda - T)(\lambda - T \mid \mathscr{E}(B, T))^{-1}x\rangle = \langle f^*(\lambda), x\rangle.$$

It follows that φ is an entire function vanishing at ∞. Thus $\varphi \equiv 0$ on $\mathbf{C}$. For $|\lambda| > \|T\|$ we obtain

$$0 = \varphi(\lambda) = \langle x^*, (\lambda - T \mid \mathscr{E}(B, T))^{-1}x\rangle = \langle x^*, (\lambda - T)^{-1}x\rangle =$$

$$= \sum_{n=}^{\infty} \lambda^{-n-1}\langle x^*, T^n x\rangle.$$

This implies $\langle x^*, T^n x\rangle = 0$ for all $n \in \mathbf{N}_0$. In particular $\langle x^*, x\rangle = 0$. Hence $x^* \in \mathscr{E}(\sigma(T)\setminus A, T)^\perp$ and the proof of (8) is complete.

Let now Ω, U be two bounded open sets in $\mathbf{C}$ with $\overline{\Omega} \subset U \subset \overline{U} \subset \mathbf{C}\setminus F$. We write $T^*_\Omega := T^* \mid \mathscr{E}(\overline{\Omega}, T^*)$. For $x \in Z$ and $y^* \in \mathscr{E}(\overline{\Omega}, T^*)$ we define $x \,\square_U\, y^* \in H^\infty(U)^*$ by

$$\langle x \,\square_U\, y^*, f\rangle := \langle f(T^*_\Omega)y^*, x\rangle \quad \text{for } f \in H^\infty(U).$$

Here $f \mapsto f(T_\Omega^*)$ denotes the analytic functional calculus for T_Ω^*. Notice, that $\sigma(T_\Omega^*) \subset \bar{\Omega}$, so that $f(T_\Omega^*)$ is defined for $f \in H^\infty(U)$. It is clear that

$$\forall x \in Z,\ y^* \in \mathcal{E}(\dot{\Omega}, T^*) : \|x \square_U y^*\| \leqslant K_{\Omega,U}\|x\| \cdot \|y^*\|$$

with a constant $K_{\Omega,U}$ independent of x and y^*.

Let us show that the definition of $x \square_U y^*$ is consistent with respect to changes in Ω and U. Indeed, if $\Omega' \subset \Omega$ then $\mathcal{E}(\bar{\Omega}', T^*) \subset \mathcal{E}(\bar{\Omega}, T^*)$ and hence $f(T_{\Omega'}^*) = f(T_\Omega^*) \mid \mathcal{E}(\bar{\Omega}', T^*)$ (notice that $\sigma(T_{\Omega'}^*) \subset \sigma(T_\Omega^*) \subset \bar{\Omega} \subset U$). It follows that

$$\langle f(T_{\Omega'}^*)y^*, x\rangle = \langle f(T_\Omega^*)y^*, x\rangle \quad \text{for all } x \in Z,\ y^* \in \mathcal{E}(\bar{\Omega}', T^*),\ f \in H^\infty(U).$$

Recall, that $H^\infty(U)$ is a dual space, $H^\infty(U) = Q(U)^*$, where $Q(U) = L^1(U, \mu)/{}^\perp H^\infty(U)$ (with respect to the planar Lebesgue measure μ). Let now also U' be a bounded open set with $U \subset U'$ and $\bar{U}' \cap F = \emptyset$. Then the restriction mapping

$$r_U^{U'} : H^\infty(U') \to H^\infty(U), \quad r_U^{U'}(f) := f \mid U \quad \text{for } f \in H^\infty(U')$$

is obviously continuous with respect to the weak* topologies. Hence, the mapping

$$J_U^{U'} : Q(U) \to Q(U') \quad \text{with } J_U^{U'}(L) := L \cdot r_U^{U'} \text{ for } L \in Q(U) \tag{9}$$

is continuous. Observe that

$$\|J_U^{U'}(L)\|_{Q(U')} \leqslant \|L\|_{Q(U)} \quad \text{for all } L \in Q(U). \tag{10}$$

Direct computation shows that

$$(r_U^{U'})^*(x \square_U y^*) = x \square_{U'} y^* \quad \text{for all } x \in Z,\ y^* \in \mathcal{E}(\dot{\Omega}, T^*). \tag{11}$$

Hence $\square_U$ is also consistent with respect to U.

Notice, that $r_U^{U'}$ and hence also $J_U^{U'}$ is compact if $\bar{U} \subset U'$. The following lemma furnishes some important slightly less elementary properties of the linear functional $x \square_U y^*$.

3.1. LEMMA. (a) *For any $x \in Z$, $y^* \in \mathcal{E}(\dot{\Omega}, T^*)$ the functional $x \square_U y^*$ is weak* continuous on $H^\infty(U)$ and hence $(x, y^*) \to x \square_U y^*$ is a continuous bilinear mapping from $Z \times \mathcal{E}(\dot{\Omega}, T^*)$ into $Q(U)$. Thus, in particular, in* (11), $(r_U^{U'})^*$ *can be replaced by $J_U^{U'}$.*

(b) *For any* $w^* \in \mathscr{E}(\bar{\Omega}, T^*)$ *and any sequence* $(x_n)_{n=1}^{\infty}$ *in* Z *converging weakly to* 0 *we have*

$$\lim_{n\to\infty} \|x_n \square_U w^*\|_{Q(U)} = 0.$$

(c) *For any* $z \in Z$ *and any sequence* $(y_n^*)_{n=1}^{\infty}$ *in* $\mathscr{E}(\bar{\Omega}, T^*)$ *converging weakly to* 0 *we have*

$$\lim_{n\to\infty} \|z \square_U y_n^*\|_{Q(U)} = 0.$$

Proof. Choose an open set V with $\bar{\Omega} \subset V \subset \bar{V} \subset U$. As mentioned earlier the mappings r_V^U and J_V^U are compact.

(a) If $f_n \to 0$ in the weak* topology of $H^\infty(U)$, then by the compactness of r_V^U, $\|r_V^U f_n\|_{H^\infty(V)} \to 0$. Hence,

$$\langle x \square_U y^*, f_n\rangle = \langle x \square_U y^*, r_V^U f_n\rangle \to 0.$$

Therefore, $x \square_U y^*$ is weak* sequentially continuous and thus weak* continuous on $H^\infty(U)$.

(b) The mapping $x \mapsto x \square_V w^*$ from Z to $Q(V)$ is continuous. Hence, the mapping $x \mapsto x \square_U w^* = J_V^U(x \square_V w^*)$ (cf. (a) and (11)) is compact and maps weak null sequences into norm null sequences.

(c) The statement follows as in (b) by the compactness of the mapping $y^* \mapsto z \square_U y^* = J_V^U(z \square_V y^*)$.

We will now use the fact that $Z = Z_0/Z_1$ is the quotient of two closed subspaces of ℓ^p. Notice, that Z^* can be identified with $Z_1^\perp/Z_0^\perp$ (and is thus a quotient of two closed subspaces of ℓ^q, where $1/p + 1/q = 1$) via

$$\langle z^*, z\rangle := \langle \xi^*, \xi\rangle \quad \text{for } \xi^* + Z_0^\perp = z^* \in Z_1^\perp/Z_0^\perp \text{ and } \xi + Z_1 = z \in Z_0/Z_1.$$

We write

$$\pi_p : \ell^p \to \ell^p/Z_1, \quad \pi_q : \ell^q \to \ell^q/Z_0^\perp$$

for the canonical epimorphisms. If $X \in \operatorname{Lat} Z$ then $X = X_0/Z_1$ with $X_0 := \pi_p^{-1}(X) \in \operatorname{Lat} \ell^p$.

To run the approximation process inherent to any variant of the S. Brown's technique we introduce some special sets in the predual $Q(U)$ of $H^\infty(U)$. Of course, as the reader will notice, they bear a strong analogy to the subset $\mathscr{X}_0(\mathscr{A})$ of the predual of a dual algebra $\mathscr{A}$ [2]. Whenever $X \in \operatorname{Lat} T$ and V is an open set such that $\bar{V} \subset U$, we define the subset $\mathscr{X}_X(V, U)$ of $Q(U)$ as the set of all those $L \in Q(U)$ such that there exist sequences $(x_n)_{n=1}^{\infty}$, $(y_n^*)_{n=1}^{\infty}$ in the unit ball of $X_0 := \pi_p^{-1}(X)$ and $\mathscr{E}_0(\bar{V}) := \pi_q^{-1}(\mathscr{E}(\bar{V}, T^*))$ respectively converging weakly to 0 and satisfying

$$\lim_{n\to\infty} \|L - \pi_p(x_n) \,\square_U\, \pi_q(y_n^*)\|_{Q(U)} = 0.$$

Obviously, the set $\mathscr{X}_X(V, U)$ is circled, i.e. for all $L \in \mathscr{X}_X(V, U)$ and $\lambda \in \mathbf{C}$ with $|\lambda| \leqslant 1$ we have $\lambda L \in \mathscr{X}_X(V, U)$. It turns out, that $\mathscr{X}_X(V, U)$ is even absolutely convex and closed. We admit and will use freely these facts for the time being. These statements will be proved (independently of the upcoming results of Sections 3 and 4) in Section 5, Proposition 5.4.

For $\lambda \in U$, we denote by E_λ the element of $Q(U)$ defined as point evaluation at λ, that is $\langle E_\lambda, f\rangle := f(\lambda)$ for all $f \in H^\infty(U)$. The following lemma establishes how spectral properties supply elements in $\mathscr{X}_X(V, U)$.

3.2. Lemma. *Let $X \in \operatorname{Lat} T$ be infinite dimensional and let Ω, V, U be bounded open sets in $\mathbf{C}$ with $\bar{\Omega} \subset V \subset \bar{V} \subset U$ and $\bar{U} \cap F = \emptyset$. Then there exists some $\delta = \delta(\Omega, V, T) > 0$ such that $\mathscr{X}_X(V, U) \supset \delta\{E_\lambda\,;\, \lambda \in \sigma_{le}(T\,|\,X) \cap \bar{\Omega}\}$.*

Proof. Fix an arbitrary $\lambda \in \sigma_{le}(T\,|\,X) \cap \bar{\Omega}$. By Proposition 2.1, there are sequences $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in $X^* = Z^*/X^\perp$ satisfying (a)–(d) in 2.1. Choose $y_n^* \in Z^*$ with $[y_n^*] = x_n^*$ in $X^* = Z^*/X^\perp$ such that $1 \leqslant \|y_n^*\| \leqslant 2$. Passing to a subsequence we may assume that $(y_n^*)_{n=1}^{\infty}$ is weakly converging to some $y^* \in Z^*$ (notice that Z is separable and reflexive). With $u_n^* := y_n^* - y^*$ we have $\|u_n^*\| \leqslant 4$ for all $n \in \mathbf{N}$ and $u_n^* \to 0$ weakly. Since $\lim_{n\to\infty}\langle [y_n^*], x_n\rangle = \eta \geqslant 1/2$ and $x_n \to 0$ weakly, by the choice of the sequences $(x_n)_{n=1}^{\infty}$ and $(x_n^*)_{n=1}^{\infty}$,

$$\lim_{n\to\infty}\langle u_n^*, x_n\rangle = \lim_{n\to\infty}\langle [y_n^*] - [y^*], x_n\rangle = \eta \geqslant \frac{1}{2}. \tag{12}$$

Let $W := \{\lambda \in V\,;\, \operatorname{dist}(\lambda, \Omega) \leqslant (1/2)\operatorname{dist}(\partial V, \partial\Omega)\} := (1/2)\inf\{|z - z'|\,;\, z \in \partial V,\ z' \in \partial\Omega\}$. Then $W = W(\Omega, V)$ is a compact neighborhood of Ω and $(\mathbf{C}\setminus W) \cup V = \mathbf{C}$. Because of the F-decomposability of T^* we obtain

$$Z^* = \mathscr{E}(\bar{V}, T^*) + \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*).$$

By the open mapping theorem there is a constant $C = C(\Omega, V, T) \geqslant 0$ only depending on T, V and W (which in turn depends only on V and Ω) such that for all $u^* \in Z^*$ there are $v^* \in \mathscr{E}(\bar{V}, T^*)$ and $w^* \in \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*)$ satisfying $u^* = v^* + w^*$ and $\|v^*\| \leqslant C\|u^*\|$, $\|w^*\| \leqslant C\|u^*\|$. In particular, we find $v_n^* \in \mathscr{E}(\bar{V}, T^*)$,

$w_n^* \in \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*)$ with

$$u_n^* = v_n^* + w_n^* \quad \text{and} \quad \|v_n^*\| \leqslant 4C, \quad \|w_n^*\| \leqslant 4C.$$

Passing again to a subsequence we may assume that v_n^* is weakly convergent to some $v^* \in \mathscr{E}(\overline{V}, T^*)$. Since $(u_n^*)_{n=1}^{\infty}$ is a weak null sequence, this also implies $w_n^* \to -v^*$ weakly and hence $v^* \in \mathscr{E}(\overline{V}, T^*) \cap \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*)$. With $a_n^* := v_n^* - v^*$ and $b_n^* := := w_n^* + v^*$ we then have $a_n^* \in \mathscr{E}(\overline{V}, T^*)$, $b_n^* \in \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*)$, $a_n^* \to 0$, $b_n^* \to 0$ weakly and $\|a_n^*\| \leqslant 8C$, $\|b_n^*\| \leqslant 8C$. Notice, that because of (7), $\lambda \notin \sigma(T^* \mid \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*))$. Using the fact that $\|(\lambda - T)x_n\| \to 0$ by the choice of $(x_n)_{n=1}^{\infty}$, we obtain therefore

$$\langle b_n^*, x_n\rangle = \langle(\lambda - T^*)(\lambda - T^* \mid \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*))^{-1} b_n^*, x_n\rangle =$$

$$= \langle(\lambda - T^* \mid \mathscr{E}(\overline{\mathbf{C}\setminus W}, T^*))^{-1} b_n^*, (\lambda - T)x_n\rangle \to 0 \quad \text{for } n \to \infty.$$

Hence, by (12),

$$\lim_{n\to\infty} \langle a_n^*, x_n\rangle = \lim_{n\to\infty} \langle u_n^*, x_n\rangle = \eta. \tag{13}$$

Choose now $c_n^* \in \pi_q^{-1}(a_n^*)$ with $\|c_n^*\|_q \leqslant 9C$. Passing again to some subsequence we may assume that $(c_n^*)_{n=1}^{\infty}$ is weakly convergent to some $c^* \in \pi^{-1}(\mathscr{E}(\overline{V}, T^*)) = = \mathscr{E}_0(\overline{V})$. If $z \in Z_0$ then

$$0 = \lim_{n\to\infty} \langle c_n^* - c^*, z\rangle = \lim_{n\to\infty} \langle a_n^*, \pi_p(z)\rangle - \langle c^*, z\rangle = -\langle c^*, z\rangle.$$

Hence, $c^* \in Z_0^{\perp}$ and thus $\pi_q(c_n^* - c^*) = \pi_q(c_n^*) = a_n^*$. With $d_n^* := (18C)^{-1}(c_n^* - c^*)$ we now have $\|d_n^*\|_q \leqslant 1$ and $d_n^* \to 0$ weakly. Choose also $e_n \in \pi_p^{-1}(x_n)$ with $\|e_n\|_p \leqslant 2$. Again passing to subsequences we may assume that $e_n \to e$ weakly for some $e \in \pi_p^{-1}(X) = X_0$. For all $z^* \in Z_1^{\perp}$ we then have

$$0 = \lim_{n\to\infty} \langle z^*, e - e_n\rangle = \langle z^*, e\rangle - \lim_{n\to\infty} \langle \pi_q(z^*), x_n\rangle = \langle z^*, e\rangle.$$

Hence $e \in (Z_1^{\perp})^{\perp} = Z_1$ and therefore $\pi_p(e_n - e) = \pi_p(e_n) = x_n$ for all $n \in \mathbf{N}$. With $f_n := (1/4)(e_n - e)$ we now have $\|f_n\|_p \leqslant 1$ and $f_n \to 0$ weakly for $n \to \infty$. Moreover, by (13),

$$\begin{aligned} \lim_{n\to\infty} \langle d_n^*, f_n\rangle &= (72C)^{-1} \lim_{n\to\infty} \langle c_n^* - c^*, e_n - e\rangle = \\ &= (72C)^{-1} \lim_{n\to\infty} \langle a_n^*, x_n\rangle = \frac{\eta}{72C}. \end{aligned} \tag{14}$$

Let now h be an arbitrary function in $H^\infty(U)$; h can be written in the form

$$h(\xi) = h(\lambda) + (\xi - \lambda)g(\xi), \quad \xi \in U$$

with $g \in H^\infty(U)$ and $\|g\| \leqslant 2\|f\|/\mathrm{dist}(\lambda, \partial U)$. Therefore,

$$h(T_V^*) = h(\lambda) + (T_V^* - \lambda)g(T_V^*)$$

and we have $\|g(T_V^*)\| \leqslant K\|h\|$ for some constant $K = K(U, V, T)$ independent of h. It follows that

$$|\langle \pi_p(f_n) \Box_U \pi_q(d_n^*) - \langle d_n^*, f_n\rangle E_\lambda, h\rangle| =$$

$$= |\langle h(T_V^*)\pi_q(d_n^*), \pi_p(f_n)\rangle - \langle \pi_q(d_n^*), \pi_p(f_n)\rangle h(\lambda)| =$$

$$= |\langle (h(T_V^*) - h(\lambda))\pi_q(d_n^*), \pi_p(f_n)\rangle| = |\langle (T^* - \lambda)g(T_V^*)\pi_q(d_n^*), \pi_p(f_n)\rangle| =$$

$$= |\langle g(T_V^*)\pi_q(d_n^*), \frac{1}{4}(T - \lambda)x_n\rangle| \leqslant \frac{K}{4}\|h\|\cdot\|(T - \lambda)x_n\|$$

and hence that

$$\|\pi_p(f_n) \Box_U \pi_q(d_n^*) - \langle d_n^*, f_n\rangle E_\lambda\|_{Q(U)} \leqslant \frac{K}{4}\|(T - \lambda)x_n\| \to 0.$$

Because of (14) this implies

$$\lim_{n\to\infty} \left\| \pi_p(f_n) \Box_U \pi_q(d_n^*) - \frac{\eta}{72C} E_\lambda \right\|_{Q(U)} = 0.$$

Thus $(\eta/72C)E_\lambda \in \mathscr{X}_X(V, U)$. As $\mathscr{X}_X(V, U)$ is circled and $\eta \geqslant 1/2$ we obtain $\delta E_\lambda \in \mathscr{X}_X(V, U)$ where $\delta = \delta(\Omega, V, T) := (144C)^{-1}$ only depends on Ω, V and T.

4. PROOF OF THEOREM 1.1

4.1. FIRST REDUCTION. In Theorem 1.1, it suffices to consider the case that S has property (β) on $\mathbf{C}\setminus F$.

Proof. Let $S \in \mathscr{L}(X)$, X as in Theorem 1.1, be such that S^* has property (β) on $\mathbf{C}\setminus F$. Since X is topologically isomorphic to X_0/X_1 $(X_0, X_1 \in \mathrm{Lat}\,\ell^p)$, X^* is topologically isomorphic to $X_1^\perp/X_0^\perp$ $(X_1^\perp, X_0^\perp \in \mathrm{Lat}\,\ell^q)$. Hence we are back in the above situation with S replaced by S^*, X by X^*, and ℓ^p by ℓ^q. Of course, if $Y \in \mathrm{Lat}\,S^*$ is nontrivial then also $Y^\perp \subset X^{**} = X$ is a nontrivial invariant subspace for $S = S^{**}$.

4.2. SECOND REDUCTION. To prove Theorem 1.1 it is sufficient to prove that Lat S is nontrivial whenever we are in the following situation

$$(15)\qquad \begin{cases} S = T\,|\,X \text{ where } T \in \mathscr{L}(Z) \text{ is } F\text{-decomposable, } F \subset \mathbf{C} \text{ closed,} \\ Z = Z_0/Z_1 \text{ with } Z_0,\ Z_1 \in \operatorname{Lat} \ell^p,\ X \in \operatorname{Lat} T \text{ and there exists} \\ \text{a sequence } (\Omega_n)_{n=1}^{\infty} \text{ of open sets satisfying (1).} \end{cases}$$

Proof. By 4.1 we may assume that $S \in \mathscr{L}(X)$ has property (β) on $\mathbf{C}\setminus F$ for some closed $F \subset \sigma(S)$ and that there exists a sequence $(\Omega_n)_{n=1}^{\infty}$ of open sets in $\mathbf{C}$ satisfying (1). As mentioned in the introduction this implies the existence of a quotient $Z = Z_0/Z_1$ of two closed subspaces $Z_1 \subset Z_0$ of ℓ^p and of a $(F \cup \partial R)$-decomposable operator $T \in \mathscr{L}(Z)$, where R is an open rectangle with $\sigma(S) \subset R$, such that S is similar to $T\,|\,Y$ for some $Y \in \operatorname{Lat} T$. Of course, Lat S is nontrivial if and only if $\operatorname{Lat} T\,|\,Y$ is nontrivial. Notice also that (by the fact that $\Omega_\infty \cap \sigma(S)$ is dominating for Ω_∞) we have $\overline{\Omega_\infty} \subset R$ and hence $\overline{\Omega_\infty} \cap (F \cup \partial R) = \emptyset$. Hence, replacing F by $F \cup \partial R$, S by $T\,|\,Y$, and X by Y we are in situation (15).

4.3. THIRD REDUCTION. We may assume that $\sigma(S) = \sigma_{\mathrm{le}}(S)$.

Proof. Indeed, for all $\lambda \in \sigma(S)\setminus\sigma_{\mathrm{le}}(S)$ the space $\ker(\lambda - S)$ or $\overline{\operatorname{ran}(\lambda - S)}$ will be a nontrivial hyperinvariant subspace for S. Note that this reduction is valid when looking for invariant subspaces of any algebra contained in the commutant of S.

Hence, in the sequel, we will assume that we are in situation (15) and that $\sigma(S) = \sigma_{\mathrm{le}}(S)$. Notice that this already implies that X is infinite dimensional. Fix now a sequence $(\Omega_n)_{n=1}^{\infty}$ satisfying (1). Let $(V_n)_{n=1}^{\infty}$ be another sequence of open sets in $\mathbf{C}$ such that $\overline{\Omega_n} \subset V_n \subset \overline{V_n} \subset \Omega_{n+1}$ for all $n \in \mathbf{N}$. To simplify notations we set $\delta_n := \delta(\Omega_n, V_n, T)$ for the constants obtained from Lemma 3.2, $Q_n := Q(\Omega_n)$, $\|\cdot\|_n := \|\cdot\|_{Q_n}$, and $\square_n$ for $\square_{\Omega_n}$. We also omit the index on $\square_n$ when there is no ambiguity on the space. Also, for $n < m$ we write J_n^m for the canonical mappings $J_n^m : Q_n \to Q_m$ given by (9). Finally, for $n \leqslant m$, let Λ_n^m denote the absolutely convex hull of $\{E_\lambda \,|\, \lambda \in \sigma_{\mathrm{le}}(S) \cap \Omega_n\}$ considered as a subset of Q_m. Hence, $J_n^m(\Lambda_n^n) = \Lambda_n^m$ for $n < m$. The dominancy of $\sigma_{\mathrm{le}}(S) \cap \Omega_n$ in Ω_n implies that Λ_n^n is dense in the unit ball of Q_n (cf. [5], Proposition 2.2). Hence, by Lemma 3.2 and the absolute convexity of $\mathscr{X}_X(V_n, \Omega_{n+1})$ we have

$$(16)\qquad \delta_n \Lambda_n^{n+1} \subset \mathscr{X}_X(V_n, \Omega_{n+1}).$$

We state now our goal which is to prove that $H^\infty(\Omega_n)$ has some sort of $(\mathbf{A}_1)$ property (in the framework of [2]) with respect to X and $\mathscr{E}(\overline{\Omega_\infty}, T^*)$.

4.4. PROPOSITION. *With the above notations let* $\rho_n > \delta_n^{-1/2}$. *Then, for any* $L \in Q_n$, *there exist* $x \in X$, $y^* \in \mathcal{E}(\overline{\Omega_\infty}, T^*)$ *with* $\|x\| \leqslant \rho_n\sqrt{\|L\|_n}$, $\|y^*\| \leqslant \rho_n\sqrt{\|L\|_n}$ *such that for all bounded open sets* U *with* $\overline{\Omega_\infty} \subset U$ *and* $\overline{U} \cap F = \emptyset$ *we have* $J^U_{\Omega_n}(L) = x \,\square_U\, y^*$.

Before proving Proposition 4.4, we first show how it can be used to finish the proof of Theorem 1.1. In fact, we will prove a little more.

4.5. THEOREM. *Suppose that* S *and* T *are given as in* (15). *Let* $\mathcal{W}$ *denote the weak operator topology closure of the algebra*

$$\{r(S)\ ;\ r \text{ rational with poles off } \sigma(S) \cup \sigma(T) \cup \overline{\Omega_\infty}\}.$$

Then Lat $\mathcal{W}$ *is nontrivial.*

Proof. By 4.3 we may assume that $\sigma(S) = \sigma_{\mathrm{le}}(S)$. Because of (1), $\sigma(T)$ is an infinite set. Hence, $\ker(\lambda - S) \neq X$ for all $\lambda \in \sigma(S)$. If $\ker(\lambda - S) \neq \{0\}$ for some $\lambda \in \sigma(S)$ then $\ker(\lambda - S)$ is a nontrivial hyperinvariant subspace for S and hence invariant for $\mathcal{W}$. Fix now an arbitrary $\lambda \in \Omega_\infty$ with $\ker(\lambda - S) = \{0\}$. Then $\lambda \in \Omega_n$ for n sufficiently large. By Proposition 4.4 there are $x \in X$ and $y^* \in \mathcal{E}(\overline{\Omega_\infty}, T^*)$ such that $x \,\square_U\, y^* = E_\lambda$ for all bounded open sets U with $\overline{\Omega_\infty} \subset U$ and $\overline{U} \cap F = \emptyset$. In particular, $1 = \langle x \,\square_U\, y^*, 1\rangle = \langle y^*, x\rangle$ and hence $x \neq 0$. We consider now $\mathcal{M}_\lambda := \bigvee\{r(S)x\ ;\ r \text{ rational with poles off } \sigma(S) \cup \sigma(T) \cup \overline{\Omega_\infty} \text{ and } r(\lambda) = 0\}$. Clearly, $\mathcal{M}_\lambda \in \mathrm{Lat}\,\mathcal{W}$. Since $(\lambda - S)x \neq 0$, we have $\mathcal{M}_\lambda \neq \{0\}$. To prove that $\mathcal{M}_\lambda \neq X$, consider an arbitrary rational function r with $r(\lambda) = 0$ and with poles off $\sigma(S) \cup \sigma(T) \cup \overline{\Omega_\infty}$. Fix a bounded open set U with $\overline{\Omega_\infty} \subset U$ and $\overline{U} \cap F = \emptyset$ such that $r \mid U \in H^\infty(U)$. From the equalities $r(S) = r(T) \mid X$ and $r(T)^* \mid \mathcal{E}(\overline{\Omega_\infty}, T^*) = r(T^* \mid \mathcal{E}(\overline{\Omega_\infty}, T^*))$ we obtain

$$0 = r(\lambda) = \langle x \,\square_U\, y^*, r\rangle = \langle r(T^*_{\overline{\Omega_\infty}})y^*, x\rangle = \langle y^*, r(S)x\rangle.$$

It follows that $y^* \in \mathcal{M}_\lambda^\perp$. As $\langle y^*, x\rangle = 1 \neq 0$ we see that $x \notin \mathcal{M}_\lambda$. Thus $\mathcal{M}_\lambda$ is a nontrivial invariant subspace for $\mathcal{W}$.

REMARK. In special cases it may happen that $\sigma(T) \subset \sigma(S)$. For instance if T is strongly decomposable [10] then $X \subset \mathcal{E}(\sigma(S), T)$ and $T_0 := T \mid \mathcal{E}(\sigma(S), T)$ is decomposable with $\sigma(T_0) \subset \sigma(S)$. So we may replace T by T_0. In particular (cf. [8]) this applies to hyponormal operators S on a Hilbert space.

Because of 4.2, Theorem 1.1 is now an immediate consequence of Theorem 4.5.

Our basic tool to prove Proposition 4.4 will be the following approximation lemma.

4.6. LEMMA. *Let* $0 \neq K \in Q_n$, $x \in X$, $y^* \in \mathscr{E}(\overline{\Omega_n}, T^*)$, *and* $\varepsilon > 0$. *Then there exist* $u \in X$, $v^* \in \mathscr{E}(\overline{V_n}, T^*)$ *such that*

$$\|J_n^{n+1}(K) - u \,\square\, v^*\|_{n+1} < \varepsilon\,; \quad \|u\|, \|v^*\| < \sqrt{\frac{\|K\|_n}{\delta_n}}$$

and

$$\|(x + u) \,\square\, (y^* + v^*) - x \,\square\, y^* - u \,\square\, v^*\|_{n+1} < \varepsilon.$$

Proof. Set $d := \|K\|_n$ and let $\varphi \in A_n^n$ satisfy $\|K - d\varphi\|_n < \varepsilon/2$. (Recall that A_n^n is dense in the unit ball of Q_n.) Since, by (16),

$$\delta_n J_n^{n+1}(\varphi) \in \delta_n A_n^{n+1} \subset \mathscr{X}_X(V_n, \Omega_{n+1})$$

we can find vectors $a \in X_0 = \pi_p^{-1}(X)$, $b^* \in \mathscr{E}_0(V_n)$ such that $\|a\|_p < 1$, $\|b^*\|_q < 1$,

$$\|\delta_n J_n^{n+1}(\varphi) - \pi_p(a) \,\square\, \pi_q(b^*)\|_{n+1} < \frac{\delta_n \varepsilon}{2d}$$

and (using Lemma 3.1 and the weak continuity of the mappings π_p and π_q),

$$\|x \,\square\, \pi_q(b^*)\|_{n+1} < \frac{\varepsilon}{2}\sqrt{\frac{\delta_n}{d}}, \quad \|\pi_p(a) \,\square\, y^*\|_{n+1} < \frac{\varepsilon}{2}\sqrt{\frac{\delta_n}{d}}.$$

Straightforward computations show that with $u := \sqrt{d/\delta_n}\,\pi_p(a)$, $v^* := \sqrt{d/\delta_n}\,\pi_q(b^*)$ the desired inequalities are satisfied.

By means of Lemma 4.6 we can now construct Cauchy sequences, whose limits will be the vectors x and y^* to be found in Proposition 4.4.

4.7. LEMMA. *With the above notations let* $\rho > \delta_n^{-1/2}$. *Then, for any* $0 \neq L \in Q_n$, *there exist Cauchy sequences* $(x_m)_{m=1}^\infty$ *in* X *and* $(y_m^*)_{m=1}^\infty$ *in* $\mathscr{E}(\overline{\Omega_\infty}, T^*)$, $y_m^* \in \mathscr{E}(\overline{V_{n+m-1}}, T^*)$ $(m \in \mathbf{N})$ *such that*

$$\|J_n^{n+m}(L) - x_m \,\square_{n+m}\, y_m^*\|_{n+m} \to 0 \quad \textit{for } n \to \infty \tag{17}$$

and

$$\|x_m\|, \|y_m^*\| < \rho\sqrt{\|L\|_n} \quad \textit{for all } m \in \mathbf{N}.$$

Proof. Fix $\varepsilon > 0$ such that $(1 + \varepsilon)\delta_n^{-1/2} < \rho$ and choose $\varepsilon_m > 0$, $\varepsilon_m \searrow 0$, with

$$\sum_{m=1}^{\infty} \sqrt{\frac{\varepsilon_m}{\delta_{n+m}}} < \varepsilon\sqrt{\frac{\varepsilon_0}{\delta_n}} \quad \text{where } \varepsilon_0 := \|L\|_n. \tag{18}$$

By Lemma 4.6 (with $K = L$, $x_0 := 0$, $y_0^* := 0$) there are $x_1 \in X$, $y_1^* \in \mathcal{E}(\dot{V}_n, T^*)$ with

$$\|J_n^{n+1}(L) - x_1 \square y_1^*\|_{n+1} < \varepsilon_1,$$

$$\|x_1\|, \|y_1^*\| < \sqrt{\frac{\varepsilon_0}{\delta_n}}.$$

Suppose, that $x_0, x_1, \ldots, x_m \in X$ and $y_i^* \in \mathcal{E}(\overline{V_{n+i-1}}, T^*)$ $(0 < 1 \leqslant m)$ have already been found with

$$\|J_n^{n+i}(L) - x_i \square y_i^*\|_{n+i} < \varepsilon_i$$

(19)

$$\|x_i - x_{i-1}\|, \|y_i^* - y_{i-1}^*\| < \sqrt{\frac{\varepsilon_{i-1}}{\delta_{n+i-1}}} \quad \text{for } 0 < i \leqslant m.$$

Applying again Lemma 4.6 (now with $n+m$ instead of n, $K := J_n^{n+m}(L) - x_m \square_{n+m} y_m^*$, $x = x_m$, $y^* = y_m^*$, $\varepsilon = \varepsilon_{m+1}/2$) we find $u \in X$ and $v^* \in \mathcal{E}(\dot{V}_{n+m}, T^*)$ with

$$\|J_n^{n+m+1}(L) - J_{n+m}^{n+m+1}(x_m \square_{n+m} y_m^*) - u \square_{n+m+1} v^*\|_{n+m+1} < \frac{\varepsilon_{m+1}}{2},$$

$$\|u\|, \|v^*\| < \sqrt{\frac{\|K\|_{n+m}}{\delta_{n+m}}} < \sqrt{\frac{\varepsilon_m}{\delta_{n+m}}},$$

and

$$\|(x_m + u) \square (y_m^* + v^*) - x_m \square y_m^* - u \square v^*\|_{n+m+1} < \frac{\varepsilon_{m+1}}{2}.$$

With $x_{m+1} := x_m + u$, $y_{m+1}^* := y_m^* + v^*$ we then obtain (recall from (11) that $J_{n+m}^{n+m+1}(x_m \square_{n+m} y_m^*) = x_m \square_{n+m+1} y_m^*$)

$$\|J_{n+m}^{n+m+1}(L) - x_{m+1} \square y_{m+1}^*\|_{n+m+1} \leqslant$$

$$\leqslant \|J_{n+m}^{n+m+1}(L) - x_m \square y_m^* - u \square v^*\|_{n+m+1} +$$

$$+ \|x_{m+1} \square y_{m+1}^* - x_m \square y_m^* - u \square v^*\|_{n+m+1} <$$

$$< \varepsilon_{m+1}/2 + \varepsilon_{m+1}/2 = \varepsilon_{m+1}.$$

Therefore, (19) is now also fulfilled for $i = m + 1$. In this way we construct two sequences $(x_m)_{m=0}^{\infty}$ in X, $(y_m^*)_{m=0}^{\infty}$ in $\mathcal{E}(\overline{\Omega^\infty}, T^*)$ with $y_m^* \in \mathcal{E}(\overline{V_{n+m-1}}, T^*)$ for $m \geqslant 1$ such that (19) holds for all $i \in \mathbf{N}$. Because of (19) and (18) these sequences must be Cauchy sequences. Moreover, for all $m \in \mathbf{N}$, we have (because of $\varepsilon_0 = \|L\|_m$)

$$\|x_m\| \leqslant \sum_{i=1}^{m} \|x_i - x_{i-1}\| \leqslant \sum_{i=1}^{m} \sqrt{\frac{\varepsilon_{i-1}}{\delta_{n+i-1}}} < \sqrt{\frac{\varepsilon_0}{\delta_n}} + \varepsilon \sqrt{\frac{\varepsilon_0}{\delta_n}} <$$

$$< (1 + \varepsilon) \sqrt{\frac{\|L\|_n}{\delta_n}} < \rho \sqrt{\|L\|_n}.$$

In the same way we obtain $\|y_m\| < \rho \sqrt{\|L\|_n}$ and the proof is complete.

We can now easily conclude the

Proof of Proposition 4.4. We may assume that $L \neq 0$. Let $(x_m)_{m=1}^{\infty}$ and $(y_m^*)_{m=1}^{\infty}$ be the sequences provided by Lemma 4.7 and define $x := \lim_{m\to\infty} x_m \in X$, $y^* := \lim_{m\to\infty} y_m^* \in \mathscr{E}(\Omega_\infty, T^*)$. Hence, $\|x\|, \|y^*\| \leqslant \rho_n \sqrt{\|L\|_n}$. Let now U be an arbitrary bounded open set in $\mathbf{C}$ with $\overline{\Omega_\infty} \subset U$ and $\overline{U} \cap F = \emptyset$. By means of (17), (10) and (11) we obtain

$$\|J_{\Omega_n}^U(L) - x_m \square_U y_m^*\|_{Q(U)} =$$

$$= \|J_{\Omega_{n+m}}^U(J_n^{n+m}(L) - x_m \square_{n+m} y_m^*)\|_{Q(U)} \leqslant$$

$$\leqslant \|J_n^{n+m}(L) - x_m \square_{n+m} y_m^*\|_{n+m} \to 0$$

and by (a) of Lemma 3.1 this implies

$$J_{\Omega_n}^U(L) = \lim_{m\to\infty} x_m \square_U y_m^* = x \square_U y^*.$$

For the proof of the fact that the operator S in Theorem 1.1 has even non-trivial rationally invariant subspaces if $\operatorname{int}(\sigma(S)\setminus F) \neq \emptyset$, we need some further facts from local spectral theory. The following lemma is a variant of Lemma IV.4.24 in [10].

4.8. LEMMA. *Let Z be an arbitrary Banach space and let $T \in \mathscr{L}(Z)$ be an F-decomposable operator. Let $A \in \mathscr{F}_F$ and suppose that G_0, G_1 are open sets in $\mathbf{C}$ such that $A \subset G_0 \cup G_1$, $\overline{G_1} \cap F = \emptyset$ and with $F \subset G_0$ or $\overline{G_0} \cap F = \emptyset$. Then*

$$\mathscr{E}(A, T) \subset \mathscr{E}(\overline{G_0}, T) + \mathscr{E}(\overline{G_1}, T).$$

Proof. Write $\hat{T}$ for the operator induced by T on $Y := Z/\mathscr{E}(\overline{G_0 \cap G_1}, T)$ and denote by $\pi : Z \to Y$ the canonical epimorphism. Because of $\sigma(\hat{T}) \subset \sigma(T)\setminus(G_0 \cap G_1)$

(cf. [10], Lemma IV. 4.23) the function f given by

$$f(\lambda) := \begin{cases} \pi((\lambda - T \,|\, \mathscr{E}(A, T))^{-1}x) & \text{for } \lambda \in \mathbf{C}\setminus A \\ (\lambda - \hat{T})^{-1}\pi(x) & \text{for } \lambda \in G_0 \cap G_1 \end{cases}$$

is well defined and analytic on $\Omega := (\mathbf{C}\setminus A) \cup (G_0 \cap G_1)$ with $(\lambda - \hat{T})f(\lambda) \equiv \pi(x)$ on Ω. Let Γ_j be admissible contours surrounding $A_j := A\setminus G_{1-j}$ in G_j $(j = 0, 1)$ and define

$$y_j := \frac{1}{2\pi i}\int_{\Gamma_j} f(\lambda)\,d\lambda.$$

Let B_j be the closure of the open set containing A_j and having Γ_j as boundary. Then the function g_j with

$$g_j(z) := \frac{1}{2\pi i}\int_{\Gamma_j}(z - \lambda)^{-1}f(\lambda)\,d\lambda \quad \text{for } z \in \mathbf{C}\setminus B_j$$

is in $\mathscr{O}(\mathbf{C}\setminus B_j, Y)$ and

$$(z - \hat{T})g_j(z) = \frac{1}{2\pi i}\int_{\Gamma_i}(z - \lambda)^{-1}(z - \lambda + (\lambda - \hat{T}))f(\lambda)d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_j} f(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_j}\pi(x)\,d\lambda = y_j$$

for all $z \in \mathbf{C}\setminus B_j$, $j = 0, 1$. Choose now $u_j \in Z$ and $h_j \in \mathscr{O}(\mathbf{C}\setminus B_j, Z)$ such that $\pi(u_j) = y_j$ and $\pi(h_j(\lambda)) \equiv g_j(\lambda)$ on $\mathbf{C}\setminus B_j$, $j = 0, 1$. For the existence of holomorphic liftings see [10], Corollary II.10.9. Then we have

$$(\lambda - T)h_j(\lambda) - u_j = v_j(\lambda) \in \mathscr{E}(\overline{G_0 \cap G_1}, T)$$

and therefore

$$u_j \equiv (\lambda - T)[h_j(\lambda) - (\lambda - T \,|\, \mathscr{E}(\overline{G_0 \cap G_1}, T))^{-1}v_j(\lambda)]$$

on $\mathbf{C}\setminus \overline{G_j}$. Thus, $u_j \in \mathscr{E}(\overline{G_j}, T)$, $j = 0, 1$. With $x_0 := u_0$, $x_1 := u_1 + (x - u_0 - u_1)$ we obtain $x_0 \in \mathscr{E}(\overline{G_0}, T)$, $x_1 \in \mathscr{E}(\overline{G_1}, T) + \mathscr{E}(\overline{G_0 \cap G_1}, T) = \mathscr{E}(\overline{G_1}, T)$ and $x = x_0 + x_1$.

4.9. LEMMA. *Let Z be a Banach space, $T \in \mathcal{L}(Z)$ F-decomposable and $B_1, B_2 \subset \mathbf{C}$ be closed sets with $B_1 \cap B_2 = \emptyset$ and $B := B_1 \cup B_2 \supset F$. Then*

$$\mathcal{E}(B, T) = \mathcal{E}(B_1, T) \oplus \mathcal{E}(B_2, T)$$

where, for $j = 1, 2$,

$$\mathcal{E}(B_j, T) = Z \cap \operatorname{ran} \alpha_T^{\mathbf{C} \setminus B_j} \in \operatorname{Lat} T$$

with

$$\sigma(T \mid \mathcal{E}(B_j, T)) \subset B_j \cap \sigma(T).$$

Proof. We have from (7), $\sigma(T \mid \mathcal{E}(B, T)) \subset (B_1 \cup B_2) \cap \sigma(T)$. By means of the analytic functional calculus we find $Y_1, Y_2 \in \operatorname{Lat} T$ with $\mathcal{E}(B, T) = Y_1 \oplus Y_2$ and $\sigma(T \mid Y_j) \subset B_j \cap \sigma(T)$. Hence $Y_j \subset \mathcal{E}(B_j, T)$. Conversely, if $x \in \mathcal{E}(B_j, T)$ then $x = y_1 + y_2$ with $y_k \in Y_k$ $(k = 1, 2)$. It follows that $y_{3-j} = x - y_j \in \mathcal{E}(B_j, T)$. Hence there is some $f \in \mathcal{O}(\mathbf{C} \setminus B_j, Z)$ with $(\lambda - T)f(\lambda) \equiv y_{3-j}$ on $\mathbf{C} \setminus B_j$. Consider now

$$h(\lambda) := \begin{cases} f(\lambda) & \text{for } \lambda \in \mathbf{C} \setminus B_j. \\ (\lambda - T \mid Y_{3-j})^{-1} y_{3-j} & \text{for } \lambda \in \mathbf{C} \setminus B_{3-j}. \end{cases}$$

Since T has the SVEP on $\mathbf{C} \setminus F \supset \mathbf{C} \setminus B$, the function h is a well defined entire function vanishing at ∞ and hence identical zero. It follows that $y_{3-j} = 0$ and thus $x = y_j \in Y_j$. This proves $Y_j = \mathcal{E}(B_j, T)$.

4.10. LEMMA. *Let Z be a Banach space, $T \in \mathcal{L}(Z)$. If $K \subset \mathbf{C}$ is closed and $x \in \mathcal{E}(K, T) := Z \cap \operatorname{ran} \alpha_T^{\mathbf{C} \setminus K}$,* i.e. *$x \equiv (\lambda - T)f(\lambda)$ on $\mathbf{C} \setminus K$ for some $f \in \mathcal{O}(\mathbf{C} \setminus K, Z)$ then $f(\lambda) \in \mathcal{E}(K, T)$ for all $\lambda \in \mathbf{C} \setminus K$.*

Proof. Indeed, the function g with

$$g(z) := \begin{cases} (\lambda - z)^{-1}(f(z) - f(\lambda)) & \text{for } \lambda \neq z \in \mathbf{C} \setminus K \\ -f'(\lambda) & \text{for } \lambda = z \end{cases}$$

is in $\mathcal{O}(\mathbf{C} \setminus K, Z)$ and satisfies $(z - T)g(z) \equiv f(\lambda)$ on $\mathbf{C} \setminus K$.

4.11. LEMMA. *Let $F_1 = F \cup F_2$ where $F, F_2 \subset \mathbf{C}$ are closed sets with $F \cap F_2 = \emptyset$. Let Z be a Banach space and $T \in \mathcal{L}(Z)$ be F_1-decomposable. Let A be a closed set with $F \subset A \subset \mathbf{C} \setminus F_2$.*

(a) *$\mathcal{E}(A, T) \in \operatorname{Lat} T$ and for $T_0 := T \mid \mathcal{E}(A, T)$ we have $\sigma(T_0) \subset A \cap \sigma(T)$.*

(b) *If also K is closed with $F \subset K \subset A$ then $\mathcal{E}(K, T) \in \operatorname{Lat} T$ by* (a). *Moreover*

$$\sigma(T \mid \mathcal{E}(K, T)) \subset \sigma(T_0) \cap K.$$

(c) *If also* $B \subset \mathbf{C}$ *is closed with* $F \subset B$ *and* $B \cap F_2 = \emptyset$ *then*

$$\mathcal{E}(A, T) \cap \mathcal{E}(B, T) = \mathcal{E}(A \cap B, T).$$

(d) T_0 *is* F_0*-decomposable with* $F_0 := (F \cup \partial A) \cap \sigma(T_0)$.

Proof. (a) follows from 4.9 with $B_1 = A$, $B_2 = F_2$.

(b) Fix an arbitrary $x \in \mathcal{E}(K, T) \subset \mathcal{E}(A, T)$. Then $x \equiv (z - T)f(z)$ for some $f \in \mathcal{O}(\mathbf{C} \setminus K, Z)$. The Z-valued function h with

$$h(\lambda) := \begin{cases} f(\lambda) & \text{for } \lambda \in \mathbf{C} \setminus K \\ (\lambda - T_0)^{-1}x & \text{for } \lambda \in \mathbf{C} \setminus \sigma(T_0) =: \rho(T_0) \end{cases}$$

is well defined and analytic on $\mathbf{C} \setminus (K \cap \sigma(T_0))$. By 4.10 we have $h(\lambda) \in \mathcal{E}(K \cap \sigma(T_0), T) \subset \mathcal{E}(K, T)$ for all $\lambda \in \rho(T_0)$. Hence, $\mathcal{E}(K, T) \in \mathrm{Lat}(\lambda - T_0)^{-1}$ for all $\lambda \in \rho(T_0)$ and thus $\sigma(T \mid \mathcal{E}(K, T)) \subset \sigma(T_0)$. Since $\sigma(T \mid \mathcal{E}(K, T)) \subset K$ by (a), we obtain (b).

(c) Fix an arbitrary $x \in \mathcal{E}(A, T) \cap \mathcal{E}(B, T)$. Hence,

$$x \equiv (z - T)a(z) \text{ on } \mathbf{C} \setminus A \quad \text{with } a \in \mathcal{O}(\mathbf{C} \setminus A, Z)$$

$$x \equiv (\lambda - T)b(\lambda) \text{ on } \mathbf{C} \setminus B \quad \text{with } b \in \mathcal{O}(\mathbf{C} \setminus B, Z)$$

and thus

$$(z - T)(a(z) - b(z)) \equiv 0 \text{ on } (\mathbf{C} \setminus A) \cap (\mathbf{C} \setminus B).$$

As T has the SVEP on $\mathbf{C} \setminus F_1$ we obtain $a \equiv b$ on $[(\mathbf{C} \setminus A) \cap (\mathbf{C} \setminus B)] \setminus F_2$. By the identity theorem we thus conclude that $a \equiv b$ on $(\mathbf{C} \setminus A) \cap (\mathbf{C} \setminus B)$. Hence, the function

$$f(z) = \begin{cases} a(z) & \text{for } z \in \mathbf{C} \setminus A \\ b(z) & \text{for } z \in \mathbf{C} \setminus B \end{cases}$$

is well defined and analytic on $\mathbf{C} \setminus (A \cap B)$ and satisfies $(z - T)f(z) \equiv x$ on $\mathbf{C} \setminus (A \cap B)$. Thus $x \in \mathcal{E}(A \cap B, T) \subset \mathcal{E}(A, T) \cap \mathcal{E}(B, T)$ and we obtain $\mathcal{E}(A, T) \cap \mathcal{E}(B, T) = \mathcal{E}(A \cap B, T)$.

(d) Let $\delta := \mathrm{dist}(A, F_2) > 0$ and set $U := \{z \in \mathbf{C}\,;\ \mathrm{dist}(z, A) < \delta/3\}$, $W_2 := \{z \in \mathbf{C};\ \mathrm{dist}(z, F_2) < \delta/3\}$. Fix an arbitrary open covering U_0, U_1 of $\sigma(T_0)$ with $F_0 \subset U_0$ and $\overline{U_1} \cap F_0 = \emptyset$. With $U_0' := U_0 \cap U$ and $U_1' := (U_1 \setminus F) \cap \mathrm{int} A$ we still have $\sigma(T_0) \subset U_0' \cup U_1'$. Hence we can choose open sets V_0, $V_1 \subset \mathbf{C}$ with $\overline{V_j} \subset U_j'$ and $\sigma(T_0) \subset V_0 \cup V_1$. Let also W_1 be an open set satisfying $A \setminus (V_1 \cup V_2) \subset W_1 \subset U$ and $\sigma(T_0) \cap \overline{W_1} = \emptyset$. It follows that $A \subset G_0 \cup G_1$ where $F_1 \subset G_0 := V_0 \cup W_1 \cup W_2$ and $G_1 := V_1$ with $\overline{G_1} \cap F_1 = \emptyset$. Fix now an arbitrary $x \in \mathcal{E}(A, T)$.

By Lemma 4.8, $x = u_0 + u_1$ with $u_j \in \mathscr{E}(\bar{G}_j, T)$ $(j = 0, 1)$. Moreover, by Lemma 4.9, $u_0 = y_1 + y_2$ with $y_1 \in \mathscr{E}(\overline{V_0 \cup W_1}, T) \subset \mathscr{E}(\bar{U}, T)$ and $y_2 \in \mathscr{E}(\bar{W}_2, T)$. Because of $\mathscr{E}(\bar{G}_1, T) \subset \mathscr{E}(A, T) \subset \mathscr{E}(\bar{U}, T)$ and 4.8 we obtain

$$y_2 = x - u_1 - y_1 \in \mathscr{E}(\bar{U}, T) \cap \mathscr{E}(\bar{W}_2, T) = \{0\}.$$

It follows that $y_1 = x - u_1 \in \mathscr{E}(\overline{V_0 \cup W_1}, T) \cap \mathscr{E}(A, T) = \mathscr{E}(A \cap \overline{V_0 \cup W_1}, T)$ (by (c)). Hence, we have shown that $x = y_1 + u_1 \in \mathscr{E}(A \cap \overline{V_0 \cup W_1}, T) + \mathscr{E}(\bar{V}_1, T)$, i.e.

$$\mathscr{E}(A) = \mathscr{E}(A \cap \overline{V_0 \cup W_1}, T) + \mathscr{E}(\bar{V}_1, T).$$

Applying (b) with $K := A \cap \overline{V_0 \cup W_1}$ we see that $\mathscr{E}(A \cap \overline{V_0 \cup W_1}, T) \in \operatorname{Lat} T_0$ and (because of $W_1 \cap \sigma(T_0) = \emptyset$) $\sigma(T_0 \mid \mathscr{E}(A \cap \overline{V_0 \cup W_1}, T)) \subset A \cap \overline{V_0 \cup W_1} \cap \cap\, \sigma(T_0) \subset \bar{V}_0 \subset U_0$. Moreover, $\mathscr{E}(\bar{V}_1, T) \in \operatorname{Lat} T_0$ and $\sigma(T_0 \mid \mathscr{E}(\bar{V}_1, T)) \subset \bar{V}_1 \subset U_1$ by (7). Hence we have proved that T_0 is F_0-decomposable.

We are now ready to prove the announced criterium for the existence of non-trivial rationally invariant subspaces.

4.12. THEOREM. *Let X be a Banach space which is topologically isomorphic to a quotient of two closed subspaces of ℓ^p $(1 < p < \infty)$ and let $S \in \mathscr{L}(X)$. If S or S^* has property (β) on $\mathbf{C} \setminus F$ for a closed set $F \subset \sigma(S)$ such that $\operatorname{int}(\sigma(S) \setminus F) \neq \emptyset$ then $\mathscr{W}_S$, the* WOT-*closure of $\{r(S)$; r rational with poles off $\sigma(S)\}$ has a nontrivial invariant subspace.*

Proof. As in the first reduction it suffices to consider the case that S has (β) on $\mathbf{C} \setminus F$ with $\operatorname{int}(\sigma(S) \setminus F) \neq \emptyset$. As mentioned in the introduction, there exist a quotient $Z = Z_0/Z_1$ of closed subspaces $Z_1 \subset Z_0$ of ℓ^p and a $(F \cup \partial R)$-decomposable operator $T \in \mathscr{L}(Z)$, where R is an open rectangle containing $\sigma(S)$, such that S is similar to $S_0 := T \mid Y$ for some $Y \in \operatorname{Lat} T$. Of course, $Y \subset \mathscr{E}(\sigma(S), T)$. Consider now $T_0 := T \mid \mathscr{E}(\sigma(S), T)$. Let us remark that also $Z := \mathscr{E}(\sigma(S), T) = \pi_p^{-1}(\mathscr{E}(\sigma(S), T))/Z_1$ is a quotient of two closed subspaces of ℓ^p. By 4.11 (d), T_0 is F_0-decomposable with $F_0 := (F \cup \partial\sigma(S)) \cap \sigma(T_0)$. Notice, that $\operatorname{int}(\sigma(S) \setminus F_0) \neq \emptyset$ since $\operatorname{int}(\sigma(S) \setminus F) \neq \emptyset$. Let now $(\Omega_n)_{n=1}^{\infty}$ be a sequence of open discs with $\emptyset \neq \bar{\Omega}_n \subset \Omega_{n+1}$ for all n and $\Omega_\infty \subset \operatorname{int}(\sigma(S) \setminus F_0)$ where $\Omega_\infty := \bigcup_{n=1}^{\infty} \Omega_n$. Then $(\Omega_n)_{n=1}^{\infty}$ satisfies (1) and we are in the situation of (15) (with S_0 instead of S, T_0 for T and F_0 for F). Because of $\Omega_\infty \subset \sigma(S) = \sigma(S_0)$ and $\sigma(T_0) = \sigma(T \mid \mathscr{E}(\sigma(S), T)) \subset \sigma(S) = \sigma(S_0)$ (by 4.11 (a)) we have

$$\sigma(S_0) = \sigma(S_0) \cup \sigma(T_0) \cup \bar{\Omega}_\infty.$$

The statement now follows from Theorem 4.5.

5. RICHNESS OF LAT S IN THE CASE OF THICK ESSENTIAL SPECTRUM

In the sequel, $Z := Z_0/Z_1$ will denote a quotient of closed subspaces $Z_1 \subset Z_0$ of ℓ^p $(1 < p < \infty)$, $T \in \mathscr{L}(Z)$ will be an F-decomposable operator, $X \in \operatorname{Lat} T$, and $S := T \mid X$. If Y is a closed subspace of ℓ^p or, more generally, a quotient of two closed subspaces of ℓ^p, then for $N \in \mathbf{N} \cup \{\aleph_0\}$ we define $\mathbf{N}(N) := \{j \in \mathbf{N}; j \leqslant N\}$ and

$$\ell^p_N(Y) := \{y = (y_j)_{j \in \mathbf{N}(N)} : y_j \in Y, \ \|y\|_p^p := \sum_{j \in \mathbf{N}(N)} \|y_j\|^p < \infty\}.$$

Endowed with the norm $\|\cdot\|_p$, $\ell^p_N(Y)$ is a Banach space. For finite $N \in \mathbf{N}$ we simply write $Y^{(N)}$ instead of $\ell^p_N(Y)$. In particular $\ell^p_N(\ell^p) = \ell^p(\mathbf{N}(N) \times \mathbf{N})$ and thus $\ell^p_N(\ell^p) \cong \ell^p$ isometrically. Moreover, if $Y \in \operatorname{Lat} \ell^p$ then $\ell^p_N(Y) \in \operatorname{Lat} \ell^p_N(\ell^p)$ and for $Z = Z_0/Z_1$ we obtain $\ell^p_N(Z) \cong \ell^p_N(Z_0)/\ell^p_N(Z_1)$ isometrically, where the isometric isomorphism is given by

$$\pi^N_p(z) \to (\pi_p(z_j))_{j \in \mathbf{N}(N)} \quad \text{for } z = (z_j)_{j \in \mathbf{N}(N)} \in \ell^p_N(Z_0).$$

Here $\pi_p : \ell^p \to \ell^p/Z_1$ and $\pi^N_p : \ell^p_N(\ell^p) \to \ell^p_N(\ell^p)/\ell^p_N(Z_1)$ denote the canonical epimorphisms. We therefore will identify $\ell^p_N(Z)$ and $\ell^p_N(Z_0)/\ell^p_N(Z_1)$ in the sequel. The space Z^* will be identified with $Z_1^\perp/Z_0^\perp$ and is thus a quotient of two closed subspaces of ℓ^q, where $1/p + 1/q = 1$. We also write $\pi_q : \ell^q \to \ell^q/Z_0^\perp$ and $\pi^N_q : \ell^q_N(\ell^q) \to \ell^q_N(\ell^q)/\ell^q_N(Z_0^\perp)$ for the canonical epimorphisms and will as above identify $\ell^q_N(Z^*)$ with $\ell^q_N(Z_1^\perp)/\ell^q_N(Z_0^\perp)$.

For a bounded open set $G \subset \mathbf{C}$ we denote by $H^\infty(G, N)$ the Banach space of all those families $H = (h_{j,k})_{j,k \in \mathbf{N}(N)}$ such that $h_{j,k} \in H^\infty(G)$ and

$$\|H\|_{H^\infty(G,N)} := \sum_{j,k \in \mathbf{N}(N)} \|h_{j,k}\|_{H^\infty(G)} < \infty.$$

Let also $Q(G, N)$ be the Banach space of all those families $\mathbf{L} = (L_{j,k})_{j,k \in \mathbf{N}(N)}$ such that $L_{j,k} \in Q(G)$ and

(20) $$\|\mathbf{L}\|_{G,N} := \sup_{j,k \in \mathbf{N}(N)} \|L_{j,k}\|_G < \infty.$$

With this norm, the unit ball of $Q(G, N)$ is precisely the set of all those $\mathbf{L} = (L_{j,k})_{j,k \in \mathbf{N}(N)} \in Q(G, N)$ such that all $L_{j,k}$, $j, k \in \mathbf{N}(N)$, are in the unit ball of $Q(G)$. It is easily seen that for $N \in \mathbf{N}$ we have $Q(G, N)^* = H^\infty(G, N)$ via the bilinear mapping $\langle \cdot , \cdot \rangle : Q(G, N) \times H^\infty(G, N) \to \mathbf{C}$ given by

$$\langle \mathbf{L}, H \rangle := \sum_{1 \leqslant j,k \leqslant N} \langle L_{j,k}, h_{j,k} \rangle$$

for $\mathbf{L} = (L_{j,k})_{1\leq j,k\leq N} \in Q(G, N)$, $H = (h_{j,k})_{1\leq j,k\leq N} \in H^\infty(G, N)$. If V, G are bounded sets with $\bar{V} \subset G$ and $\bar{G} \cap F = \emptyset$, $N \in \mathbf{N} \cup \{\aleph_0\}$, then for $x = (x_j)_{j\in \mathbf{N}(N)} \in \ell^p_N(X)$, $y^* = (y^*_k)_{k\in \mathbf{N}(N)} \in \ell^q_N(\mathscr{E}(\bar{V}, T^*))$ the element

$$x \square_G y^* := (x_j \square_G y^*_k)_{j,k\in\mathbf{N}(N)} \tag{21}$$

of $Q(G, N)$ operates on $H^\infty(G, N)$ via

$$\begin{aligned}\langle x \square_G y^*, H\rangle &= \sum_{j,k\in \mathbf{N}(N)} \langle x_j \square_G y^*_k, h_{jk}\rangle = \\ &= \sum_{j,k\in \mathbf{N}(N)} \langle h_{j,k}(T^*) y^*_k, x_j\rangle.\end{aligned} \tag{22}$$

We also introduce the following notations: $X_0 := \pi_p^{-1}(X)$, $\mathscr{E}_0(\bar{V}) := \pi_q^{-1}(\mathscr{E}(\bar{V}, T^*))$. Then, for all $N \in \mathbf{N}$, $X_0^{(N)} = (\pi_p^N)^{-1}(X^{(N)})$, $\mathscr{E}_0^N(\bar{V}) := \mathscr{E}_0(\bar{V})^{(N)} = (\pi_q^N)^{-1}(\mathscr{E}(\bar{V}, T^*)^{(N)})$ where we identify $X^{(N)}$ with $X_0^{(N)}/Z_1^{(N)}$ and $\mathscr{E}(\bar{V}, T^*)^{(N)}$ with $\mathscr{E}_0^N(\bar{V})/Z_0^{\perp(N)}$ by means of the canonical isometrical isomorphisms.

Let now $N \in \mathbf{N}$ be finite. We introduce the set $\mathscr{X}_X^N(V, G)$ as follows.

5.1. Definition. With the preceeding notations, let $\mathscr{X}_X^N(V, G)$ be the set of all those $\mathbf{L} \in Q(G, N)$ such that there exist sequences $(x_n)_{n=1}^\infty$ in the unit ball of $X_0^{(N)}$ and $(y_n^*)_{n=1}^\infty$ in the unit ball of $\mathscr{E}_0^N(\bar{V})$ converging weakly to 0 in ℓ^p resp. ℓ^q, and satisfying

$$\|\mathbf{L} - \pi_p^N(x_n) \square_G \pi_q^N(y_n^*)\|_{G,N} \to 0 \quad \text{for } n \to \infty.$$

We shall need the following version of the vanishing lemma which, by means of (20)–(22), follows immediately from the case $N = 1$ (see Lemma 3.1 (b), (c)).

5.2. Lemma. (a) *For any $w^* \in \mathscr{E}(\bar{V}, T^*)^{(N)}$ and any sequence $(u_n)_{n=1}^\infty$ in $Z^{(N)}$ converging weakly to 0 we have*

$$\lim_{n\to\infty} \|u_n \square_G w^*\|_{G,N} = 0.$$

(b) *For any $z \in Z^{(N)}$ and any sequence $(v_n^*)_{n=1}^\infty$ in $\mathscr{E}(\bar{V}, T^*)^{(N)}$ converging weakly to 0 we have*

$$\lim_{n\to\infty} \|z \square_G v_n^*\|_{G,N} = 0.$$

The following will be needed for the proof of the convexity of $\mathscr{X}_X^N(V, G)$.

5.3. Lemma. *Let $x = (x_j)_{j=1}^\infty \in \ell^r$, $1 < r < \infty$, with $\|x\|_r < 1$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. If $(y_n)_{n=1}^\infty$ is a sequence of elements $y_n = (y_{n,j})_{j=1}^\infty$ in the open unit ball of ℓ^r tending weakly to 0 in ℓ^r, then there is some $n_0 \in \mathbf{N}$ such that for all*

$n \geqslant n_0$ *we have*

$$\|\alpha^{1/r}x + \beta^{1/r}y_n\|_r < 1.$$

Proof. (α) First recall that a sequence $(z_n)_{n=1}^{\infty}$ of elements $(z_{n,j})_{j=1}^{\infty} \in \ell^s$, $1 < s < \infty$, is a weak null sequence in ℓ^s if and only if $(\|z_n\|_s)_{n=1}^{\infty}$ is bounded and $|z_{n,j}| \to 0$ for all $j \in \mathbf{N}$ when $n \to \infty$.

(β) For $0 < \rho < r$ we define $\xi^\rho := (|x_j|^\rho)_{j=1}^{\infty}$ and $\eta_n^\rho := (|y_{n,j}|^\rho)_{j=1}^{\infty}$, $n \in \mathbf{N}$. Notice that $\xi^{r-\rho} \in \ell^t$ with $1 < t := r/(r-\rho) < \infty$ and $\eta_n^\rho \in \ell^s$ with $1 < s := r/\rho < \infty$. Moreover, $\|\eta_n^\rho\|_s = \|y_n\|_r^\rho < 1$, so that $(\|\eta_n^\rho\|_s)_{n=1}^{\infty}$ is bounded. As $(y_n)_n$ is a weak null sequence in ℓ^r, we have by (α), $|y_{n,j}|^\rho \to 0$ for all $j \in \mathbf{N}$. Again by (α), this implies that $(\eta_n^\rho)_{n=1}^{\infty}$ is a weak null sequence in ℓ^s. Because of $1/s + 1/t = 1$ we conclude that

$$(23) \qquad \sum_{j=1}^{\infty} |x_j|^{r-\rho} |y_{n,j}|^\rho = \langle \xi^{r-\rho}, \eta_n^\rho \rangle \to 0 \quad \text{for } n \to \infty.$$

(γ) If $\alpha = 0$ then $\beta = 1$ and we may choose $n_0 = 1$. Hence suppose now that $\alpha > 0$. Let m be the unique natural number with $m < r$ and $r - m \leqslant 1$. Then,

$$\|\alpha^{1/r}x + \beta^{1/r}y_n\|_r^r = \sum_{j=1}^{\infty} |\alpha^{1/r}x_j + \beta^{1/r}y_{n,j}|^r \leqslant$$

$$\leqslant \sum_{j=1}^{\infty} (\alpha^{1/r}|x_j| + \beta^{1/r}|y_{n,j}|)^{m+(r-m)} \leqslant$$

$$\leqslant \sum_{j=1}^{\infty} \sum_{\mu=0}^{m} \binom{m}{\mu} \alpha^{\mu/r} |x_j|^\mu \beta^{(m-\mu)/r} |y_{n,j}|^{m-\mu} \cdot$$

$$\cdot (|\alpha^{1/r}x_j|^{r-m} + |\beta^{1/r}y_{n,j}|^{r-m}) \leqslant$$

$$\leqslant \alpha\|x\|_r^r + \beta\|y_n\|_r^r + R_n \leqslant \alpha\|x\|_r^r + \beta + R_n$$

with

$$R_n := \sum_{\mu=0}^{m-1} \binom{m}{\mu} \alpha^{(r-m+\mu)/r} \beta^{(m-\mu)/r} \langle \xi^{r-m+\mu}, \eta_n^{m-\mu} \rangle +$$

$$+ \sum_{\mu=1}^{m} \binom{m}{\mu} \alpha^{\mu/r} \beta^{(r-\mu)/r} \langle \xi^\mu, \eta_n^{r-\mu} \rangle.$$

Hence, using (23) with $\rho = m - \mu$ resp. $\rho = r - \mu$, we obtain $R_n \to 0$ for $n \to \infty$. Because of $\alpha > 0$ we have $\alpha\|x\|_r^r + \beta < \alpha + \beta = 1$ and the lemma follows.

The next proposition has already been used (without proof) in the special case $N = 1$.

5.4. PROPOSITION. *$\mathscr{X}^N_X(V, G)$ is an absolutely convex closed subset of $Q(G, N)$.*

Proof. (α) $\ell^{p(N)}$ and $\ell^{q(N)}$ are separable Banach spaces because of $1 < p < \infty$, $1/p + 1/q = 1$. Hence we can choose dense sequences $(z_n)_{n=1}^{\infty}$ in $\ell^{p(N)}$ and $(w_n^*)_{n=1}^{\infty}$ in $\ell^{q(N)}$.

(β) $\mathscr{X}^N_X(V, G)$ is closed.

To prove this, consider a sequence $(\mathbf{L}_r)_{r=1}^{\infty}$ in $\mathscr{X}^N_X(V, G)$ converging to $\mathbf{L}$ in $Q(G, N)$. For each $r \in \mathbf{N}$ let $(x_n^{(r)})_{n=1}^{\infty}$, $(y_n^{*(r)})_{n=1}^{\infty}$ be sequences associated to $\mathbf{L}_r$ via the property in Definition 5.1 defining $\mathscr{X}^N_X(V, G)$. For $r = 1$ we choose n_1 large enough so that

$$\|\mathbf{L}_1 - \pi_p^N(x_{n_1}^{(1)}) \square_G \pi_q^N(y_{n_1}^{*(1)})\|_{G,N} < 1$$

and

$$|\langle w_1^*, x_{n_1}^{(1)}\rangle| + |\langle y_{n_1}^{*(1)}, z_1\rangle| < 1.$$

Suppose, $n_1, \ldots, n_k$ have been selected so that for $j = 1, \ldots, k$,

$$\|\mathbf{L}_j - \pi_p^N(x_{n_j}^{(j)}) \square_G \pi_q^N(y_{n_j}^{*(j)})\|_{G,N} < 1/j$$

and for $1 \leqslant i \leqslant j \leqslant k$

$$|\langle w_i^*, x_{n_j}^{(j)}\rangle| + |\langle y_{n_j}^{*(j)}, z_i\rangle| < 1/j.$$

One can select n_{k+1} so as to ensure

$$\|\mathbf{L}_{k+1} - \pi_p^N(x_{n_{k+1}}^{(k+1)}) \square_G \pi_q^N(y_{n_{k+1}}^{*(k+1)})\|_{G,N} < \frac{1}{k+1}$$

and, for $1 \leqslant i \leqslant k + 1$,

$$|\langle w_i^*, x_{n_{k+1}}^{(k+1)}\rangle| + |\langle y_{n_{k+1}}^{*(k+1)}, z_i\rangle| < \frac{1}{k+1}.$$

With $u_k := x_{n_k}^{(k)}$, $v_k^* := y_{n_k}^{*(k)}$ we clearly have

$$\lim_{k\to\infty} \|\mathbf{L} - \pi_p^N(u_k) \square_G \pi_q^N(v_k^*)\|_{G,N} = 0.$$

Moreover, $\lim_{k\to\infty} \langle w^*, u_k\rangle = 0$ whenever $w^* = w_i^*$ for some $i \in \mathbf{N}$. Since $(u_k)_{k=1}^{\infty}$ is bounded in $\ell^{p(N)}$ and $\{w_i^* \,; i \in \mathbf{N}\}$ is dense in $\ell^{q(N)}$, the sequence $(u_k)_{k=0}^{\infty}$ tends weakly to 0. Similarly one shows that $(v_k^*)_{k=1}^{\infty}$ enjoys the same property.

(γ) $\mathscr{X}^N_X(V, G)$ is absolutely convex.

Since $\mathscr{K}_X^N(V, G)$ is obviously circled it suffices to prove that it is convex. Hence, fix $\mathbf{L}, \tilde{\mathbf{L}} \in \mathscr{K}_X^N(V, G)$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. We want to show that $\mathbf{M} := \alpha\mathbf{L} + \beta\tilde{\mathbf{L}} \in \mathscr{K}_X^N(V, G)$. Let $(x_n)_{n=1}^\infty$, $(y_n^*)_{n=1}^\infty$ resp. $(\tilde{x}_n)_{n=1}^\infty$, $(\tilde{y}_n^*)_{n=1}^\infty$ be sequences associated to $\mathbf{L}$ resp. $\tilde{\mathbf{L}}$ by the definition (5.1) of $\mathscr{K}_X^N(V, G)$. We may assume that $\|x_n\|_p$, $\|\tilde{x}_n\|_p$, $\|y_n^*\|_q$, $\|\tilde{y}_n^*\|_q$ are all strictly less than 1 for all $n \in \mathbf{N}$. Let

$$\xi_{m,n} := \alpha^{1/p} x_m + \beta^{1/p} \tilde{x}_n$$

$$\eta_{m,n}^* := \alpha^{1/q} y_m^* + \beta^{1/q} \tilde{y}_n^*$$

for $m, n \in \mathbf{N}$. We first choose m_1 large enough so that

$$|\langle w_1^*, x_{m_1}\rangle| + |\langle y_{m_1}^*, z_1\rangle| < 1/2$$

and

$$\|\mathbf{L} - \pi_p^N(x_{m_1}) \,\square_G\, \pi_q^N(y_{m_1}^*)\|_{G,N} < 1/3.$$

Since $\tilde{x}_n \to 0$ weakly in $\ell^{p(N)}$ and $\tilde{y}_n^* \to 0$ weakly in $\ell^{q(N)}$, for $n \geqslant n_1$ sufficiently large, we have

$$\|\xi_{m_1,n}\|_p < 1, \quad \|\eta_{m_1,n}^*\|_q < 1 \qquad \text{(by Lemma 5.3)},$$

$$\|\tilde{\mathbf{L}} - \pi_p^N(\tilde{x}_n) \,\square_G\, \pi_q^N(\tilde{y}_n^*)\|_{G,N} < 1/3 \qquad \text{(by the definition of the sequences)},$$

$$|\langle w_1^*, \tilde{x}_n\rangle| + |\langle \tilde{y}_n^*, z_1\rangle| < 1/2.$$

and (using Lemma 5.2 and the fact that $\pi_p^N(\tilde{x}_n) \to 0$ weakly and $\pi_q^N(\tilde{y}_n^*) \to 0$ weakly)

$$\|\pi_p^N(\tilde{x}_n) \,\square_G\, \pi_q^N(y_{m_1}^*)\|_{G,N} + \|\pi_p^N(x_{m_1}) \,\square_G\, \pi_q^N(\tilde{y}_n^*)\|_{G,N} < 1/3.$$

We set $\xi_1 := \xi_{m_1,n_1}$, $\eta_1^* := \eta_{m_1,n_1}^*$ and obtain

$$\|\xi_1\|_p < 1, \quad \|\eta_1^*\|_q < 1,$$

$$\|\mathbf{M} - \pi_p^N(\xi_1) \,\square_G\, \pi_q^N(\eta_1^*)\| < 1,$$

$$|\langle w_1^*, \xi_1\rangle| + |\langle \eta_1^*, z_1\rangle| < 1.$$

In the same way, we obtain by induction two strictly increasing sequences $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$ in $\mathbf{N}$ such that

$$|\langle w_j^*, x_{m_k}\rangle| + |\langle y_{m_k}^*, z_j\rangle| < \frac{1}{2k} \quad \text{for } 1 \leqslant j \leqslant k$$

$$\|\mathbf{L} - \pi_p^N(x_{m_k}) \square_G \pi_q^N(y_{m_k}^*)\|_{G,N} < \frac{1}{3k}$$

and (after m_k has been chosen this way) we can find n_k sufficiently large such that

$$\|\xi_{m_k,n_k}\|_p < 1, \quad \|\eta_{m_k,n_k}^*\|_q < 1,$$

$$|\langle w_j^*, \tilde{x}_{n_k}\rangle| + |\langle \tilde{y}_{n_k}^*, z_j\rangle| < \frac{1}{2k} \quad \text{for } 1 \leqslant j \leqslant k,$$

$$\|\tilde{\mathbf{L}} - \pi_p^N(\tilde{x}_{n_k}) \square_G \pi_q^N(\tilde{y}_{n_k}^*)\|_{G,N} < \frac{1}{3k},$$

and

$$\|\pi_p^N(x_{m_k}) \square_G \pi_q^N(\tilde{y}_{n_k}^*)\|_{G,N} + \|\pi_p^N(\tilde{x}_{n_k}) \square_G \pi_q^N(y_{m_k}^*)\|_{G,N} < \frac{1}{3k}.$$

With $\xi_k := \xi_{m_k,n_k}$, $\eta_k^* := \eta_{m_k,n_k}^*$ we then have

$$\|\xi_k\|_p < 1, \quad \|\eta_k^*\|_q < 1,$$

$$\|\mathbf{M} - \pi_p^N(\xi_k) \square_G \pi_q^N(\eta_k^*)\|_{G,N} < 1/k,$$

$$|\langle w_j^*, \xi_k\rangle| + |\langle \eta_k^*, z_j\rangle| < \frac{1}{k} \quad \text{for } 1 \leqslant j \leqslant k.$$

As we have seen in the proof of part (β), this implies $\mathbf{M} \in \mathscr{X}_X^N(V, G)$. Thus, $\mathscr{X}_X^N(V, G)$ is convex and the proof is complete.

5.5. Proposition. *If* $\mathbf{L} = (L_{j,k})_{1 \leqslant j,k \leqslant N} \in Q(G, N)$ *with* $L_{j,k} \in \mathscr{X}_X(V, G)$ *for* $1 \leqslant j, k \leqslant N$, *then* $N^{-2}\mathbf{L} \in \mathscr{X}_X^N(V, G)$.

Proof. (α) Let us first show that, for all $L \in \mathscr{X}_X(V, G)$, we have $L^{(\mu,\nu)} := (\delta_{\mu,j}\delta_{\nu,k}L)_{1 \leqslant j,k \leqslant N} \in \mathscr{X}_X^N(V, G)$, where

$$\delta_{\mu,j} := \begin{matrix} 0 \\ 1 \end{matrix} \begin{cases} \text{for } \mu \neq j \\ \text{for } \mu = j \end{cases}$$

is the Kronecker symbol. Indeed, if $L \in \mathcal{T}_N(V, G)$, then there are weak null sequences $(x_n)_{n=1}^\infty$ in the unit ball of X_0 and $(y_n^*)_{n=1}^\infty$ in the unit ball of $\mathcal{E}_0(\overline{V})$ such that

$$\lim_{n\to\infty} \|L - \pi_p(x_n) \square_G \pi_q(y_n^*)\|_G = 0.$$

Set now $\vec{x}_n := (\delta_{\mu,j} x_n)_{j=1}^N$, $\vec{y}_n^* := (\delta_{\nu,k} y_n^*)_{k=1}^N$. Then $(\vec{x}_n)_{n=1}^\infty$ resp. $(\vec{y}_n^*)_{n=1}^\infty$ are weak null sequences in the unit balls of $X_0^{(N)}$ resp. $\mathcal{E}_0^N(\overline{V})$. Moreover,

$$\|L^{(\mu,\nu)} - \pi_p^N(\vec{x}_n) \square_G \pi_q^N(\vec{y}_n^*)\|_{G,N} = \|L - \pi_p(x_n) \square_G \pi_q(y_n^*)\|_G \to 0$$

for $n \to \infty$. Therefore, $L^{(\mu,\nu)} \in \mathcal{T}_N^N(V, G)$.

(β) Fix now $\mathbf{L} = (L_{\mu,\nu})_{1\le\mu,\nu\le N} \in Q(G, N)$ with $L_{\mu,\nu} \in \mathcal{T}_N(V, G)$ for $1 \le \mu, \nu \le N$. Then, because of (α), $N^{-2}\mathbf{L} = \sum_{1\le\mu,\nu\le N} N^{-2} L_{\mu,\nu}^{(\mu,\nu)}$ is a convex combination of elements in $\mathcal{T}_N^N(V, G)$ and hence itself in $\mathcal{T}_N^N(V, G)$ (by Proposition 5.4).

For a bounded open set Ω let Λ_Ω denote the absolutely convex hull of $\{E_\lambda ; \lambda \in \Omega \cap \sigma_{le}(S)\}$ and set $\Lambda_\Omega^N := \{\mathbf{L} = (L_{\mu,\nu})_{\mu,\nu\in \mathbf{N}(N)} ; L_{\mu,\nu} \in \Lambda_\Omega$ for $\mu, \nu \in \mathbf{N}(N)\}$, where $N \in \mathbf{N} \cup \{\aleph_0\}$. It follows that Λ_Ω (resp. Λ_Ω^N) is dense in the unit ball of $Q(\Omega)$ (resp. $Q(\Omega, N)$) whenever $\Omega \cap \sigma_{le}(S)$ is dominating for Ω and $N \in \mathbf{N}$. If $\Omega \subset G$ then we write again $J_\Omega^G : Q(\Omega, N) \to Q(G, N)$ for the canonical mapping induced by $J_\Omega^G: Q(\Omega) \to Q(G)$ (see (9)):

$$J_\Omega^G(\mathbf{L}) := (J_\Omega^G(L_{\mu,\nu}))_{\mu,\nu\in\mathbf{N}(N)} \quad \text{for } \mathbf{L} = (L_{\mu,\nu})_{\mu,\nu\in\mathbf{N}(N)} \in Q(\Omega, N).$$

The following result is an immediate consequence of Lemma 3.2, Proposition 5.5 and Proposition 5.4.

5.6. LEMMA. *Let Ω, V, G be bounded open sets in $\mathbf{C}$ with $\overline{\Omega} \subset V \subset \overline{V} \subset G$ and $\overline{G} \cap F = \emptyset$. Then, for all $N \in \mathbf{N}$, we have the inclusion*

$$N^{-2}\delta(\Omega, V, T) J_\Omega^G(\Lambda_\Omega^N) \subset \mathcal{T}_N^N(V, G)$$

where $\delta(\Omega, V, T)$ is the constant from Lemma 3.2.

The next statement is the counterpart of Lemma 4.6.

5.7. LEMMA. *Let Ω, V, G be as in Lemma 5.6 and assume in addition that $\Omega \cap \sigma_{le}(X)$ is dominating for Ω. Let also $N \in \mathbf{N}$, $\mathbf{K}\in Q(\Omega, N)$, $a\in X^{(N)}$, $b^* \in \mathcal{E}(\overline{V}, T^*)^{(N)}$, and $\varepsilon > 0$ be given. Then there exist $u \in X^{(N)}$, $v^* \in \mathcal{E}(\overline{V}, T^*)^{(N)}$ such that*

$$\|J_\Omega^G(\mathbf{K}) - u \square_G v^*\|_{G,N} < \varepsilon$$

$$\|u\|, \|v^*\| < N \sqrt{\frac{\|\mathbf{K}\|_{\Omega,N}}{\delta}} \quad \textit{where } \delta := \delta(\Omega, V, T), \tag{24}$$

$$\|(a + u) \square_G (b^* + v^*) - a \square_G b^* - u \square_G v^*\|_{G,N} < \varepsilon.$$

Proof. Since $A(\Omega, N)$ is dense in the unit ball of $Q(\Omega, N)$ there is some $\mathbf{L} \in A_\Omega^N$ such that $\|\mathbf{K} - \|\mathbf{K}\|_{\Omega,N}\mathbf{L}\|_{\Omega,N} < \varepsilon/2$. Then we also have $\|J_\Omega^G(\mathbf{K}) - \|\mathbf{K}\|_{\Omega,N}J_\Omega^G(\mathbf{L})\|_{\Omega,N} < \varepsilon/2$ as inequality (10) carries over to the present case. By Lemma 5.6 we have $N^{-2}\delta J_\Omega^G(\mathbf{L}) \in \mathscr{X}_X^N(V, G)$. Therefore, there are $\tilde{u}$, $\tilde{v}^*$ in the open unit balls of $X_0^{(N)}$ and $\mathscr{E}_0^{(N)}(\overline{V})$ satisfying

$$\|N^{-2}\delta J_\Omega^G(\mathbf{L}) - \pi_p^N(\tilde{u}) \,\Box_G\, \pi_q^N(\tilde{v}^*)\|_{G,N} < \frac{N^2\varepsilon}{2\delta\|\mathbf{K}\|_{\Omega,N}}$$

and (applying Lemma 5.2 in the same way as before),

$$\|\pi_p^N(\tilde{u}) \,\Box_G\, b^*\|_{G,N} + \|a \,\Box_G\, \pi_q^N(\tilde{v}^*)\|_{G,N} < \frac{\varepsilon}{N}\sqrt{\frac{\delta}{\|\mathbf{K}\|_{\Omega,N}}}.$$

With

$$u := N\sqrt{\frac{\|\mathbf{K}\|_{\Omega,N}}{\delta}}\,\pi_p^N(\tilde{u}), \quad v^* := N\sqrt{\frac{\|\mathbf{K}\|_{\Omega,N}}{\delta}}\,\pi_q^N(\tilde{v}^*)$$

we obtain the desired inequalities.

The following theorem is the main tool needed for the proof of Theorem 1.4. It states that the dual algebra $H^\infty(\Omega_1)$ (in situation (15) with (1) satisfied for $\sigma_{\mathrm{le}}(S)$ instead of $\sigma(S)$) has a sort of $\mathbf{A}_{\aleph_0}$ property in $H^\infty(G)$ relative to the bilinear mapping $\Box_G$ and the spaces X and $\mathscr{E}(\overline{\Omega^\infty}, T^*)$.

5.8. THEOREM. *Let $Z = Z_0/Z_1$ be a quotient of two closed subspaces of ℓ^p, $T \in \mathscr{L}(Z)$ be F-decomposable, $X \in \operatorname{Lat} T$, $S := T\,|\,X$, and let $(\Omega_n)_{n=1}^\infty$ be a sequence of open sets such that (1) is fulfilled with $\sigma(S)$ replaced by $\sigma_{\mathrm{le}}(S)$. Given any family $(L_{j,k})_{j,k\in\mathbf{N}}$ of elements in $Q(\Omega_1)$ there exist sequences $(x_j)_{j=1}^\infty$ in X, $(y_k^*)_{k=1}^\infty$ in $\mathscr{E}(\overline{\Omega_\infty}, T^*)$ such that*

$$\forall\, j, k \in \mathbf{N} : J_{\Omega_1}^G(L_{j,k}) = x_j \,\Box_G\, y_k^* \tag{25}$$

for any bounded open set $G \supset \overline{\Omega_\infty}$ with $F \cap \overline{G} = \emptyset$.

Proof. (α) We first introduce some notations. Let V be a bounded open set in $\mathbf{C}$ with $\overline{V} \cap F = \emptyset$. For $\mathbf{L} = (L_{j,k})_{j,k\in\mathbf{N}} \in Q(V, \aleph_0)$ and $N \in \mathbf{N}$ we define $\mathbf{L}_N := (L_{j,k})_{1\leqslant j,k\leqslant N} \in Q(V, N)$. The mapping $\mathbf{L} \to \mathbf{L}_N$ is thus the canonical epimorphism from $Q(V, \aleph_0)$ onto $Q(V, N)$. For $\mathbf{L} = (L_{j,k})_{1\leqslant j,k\leqslant N}$ we define $\hat{\mathbf{L}} = (\hat{L}_{j,k})_{1\leqslant j,k\leqslant N+1} \in Q(V, N+1)$ and $\tilde{\mathbf{L}} = (\tilde{L}_{j,k})_{j,k\in\mathbf{N}}$ by $\tilde{L}_{j,k} = \hat{L}_{j,k} := L_{j,k}$ for $1 \leqslant j,\ k \leqslant N$ and $\hat{L}_{j,N+1} := \hat{L}_{N+1,j} := 0$ for $j = 1, \ldots, N+1$, $\tilde{L}_{j,k} := 0$ if $j > N$ or $k > N$. The mappings $\mathbf{L} \mapsto \hat{\mathbf{L}}$ resp. $\mathbf{L} \mapsto \tilde{\mathbf{L}}$ are isometric embeddings of $Q(V, N)$

into $Q(V, N+1)$ resp. $Q(V, \aleph_0)$. If Y is a Banach space and $1 < r < \infty$ then the canonical isometric injections $\hat{}: \ell^r_N(Y) \to \ell^r_{N+1}(Y)$, $\tilde{}: \ell^r_N(Y) \to \ell^r_{\aleph_0}(Y)$ are defined by $\hat{y} := (y_1, \ldots, y_N, 0)$, $\tilde{y} := (y_1, \ldots, y_N, 0, 0, \ldots)$ for $y = (y_j)_{j=1}^N \in \ell^r_N(Y) = Y^{(N)}$.

Fix now a sequence $(V_n)_{n=1}^\infty$ of open sets with $\overline{\Omega_n} \subset V_n \subset \overline{V_n} \subset \Omega_{n+1}$ and write $\delta_n := \delta(\Omega_n, V_n, T)$ for the constant obtained from Lemma 3.2. We also set $Q_n := Q(\Omega_n)$, $Q_{n,N} := Q(\Omega_n, N)$, $\|\cdot\|_n := \|\cdot\|_{Q_n}$, $\|\cdot\|_{n,N} := \|\cdot\|_{Q_n,N}$, $\Box_n := \Box_{\Omega_n}$, $J^n_m := J^{\Omega_n}_{\Omega_m}$ for $m, n \in \mathbf{N}$ with $m \leqslant n$ and $N \in \mathbf{N} \cup \{\aleph_0\}$.

(β) Before solving the general problem we first consider the special case that $\mathbf{L} = (L_{j,k})_{j,k \in \mathbf{N}} \in Q(\Omega_1, \aleph_0)$ with

$$\sum_{n=1}^{\infty} n \sqrt{\frac{d_n}{\delta_n}} < \infty \tag{26}$$

where $d_n = d_n(\mathbf{L}) := \sum_{k=1}^{n} \|L_{n,k}\|_1 + \sum_{j=1}^{n-1} \|L_{j,n}\|_1$.

Straightforward computations show that (26) is satisfied if, for instance,

$$\|L_{j,k}\|_1 \leqslant (jk)^{-4} \min\{1, \sqrt{\delta_m}\,; 1 \leqslant m < \max\{j, k\}\} \tag{27}$$

for all $j, k \in \mathbf{N}$. Let now $(\varepsilon_n)_{n=1}^\infty$ be a sequence of positive real numbers with $\varepsilon_n \searrow 0$ and

$$\sum_{n=2}^{\infty} n \sqrt{\frac{\varepsilon_{n-1}}{\delta_n}} < \infty. \tag{28}$$

We will now construct sequences $(u_n)_{n=1}^\infty$ and $(v_n^*)_{n=1}^\infty$ such that (with $u_0 := 0 \in X$, $v_0^* := 0 \in \mathscr{E}(\overline{V_1}, T^*)$, $\varepsilon_0 := 0$)

$$u_n \in X^{(n)}, \quad v_n^* \in \mathscr{E}(\overline{V_n}, T^*)^{(n)}$$

$$\|J_1^{n+1}(\mathbf{L}_n) - u_n \Box_{n+1} v_n^*\|_{n+1,n} < \varepsilon_n \tag{29}$$

$$\|\tilde{u}_n - \tilde{u}_{n-1}\|_p, \quad \|\tilde{v}_n^* - \tilde{v}_{n-1}^*\|_q < n \sqrt{\frac{d_n + \varepsilon_{n-1}}{\delta_n}}$$

for all $n \in \mathbf{N}$. For $n = 1$ we apply Lemma 5.7 with $N = 1$, $\varepsilon := \varepsilon_1$, $\Omega := \Omega_1$, $V := V_1$, $G := \Omega_2$, $\mathbf{K} := \mathbf{L}_1$, $a := u_0 = 0$, $b^* := v_0^* = 0$, to obtain vectors $u_1 \in X = X^{(1)}$, $v_1^* \in \mathscr{E}(\overline{V_1}, T^*) = \mathscr{E}(\overline{V_1}, T^*)^{(1)}$ satisfying

$$\|J_1^2(\mathbf{L}_1) - u_1 \Box_2 v_1^*\|_{2,1} < \varepsilon_1$$

$$\|\tilde{u}_1\|_p = \|u_1\| \leqslant \sqrt{\frac{\|L_1\|_1}{\delta_1}} = \sqrt{\frac{d_1}{\delta_1}}, \quad \|\tilde{v}_1^*\|_q = \|v_1^*\| \leqslant \sqrt{\frac{d_1}{\delta_1}}.$$

Thus (29) is fulfilled for $n = 1$. Suppose now we have obtained u_j, v_j^* for $j = 1, \ldots, n$ such that (29) holds for $j = 1, \ldots, n$. Then, applying Lemma 5.7 with $N = n + 1$, $\Omega := \Omega_{n+1}$, $V := V_{n+1}$, $G := \Omega_{n+1}$, $\mathbf{K} := J_1^{n+1}(\mathbf{L}_{n+1}) - \hat{u}_n \Box_{n+1} \hat{v}_n^*$, $a := \hat{u}_n$, $b^* := \hat{v}_n^*$, $\varepsilon := (1/2)\varepsilon_{n+1}$, we obtain $u \in X^{(n+1)}$, $v^* \in \mathscr{E}(\overline{V_{n+1}}, T^*)^{(n+1)}$ satisfying (24). With $u_{n+1} := \hat{u}_n + u$, $v_{n+1}^* := v_n^* + v^*$ we then have

$$\|J_1^{n+2}(\mathbf{L}_{n+1}) - u_{n+1} \Box_{n+2} v_{n+1}^*\|_{n+2,n+1} < \varepsilon_{n+1}$$

and

$$\|\tilde{u}_{n+1} - \tilde{u}_n\|_p = \|u_{n+1} - \hat{u}_n\|_p \leqslant (n + 1) \sqrt{\frac{d_{n+1} + \varepsilon_n}{\delta_{n+1}}},$$

$$\|\tilde{v}_{n+1}^* - \tilde{v}_n^*\|_q = \|v_{n+1}^* - \hat{v}_n^*\|_p \leqslant (n + 1) \sqrt{\frac{d_{n+1} + \varepsilon_n}{\delta_{n+1}}}$$

because of

$$\|\mathbf{K}\|_{n+1,n+1} \leqslant \|J_1^{n+1}(\mathbf{L}_{n+1} - \hat{\mathbf{L}}_n)\|_{n+1,n+1} +$$

$$+ \|J_1^{n+1}(\hat{\mathbf{L}}_n) - \hat{u}_n \Box_{n+1} \hat{v}_n^*\|_{n+1,n+1} \leqslant$$

$$\leqslant \|\mathbf{L}_{n+1} - \hat{\mathbf{L}}_n\|_{1,n+1} + \|J_1^{n+1}(\mathbf{L}_n) - u_n \Box_{n+1} v_n^*\|_{n+1,n} \leqslant d_{n+1} + \varepsilon_n.$$

Therefore (29) is now also fulfilled for $n + 1$ and our inductive construction of $(u_n)_{n=1}^\infty$, $(v_n^*)_{n=1}^\infty$ is complete. Because of (26) and (28) we have

$$\sum_{n=1}^{\infty} n \sqrt{\frac{d_{n+1} + \varepsilon_n}{\delta_{n+1}}} < \infty.$$

This implies that $(\tilde{u}_n)_{n=1}^\infty$, $(\tilde{v}_n^*)_{n=1}^\infty$ are Cauchy sequences in $\ell^p_{\aleph_0}(X)$ resp. $\ell^q_{\aleph_0}(\mathscr{E}(\overline{\Omega_\infty}, T^*))$. Define now $x = (x_j)_{n=1}^\infty := \lim_{n\to\infty} \tilde{u}_n \in \ell^p_{\aleph_0}(X)$, $y^* = (y_k^*)_{k=1}^\infty := \lim_{n\to\infty} \tilde{v}_n^* \in \ell^q_{\aleph_0}(\mathscr{E}(\overline{\Omega_\infty}, T^*))$. Fix now an arbitrary bounded open set $G \supset \overline{\Omega_\infty}$ with $\overline{G} \cap F = \emptyset$. Notice that for all $N \in \mathbf{N}$ the mapping $(a, b^*) \mapsto (a \Box_G b^*)_N$ is continuous from $\ell^p_{\aleph_0}(X) \times \ell^q_{\aleph_0}(\mathscr{E}(\overline{\Omega}_\infty, T^*))$ to $Q(G, N)$. Hence, for $N < n \to \infty$, (using $\|J^G_{\Omega_{n+1}}\| \leqslant 1$),

$$\|J^G_{\Omega_1}(\mathbf{L})_N - (x \Box_G y^*)_N\| \leqslant$$

$$\leqslant \|J^G_{\Omega_1}(\mathbf{L}_n)_N - (\tilde{u}_n \Box_G \tilde{v}_n^*)_N\|_{G,N} + \|(\tilde{u} \Box_G \tilde{v}_n^*)_N - (x \Box_G y^*)_N\|_{G,N} \leqslant$$

$$\leqslant \|(J_1^{n+1}(\mathbf{L}_n) - u_n \Box_{n+1} v_n^*)_N\|_{n+1,N} + \|(\tilde{u}_n \Box_G v_n^*)_N - (x \Box_G y^*)_N\|_{G,N} \leqslant$$

$$\leqslant \|J_1^{n+1}(\mathbf{L}_n) - u_n \Box_{n+1} v_n^*\|_{n+1,n} + \|(\tilde{u}_n \Box_G \tilde{v}_n^*)_N - (x \Box_G y^*)_N\|_{G,N} \leqslant$$

$$\leqslant \varepsilon_n + \|(\tilde{u}_n \Box_G \tilde{v}_n^*)_N - (x \Box_G y^*)_N\|_{G,N} \to 0.$$

Hence we obtain (25). This completes the proof of the theorem when $\mathbf{L} = (L_{j,k})_{j,k\in\mathbf{N}}$ is in $Q(\Omega_1, \aleph_0)$ and satisfies (26).

(γ) For an arbitrary family $(L_{j,k})_{j,k\in\mathbf{N}}$ with $L_{j,k} \in Q(\Omega_1)$ for all $j, k \in \mathbf{N}$, it is easy to build sequences $\alpha_j > 0$, $\beta_k > 0$ so that for $\mathbf{L} := ((1/\alpha_j\beta_k)L_{jk})_{j,k\in\mathbf{N}}$ inequalities (26) and (27) are satisfied. As we noticed earlier, this implies now that $\mathbf{L} \in Q(\Omega_1, \aleph_0)$. Hence, by ($\beta$) there are $(u_j)_{j=1}^{\infty} \in \ell^p_{\aleph_0}(X)$, $(v_k^*)_{k=1}^{\infty} \in \ell^q_{\aleph_0}(\mathscr{E}(\overline{\Omega_\infty}, T^*))$ such that $(\alpha_j\beta_k)^{-1}J_\Omega^G(L_{j,k}) = u_j \,\square_G\, v_k^*$ for all $j, k \in \mathbf{N}$ and any bounded open set G with $\bar{G} \cap F = \emptyset$. With $x_j := \alpha_j u_j$, $y_k^* := \beta_k v_k^*$ we obtain (25).

Theorem 1.4 will now be an easy consequence of

5.9. THEOREM. *Under the assumptions of Theorem 5.8 denote by $\mathscr{W}$ the* WOT-*closure of*

$$\{r(S)\,;\, r \text{ rational with poles off } \sigma(S) \cup \sigma(T) \cup \overline{\Omega_\infty}\}.$$

Then there exist $\mathscr{M}, \mathscr{N} \in$ Lat $\mathscr{W}$ such that $\mathscr{M}/\mathscr{N}$ is infinite dimensional and such that Lat $\mathscr{W}$ *contains a sublattice order isomorphic to* Lat $\mathscr{M}/\mathscr{N}$.

Proof. Choose $\lambda \in \Omega_1$ and set $L_{j,k} := \delta_{j,k}E_\lambda$ $(j, k \in \mathbf{N})$. By Theorem 5.8 there exist sequences $(x_j)_{j=1}^{\infty}$ in X and $(y_k^*)_{k=1}^{\infty}$ in $\mathscr{E}(\overline{\Omega_\infty}, T^*)$ such that for any bounded open set $G \supset \overline{\Omega_\infty}$ with $\bar{G} \cap F = \emptyset$,

$$x_j \,\square_G\, y_k^* = \delta_{j,k}E_\lambda \quad \text{for all } j, k \in \mathbf{N},$$

where now E_λ is the point evaluation at λ considered on $H^\infty(G)$. For any rational function r with poles off $K := \sigma(T) \cup \sigma(S) \cup \overline{\Omega_\infty}$ we can find $G \supset K$ such that $r \in H^\infty(G)$. It follows from (30) that

$$\langle r(T^*_{\Omega_\infty})y_k^*, x_j\rangle = \delta_{j,k}r(\lambda) \quad \text{for all } j, k \in \mathbf{N}.$$

Because of $\mathscr{E}(\overline{\Omega_\infty}, T^*) \subset Z^*$ and $\sigma(T^*_{\Omega_\infty}) \subset \overline{\Omega_\infty} \subset K$, $\sigma(T^*) = \sigma(T) \subset K$, $\sigma(S) \subset K$, this implies

$$\langle y_k^*, r(S)x_j\rangle = \langle y_k^*, r(T)x_j\rangle = \langle r(T)^*y_k^*, x_j\rangle =$$

$$= \langle r(T^*)y_k^*, x_j\rangle = \langle r(T^*_{\Omega_\infty})y_k^*, x_j\rangle = \delta_{j,k}r(\lambda).$$

Let u_k^* be the class of y_k^* in $X^* = Z^*/X^\perp$. Thus

$$\langle u_k^*, r(S)x_j\rangle = \begin{cases} r(\lambda) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}. \tag{31}$$

Since $\mathscr{R} := \{r\,;\, r$ rational with poles off $K\}$ is WOT-dense in $\mathscr{W}$, we conclude that for all $A \in \mathscr{W}$

$$\langle u_k^*, Ax_j \rangle = 0 \quad \text{if } j \neq k.$$

For $j = k \in \mathbf{N}$ we conclude from (31) that the WOT-continuous functional $\varphi_k : A \mapsto \langle u_k^*, Ax_k \rangle$ is multiplicative on $\mathscr{W}$ being multiplicative on the WOT-dense subalgebra $\mathscr{R}$. Again by the WOT-density of $\mathscr{R}$ in $\mathscr{W}$, we have $\varphi(A) := \varphi_1(A) = \varphi_k(A)$ for all $k \in \mathbf{N}$. We now define $\mathscr{M} := \bigvee\{Ax_j, A \in \mathscr{W}, j \in \mathbf{N}\}$, $\mathscr{M}_* := \bigvee\{B^*w_k^*\,;\, B \in \mathscr{W}, k \in \mathbf{N}\} \subset X^*$ and $\mathscr{N} := \mathscr{M} \cap \mathscr{M}_*^{\perp}$. Clearly $\mathscr{M}, \mathscr{N} \in \operatorname{Lat}\mathscr{W}$ with $\mathscr{N} \subset \mathscr{M}$. Because of $u_k^* \in \mathscr{M}_* \subset \mathscr{N}^{\perp}$ the functional f_k^* with $\langle f_k^*, [v] \rangle := \langle u_k^*, v \rangle$ for $[v] = v + \mathscr{N} \in \mathscr{M}/\mathscr{N}$ is well defined and continuous. Since we have $\langle f_k^*, [x_j] \rangle = \delta_{j,k}$ (by (31)), the vectors $([x_j])_{j=1}^{\infty}$ are linearly independent in $\mathscr{M}/\mathscr{N}$. Hence, $\mathscr{M}/\mathscr{N}$ is infinite dimensional. From the definition of φ we also obtain for all $k \in \mathbf{N}$, $A, B, C \in \mathscr{W}$

$$\langle C^*u_k^*, BAx_j \rangle = \varphi(CBA) = \varphi(C)\varphi(B)\varphi(A).$$

This shows that $((\ker\varphi)\mathscr{M})^- = \{Ax\,;\, A \in \ker\varphi, x \in \mathscr{M}\}^-$ belongs to $\operatorname{Lat}\mathscr{W}$ and is contained in $\mathscr{N}$. Thus, for any $A \in \mathscr{W}$,

$$(A - \varphi(A))\mathscr{M} \subset \mathscr{N}.$$

Hence, if $\mathscr{E} \in \operatorname{Lat}\mathscr{M}/\mathscr{N}$, we have $(A - \varphi(A))\pi^{-1}(\mathscr{E}) \subset \mathscr{N} \subset \pi^{-1}(\mathscr{E})$ and hence $\pi^{-1}(\mathscr{E}) \in \operatorname{Lat}\mathscr{W}$ where $\pi : \mathscr{M} \to \mathscr{M}/\mathscr{N}$ is the canonical epimorphism.

Proof of Theorem 1.4. If the situation of Theorem 1.4 is given, then by the proof of 4.2 we can always assume (up to similarity) that S is as in the assumptions of Theorem 5.8. Note that if $\sigma_e(S)\setminus\sigma_{le}(S) \neq \emptyset$ then the statement of the theorem is trivial. Thus, Theorem 1.4 is now an immediate consequence of Theorem 5.9.

5.10. THEOREM. *Let X be a Banach space which is topologically isomorphic to a quotient of two closed subspaces of ℓ^p $(1 < p < \infty)$. If $S \in \mathscr{L}(X)$ has property (β) on $\mathbf{C}\setminus F$ for some closed $F \subset \sigma(S)$ such that $\operatorname{int}(\sigma_e(S)\setminus F) \neq \emptyset$, then there are $\mathscr{M}, \mathscr{N} \in \operatorname{Lat}\mathscr{W}_S$ with $\mathscr{N} \subset \mathscr{M}$ such that $\mathscr{M}/\mathscr{N}$ is infinite dimensional and such that $\operatorname{Lat}\mathscr{W}_S$ contains a sublattice order isomorphic to $\operatorname{Lat}\mathscr{M}/\mathscr{N}$.*

Proof. As in the proof of Theorem 4.12, S is similar to $S_0 = T_0 \mid Y$ where $T_0 \in \mathscr{L}(\tilde{Z})$ is F_0-decomposable with $\sigma(T_0) \subset \sigma(S_0)$, F_0 a closed subset of $F \cup \partial\sigma(S)$, $\tilde{Z}$ a quotient of two closed subspaces of ℓ^p, and $Y \in \operatorname{Lat} T_0$. As above we may assume that $\sigma_e(S) = \sigma_{le}(S)$. Because of $\operatorname{int}(\sigma_{le}(S)\setminus F) \neq \emptyset$, $\sigma_{le}(S_0) = \sigma_{le}(S)$, $F_0 \subset F$, we have $\operatorname{int}(\sigma_{le}(S_0)\setminus F_0) \neq \emptyset$. Let now $(\Omega_n)_{n=1}^{\infty}$ be a sequence of open discs in $\operatorname{int}(\sigma_{le}(S_0)\setminus F_0)$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$ for all n and with $\overline{\Omega_\infty} \cap F_0 = \emptyset$ where

$\Omega_\infty := \bigcup_{n=1}^{\infty} \Omega_n$. Then $(\Omega_n)_{n=1}^{\infty}$ fulfils (1) with $\sigma(S)$ replaced by $\sigma_{le}(S_0)$. Notice that $\sigma(S_0) = \sigma(S_0) \cup \sigma(T_0) \cup \overline{\Omega_\infty}$ so that we have $\mathcal{W} = \mathcal{W}_{S_0}$ in Theorem 5.9. The statement now follows immediately from Theorem 5.9.

5.11. COROLLARY. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $S \in \mathcal{L}(\mathcal{H})$ have property (β) on $\mathbf{C}\setminus F$ for a closed set $F \subset \sigma(S)$. If* $\mathrm{int}(\sigma_e(S)\setminus F) \neq \emptyset$ *then* $\mathrm{Lat}\,\mathcal{W}_S$ *contains a sublattice order isomorphic to* $\mathrm{Lat}\,\mathcal{H}$.

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