J. OPERATOR THEORY
19(1988), 25–42

ECONOMICAL COMPACT PERTURBATIONS. II: FILLING IN THE HOLES

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1. INTRODUCTION

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all (bounded linear) operators acting on a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$, and let $\mathscr{K}(\mathscr{H})$ denote the ideal of all compact operators.

If $\sigma(T)$ denotes the spectrum of T and $\sigma_e(T)$ is the essential spectrum (i.e., the spectrum of the canonical projection $\tilde{T}$ of T in the quotient Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$), then

$$\sigma_0(T) = \{\lambda \in \mathbf{C} : \lambda \text{ is an isolated point of } \sigma(T) \setminus \sigma_e(T)\}$$

is the set of all normal eigenvalues of T. An isolated point λ of $\sigma(T)$ is a normal eigenvalue if and only if the Riesz spectral subspace $\mathscr{H}(T; \lambda)$ (corresponding to the clopen subset $\{\lambda\}$ of $\sigma(T)$) is finite dimensional; $\sigma_0(T)$ is an at most denumerable set, all whose limit points belong to $\partial\sigma_e(T)$, the boundary of $\sigma_e(T)$, and there exists a compact operator K_0 such that $\sigma_0(T - K_0) = \emptyset$ [18].

Recall that $T \in \mathscr{L}(\mathscr{H})$ is semi-Fredholm if $\operatorname{ran} T := T\mathscr{H}$ is closed and at least one of $\operatorname{nul} T := \dim \ker T$ or $\operatorname{nul} T^*$ is finite dimensional. In this case, the index of T is defined by

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

The reader is referred to [17] for the stability properties of the semi-Fredholm operators. Recall, in particular, that if T is semi-Fredholm and $K \in \mathscr{K}(\mathscr{H})$, then $T - K$ is also semi-Fredholm, and $\operatorname{ind}(T - K) = \operatorname{ind} T$.)

Let $\rho_{s\text{-}F}(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is semi-Fredholm}\}$. The Weyl spectrum of T, $\sigma_W(T) := \sigma(T) \setminus \{\lambda \in \rho_{s\text{-}F}(T) : \operatorname{ind}(\lambda - T) = 0\}$ $(= \sigma_e(T) \cup \{\lambda \in \rho_{s\text{-}F}(T) : : \operatorname{ind}(\lambda - T) \neq 0\})$ is the largest part of the spectrum that is invariant under compact perturbations:

$$\sigma_W(T) = \{\sigma(T - K) : K \in \mathscr{K}(\mathscr{H})\};$$

moreover, J. G. Stampfli proved that there exists K_W in $\mathcal{K}(\mathcal{H})$ such that $\sigma_W(T) = \sigma_W(T - K_W)$ [18].

A more general theorem was later obtained by C. Apostol, C. M. Pearcy and and N. Salinas in [3] (see also [9, Section 4.3]). In particular, given any bounded sequence $\{\lambda_n\} \subset \mathbf{C} \setminus \sigma_W(T)$ such that the λ_n's only accumulate of $\partial\sigma_e(T)$, and a subfamily $\{\Omega_k\}$ of the "holes" of $\sigma_W(T)$ such that $\{\lambda_n\} \cap (\bigcup_k \Omega_k) = \emptyset$, there exists $K \in \mathcal{K}(\mathcal{H})$ such that

$$\sigma(T - K) = \sigma_W(T) \cup \{\lambda_n\} \cup (\bigcup_k \Omega_k), \quad \sigma_0(T - K) = \{\lambda_n\}.$$

(A "hole" of a compact subset Γ of $\mathbf{C}$ is a *bounded* component of $\mathbf{C} \setminus \Gamma$.)

In [1], C. Apostol introduced the notion of "triangular representation" of an operator, which strongly simplified the analysis of these problems. In [12], the author began to analyze the problem of computing the infimum of the norms of the compact operators K that produce a modification of the spectrum of the above described type. For instance, if the only information that we possess about T is $\sigma(T)$, $\sigma_0(T)$ and $\sigma_W(T)$, then the best possible result for K_0 (such that $\sigma_0(T - K_0) = \emptyset$) is this one: given $\varepsilon > 0$ there exists K_0 as above with

$$\|K_0\| < \frac{1}{2} \max\{\text{dist}[\lambda, \sigma_W(T)] : \lambda \in \sigma_0(T)\} + \varepsilon.$$

(Exactly the same estimate holds for K_W such that $\sigma(T - K_W) = \sigma_W(T)$.)

If

$$m(\lambda - T) = \min\{r \in \sigma([(\lambda - T)^*(\lambda - T)]^{1/2})\}$$

and

$$m_e(\lambda - T) = \min\{r \in \sigma_e([(\lambda - T)^*(\lambda - T)]^{1/2})\},$$

then we define

$$\Delta_\gamma(T) = \{\lambda \in \mathbf{C} : m_e(\lambda - T) \leqslant \gamma\} \quad (\gamma \geqslant 0).$$

If, in addition to $\sigma(T)$, $\sigma_0(T)$ and $\sigma_W(T)$, we have information about the function $m_e(\lambda - T)$ $(\lambda \in \mathbf{C})$, then the best possible estimate for the size of the compact operator K_0 $(\sigma_0(T - K_0) = \emptyset)$ is given by

$$\|K_0\| < \max\{m_e(T; \lambda) : \lambda \in \sigma_0(T)\} + \varepsilon,$$

where $m_e(T; \lambda) = \min\{\gamma \geqslant 0 : \text{dist}[\lambda, \Delta_\gamma(T)] \leqslant \gamma\}$. (Once again, exactly the same estimate holds for K_W such that $\sigma_W(T) = \sigma(T - K_W)$.)

On the other hand, in many interesting cases, the distance from a given operator T to a class of operators $\mathscr{W}$ that is invariant under similarities, but not under compact perturbations, is given by a formula of the type

$$\mathrm{dist}[T, \mathscr{W}] = \max\{\mathrm{dist}[T, \mathscr{W} + \mathscr{K}(\mathscr{H})], \delta_0(T)\},$$

where

$$\delta_0(T) := \inf\{\|B\| : B \in \mathscr{L}(\mathscr{H}), \sigma_0(T - B) = \emptyset\}.$$

(dist $[T, \mathscr{W} + \mathscr{K}(\mathscr{H})]$ is usually determined in terms of the structures of $[\mathscr{W} + \mathscr{K}(\mathscr{H})]^-$, the different pieces of the Weyl spectrum of T, and the sets $\Delta_\gamma(T)$; see [2, Chapter 12], [8].)

The above result applies, in particular, to the cases when $\mathscr{W}$ is the set of all nilpotent operators [9, Section 12.7.3], [12], or $\mathscr{W}$ is one of the Cowen-Douglas classes $\mathscr{B}_n(\Omega)$ [6] (see [13], [14, Section 5]).

The reader with some expertise in these problems will intuitively find something odd in the definition of $\delta_0(T)$: the idea of "erasing" the normal eigenvalues of T with a *non-compact perturbation* sounds atrocious!

In the first part of this article, it will be shown that (at least in the present case) intuition and rigourous analysis can go along "hand-in-hand":

$$\delta_0(T) \text{ coincides with } \inf\{\|K_0\| : K_0 \in \mathscr{K}(\mathscr{H}), \sigma_0(T - K_0) = \emptyset\}.$$

(This affirmatively answers Conjecture 3.3 of [12].)

The second result is the analogous of the estimate of [12], for the problem of "filling in the holes":

Let $\{\Omega_k\}$ be a finite or denumerable family of "holes" of $\sigma_W(T)$ and let $\varepsilon > 0$; then there exists $K \in \mathscr{K}(\mathscr{H})$ such that

$$\sigma(T - K) = \sigma(T) \cup (\bigcup_k \Omega_k)$$

and

$$\|K\| < \max\{m_e(\lambda - T) : \lambda \in \bigcup_k \Omega_k\} + \varepsilon.$$

Several related problem are analyzed, in both cases.

2. ERASING NORMAL EIGENVALUES

THEOREM 2.1. *Let $T \in \mathscr{L}(\mathscr{H})$; then*

$$\inf\{\|B\| : B \in \mathscr{L}(\mathscr{H}), \sigma_0(T - B) = \emptyset\} =$$

$$= \inf\{\|K\| : K \in \mathscr{K}(\mathscr{H}), \sigma_0(T - K) = \emptyset\}.$$

Clearly, it suffices to show that if $B \in \mathscr{L}(\mathscr{H})$, $\|B\| < C$ and $\sigma_0(T - B) = \emptyset$, then there exists $K \in \mathscr{K}(\mathscr{H})$ such that

$$\|K\| < C \quad \text{and} \quad \sigma_0(T - K) = \emptyset.$$

If $\sigma_0(T - B) = \emptyset$, we cannot expect, in general, to find a *compact* operator K such that $\|K\| = \|B\|$ and $\sigma_0(T - K) = \emptyset$. For example, if $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathscr{H}$ and $P_0 = e_0 \otimes e_0$ is the orthogonal projection onto "the first coordinate", then

$$P_0 + \left(\frac{1}{2} I - P_0\right) = \frac{1}{2} I$$

and

$$\sigma_0(P_0 - A) \cap \left\{\lambda : |\lambda - 1| < \frac{1}{2}\right\} \neq \emptyset$$

for all $A \in \mathscr{L}(\mathscr{H})$ such that $\|A\| < 1/2$. (This is an easy consequence of the continuity properties of the functional calculus; see, e.g., [9, Chapter 1]. Here $(e \otimes f)g = (g, f)e$ for all e, f, g in $\mathscr{H}$.)

Thus, if $B = -((1/2) I - P_0)$, then $\sigma_0(P_0 - B) = \emptyset$, whence we obtain $\delta_0(P_0) = \|B\| = 1/2$; but

$$\sigma_0(P_0 - K) \cap \left\{\lambda : |\lambda - 1| \leqslant \frac{1}{2}\right\} \neq \emptyset$$

for all $K \in \mathscr{K}(\mathscr{H})$ such that $\|K\| \leqslant 1/2$. Indeed, if $\|K\| \leqslant 1/2$ then for each $n = 1, 2, \ldots$, $\|(1 - 1/n)K\| < 1/2$ and we can find a point $\lambda_n \in \sigma_0(P_0 - (1 - 1/n)K) \cap \{\lambda : |\lambda - 1| < 1/2\}$. Let $\lambda_0 \in \{\lambda : |\lambda - 1| \leqslant 1/2\}$ be a limit point of the sequence $\{\lambda_n\}_{n=1}^{\infty}$; the upper semicontinuity of separate parts of the spectrum (same reference as above) implies that $\lambda_0 \in \sigma(P_0 - K)$. Since $P_0 - K$ is compact and $\lambda_0 \neq 0$, we see that $\lambda_0 \in \sigma_0(P_0 - K)$, and therefore $\sigma_0(P_0 - K) \cap \{\lambda : |\lambda - 1| \leqslant 1/2\}$ is a nonempty set.

Hence, we cannot expect $\sigma_0(P_0 - K) = \emptyset$ and $\|K\| = \delta_0(P_0)$ in this case.

However, we can easily construct a *finite rank* operator F, with $\|F\| = 1/2$, such that

$$P_0 - F = \frac{1}{2} e_0 \otimes e_0 + \sum_{j=1}^{N} r_j e_j \otimes e_j,$$

for any finite sequence $\{r_j\}_{j=1}^{N}$ such that $1/2 > r_1 > r_2 > \ldots > r_N > 0$. If the r_j's are carefully chosen, then given $\varepsilon > 0$ there exists C_ε of finite rank, with $\|C_\varepsilon\| < \varepsilon$, such that

$$Q_\varepsilon = P_0 - (F + C_\varepsilon)$$

is a finite rank *nilpotent* operator, and therefore $\sigma_0(Q_\varepsilon) = \emptyset$ [9, Chapter 2]. Thus, the *compact* operator $K_0 = F + C_\varepsilon$ satisfies

$$\|K_0\| < \frac{1}{2} + \varepsilon \quad \text{and} \quad \sigma_0(T - K_0) = \emptyset.$$

The proof of Theorem 2.1 is nothing but a glorified version of the same construction.

We shall need an auxiliary result. Following [1], we define

$$\text{min.ind}\, T = \min\{\text{nul}\, T,\ \text{nul}\, T^*\}.$$

The proof of the main result of [12] yields the following.

COROLLARY 2.2. *Let $T \in \mathscr{L}(\mathscr{H})$. Given $\varepsilon > 0$ and a finite dimensional subspace $\mathscr{M}$ of $\mathscr{H}$, there exists $K \in \mathscr{K}(\mathscr{H})$ such that*

$$\text{min.ind}(\lambda - [T - K])^k = \text{min.ind}(\lambda - T)^k$$

for all $\lambda \in \rho_{s\text{-}F}(T) \setminus \sigma_0(T)$ and all $k = 1, 2, \ldots$, $\sigma_0(T - K) = \emptyset$

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{matrix} \mathscr{M} \\ \mathscr{H} \ominus \mathscr{M} \end{matrix}, \qquad \max\{\|K_{11}\|, \|K_{12}\|, \|K_{21}\|\} < \varepsilon$$

and

$$\|K_{22}\| < \max\{m_e(\lambda - T) : \lambda \in \sigma_0(T)\} + \varepsilon.$$

Sketch of the proof. If $A \in \mathscr{L}(\mathscr{H}_1)$, $B \in \mathscr{L}(\mathscr{H}_2)$, then $A \oplus B$ is the direct sum of A and B acting in the usual fashion on the orthogonal direct sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ of $\mathscr{H}_1$ and $\mathscr{H}_2$. For each cardinal, $0 \leqslant \alpha \leqslant \infty$, $A^{(\alpha)}$ denotes the direct sum of α copies of A acting on the direct sum of α copies of the underlying space.

As in Lemma 2.10 of [12], we first consider the case when $T = \lambda_0 \oplus A^{(\infty)} \in \mathscr{L}(\mathbf{C}^1 \oplus \mathscr{H}^{(\infty)})$ $(\lambda_0 \notin \sigma(A))$. We can obviously assume that $\mathbf{C}^1 \subset \mathscr{M}$. In this case, the proof follows exactly as in that of [12, Lemma 2.10], with $\gamma_0 = m_e(T; \lambda_0)$ replaced by $\gamma_0 = m_e(\lambda_0 - T)$ $(= m(\lambda_0 - A))$; then we only have to change "$-(\lambda_0 - \mu_0) \oplus \ldots$" in page 296 of [12] (second line from the bottom) by "$0 \oplus \ldots$".

This guarantees that the modification will be small in "the coordinate of $\mathbf{C}^1$", in the sense that the modified operator, $T - K$, will satisfy $\|Ke_0\| \leqslant \varepsilon$, $\|K^* e_0\| \leqslant \varepsilon$ (where e_0 is a unit vector in $\mathbf{C}^1$). Since $\mathscr{M}$ is finite dimensional, in order to guarantee that $\|K|\mathcal{M}\|$ and $\|K^*|\mathcal{M}\|$ will be small, it suffices to leave untouched a sufficiently large number of the first direct summands of $A^{(\infty)} = A \oplus A \oplus A \oplus \dots$.

Now the general case follows exactly as in the proof of Theorem 2.1 of [12]. ▨

Proof of Theorem 2.1. Suppose $\sigma_0(T - B) = \emptyset$ for some operator B and let $C > \|B\|$ and $0 < \varepsilon < (C - \|B\|)/6$. We can (and shall) assume, without loss of generality, that $C = 1$.

Preparation. Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of finite rank orthogonal projections converging strongly to 1.

For each $\lambda \in \mathbf{C}$, we have $(T - B - \lambda) = U_\lambda H_\lambda$ (polar decomposition) with

$$m(T - B - \lambda) = \min\{r \in \sigma(H_\lambda)\}$$

and

$$m_e(T - B - \lambda) = \min\{r \in \sigma_e(H_\lambda)\}$$

$(0 \leqslant m(T - B - \lambda) \leqslant m_e(T - B - \lambda))$.

If $H_\lambda = \int t \, dE_\lambda(t)$ (spectral decomposition) and

$$F_\lambda = \int \max[m_e(T - B - \lambda) - t, 0] \, dE_\lambda(t),$$

then $U_\lambda F_\lambda$ is compact and $m(T - B - U_\lambda F_\lambda - \lambda) = m_e(T - B - \lambda)$.

For each λ, we define $C_\lambda = P_n U_\lambda F_\lambda P_n$, for some n large enough to guarantee that

$$m(T - B - C_\lambda - \lambda) > m_e(T - B - \lambda) - \varepsilon.$$

Since $\{\lambda \in \mathbf{C} : m(T - B - \lambda) \leqslant \varepsilon\}$ is a compact set, it is not difficult to deduce from the above construction that there exist a finite family of finite rank operators $\{C_i\}_{i=1}^{p}$ ($C_i = C_{\lambda_i}$ for a suitable λ_i in the above set) and an index m such that

$$C_i = P_m C_i = C_i P_m \quad \text{and} \quad \|C_i\| \leqslant \|T\| \quad \text{for all } i = 1, 2, \dots, p,$$

and

$$\max_i m(T - B - C_i - \lambda) > m_e(T - B - \lambda) - \varepsilon$$

for all λ such that $m(T - B - \lambda) \leqslant \varepsilon$.

Another elementary argument of compactness shows that

$$\|P_m(T - B - \lambda)^{-1}(1 - P_n)\| \to 0 \quad (n \to \infty),$$

uniformly for $\lambda \notin \Delta_\varepsilon(T - B)$.

First Step. Consider the operators

$$T_n = T - [B - \varepsilon(1 - P_n)B] \quad (n \geqslant m).$$

Clearly, $\|(B - \varepsilon(1 - P_n)B + K)^{\sim}\| = (1 - \varepsilon)\|\tilde{B}\|$ for all compact K. It will be shown that, for a sufficiently large n, there exists a compact operator K_1, with $\|K_1\| < (5 + 1/2)\varepsilon$, such that

$$\sigma_0(T - [B - \varepsilon(1 - P_n)B + K_1]) = \emptyset.$$

Suppose $\lambda \in \sigma_0(T_n)$. If $\lambda \in \Delta_\varepsilon(T - B)$, then $\lambda \in \Delta_{2\varepsilon}(T_n)$ because

$$\|\tilde{T}_n - (\tilde{T} - \tilde{B})\| \leqslant \|T_n - (T - B)\| = \varepsilon\|(1 - P_n)B\| \leqslant \varepsilon\|B\| < \varepsilon.$$

Assume that $\lambda \notin \Delta_\varepsilon(T - B)$, and let x be a unit vector such that $T_n x = \lambda x$; then

$$0 = (T_n - \lambda)x = (T - B - \lambda)x + \varepsilon(1 - P_n)Bx,$$

so that $(T - B - \lambda)x = -\varepsilon(1 - P_n)Bx$, whence we obtain

$$m(T - B - \lambda) \leqslant \|(T - B - \lambda)x\| \leqslant \varepsilon\|B\| < \varepsilon,$$

and

$$x = -\varepsilon(T - B - \lambda)^{-1}(1 - P_n)Bx.$$

It follows that

$$\|P_m x\| \leqslant \varepsilon\|P_m(T - B - \lambda)^{-1}(1 - P_n)\| \cdot \|B\| < \varepsilon/(\|T\| + 1)$$

provided n is large enough; that is, if $n \geqslant n(m, \varepsilon)$, then $\|P_m x\| < \varepsilon/(\|T\| + 1)$, and therefore x is "almost orthogonal" to $\operatorname{ran} P_m$.

Thus, we have

$$m_e(T - B - \lambda) < \max_i \|(T - B - C_i - \lambda)x\| + \varepsilon \leqslant$$

$$\leqslant \|(T - B - \lambda)x\| + \max_i \|C_i x\| + \varepsilon <$$

$$< \varepsilon + \|C_i\|\varepsilon/(\|T\| + 1) + \varepsilon < 3\varepsilon,$$

so that $\lambda \in \Delta_{3\varepsilon}(T - B) \subset \Delta_{4\varepsilon}(T_{n(m,\varepsilon)})$.

By Corollary 2.2, there exists $K_1 \in \mathscr{K}(\mathscr{H})$, such that

$$K_1 = \begin{pmatrix} K'_{11} & K'_{12} \\ K'_{21} & K'_{22} \end{pmatrix} \begin{matrix} \operatorname{ran} P_m \\ \ker P_m \end{matrix}, \quad \left\| \begin{pmatrix} K'_{11} & K'_{12} \\ K'_{21} & 0 \end{pmatrix} \right\| < \varepsilon/2, \quad \|K'_{22}\| < 5\varepsilon$$

and

$$\sigma_0(T_{n(m,\varepsilon)} - K_1) = \emptyset.$$

Since K_1 is compact, there exists $n_1 > n(m, \varepsilon)$ such that

$$K_1 = \begin{pmatrix} * & * & * \\ * & K_{22}'' & * \\ * & * & * \end{pmatrix} \begin{matrix} \operatorname{ran} P_m \\ \operatorname{ran}(P_{n_1} - P_m), \\ \ker P_{n_1} \end{matrix}$$

where $F_1 = (P_{n_1} - P_m)K_1(P_{n_1} - P_m) = 0 \oplus K_{22}'' \oplus 0$ and $G_1 = K_1 - F_1$ satisfy $\|F_1\| < 5\varepsilon$ and, respectively, $\|G_1\| < \varepsilon/2$.

Thus, if $Q_1 = P_m$, $R_1 = P_{n(m,\varepsilon)}$, $S_1 = P_{n_1}$ $(Q_1 \leqslant R_1 \leqslant S_1)$ and

$$B_1 = B - \varepsilon(1 - R_1)B + K_1 \quad (= [1 - \varepsilon(1 - R_1)]B + F_1 + G_1),$$

then

$$T - B_1 = T_{n(m,\varepsilon)} - K_1, \quad \sigma_0(T - B_1) = \emptyset,$$

$$\|B_1\| \leqslant \|B\| + \|K_1\| < \|B\| + \left(5 + \frac{1}{2}\right)\varepsilon,$$

and

$$\|\tilde{B}_1\| = (1 - \varepsilon)\|\tilde{B}\|.$$

Inductive Step. Repeat the same argument with B replaced by B_1 and $\{P_n\}_{n=1}^{\infty}$ replaced by $\{P_n'\}_{n=1}^{\infty}$, where $P_n' = P_{n_1+}$ $(n = 1, 2, \ldots)$.

We obtain finite rank orthogonal projections Q_2, R_2, S_2 $(Q_1 \leqslant R_1 \leqslant S_1 \leqslant \leqslant Q_2 \leqslant R_2 \leqslant S_2)$ and a compact operator K_2 such that, if

$$B_2 = B_1 - \varepsilon(1 - R_2)B + K_2$$

$$(= [R_1 - \varepsilon(R_2 - R_1) - 2\varepsilon(1 - R_2)]B + F_1 + G_1 + K_2),$$

then

$$K_2 = F_2 + G_2, \quad \text{where } F_2 = (S_2 - Q_2)K_2(S_2 - Q_2),\ G_2 = K_2 - F_2,$$

$$\|F_2\| < 5\varepsilon, \quad \|G_2\| < \varepsilon/4,$$

$$\|B_2\| \leqslant \|B\| + \max\{\|F_1\|, \|F_2\|\} + \|G_1\| + \|G_2\| < \|B\| + \left(5 + \frac{1}{2} + \frac{1}{4}\right)\varepsilon$$

(because $F_1 = (S_1 - Q_1)F_1 = F_1(S_1 - Q_1)$ and $F_2 = (S_2 - Q_2)F_2 = F_2(S_2 - Q_2)$, so that $\|F_1 + F_2\| = \max\{\|F_1\|, \|F_2\|\}$), $\|\widetilde{B}_2\| = (1 - 2\varepsilon)\|\widetilde{B}\|$, and $\sigma_0(T - B_2) = \emptyset$.

By replacing, if necessary, ε by some ε', $0 < \varepsilon' \leqslant \varepsilon$, we can assume that $1/\varepsilon = N$ is an integer. After N steps, a formal inductive repetition of the same argument will produce finite rank orthogonal projections $\{Q_j\}_{j=1}^N$, $\{R_j\}_{j=1}^N$ and $\{S_j\}_{j=1}^N$ with

$$Q_1 \leqslant R_1 \leqslant S_1 \leqslant Q_2 \leqslant R_2 \leqslant S_2 \leqslant \ldots \leqslant Q_N \leqslant R_N \leqslant S_N,$$

and compact operators $K_1, K_2, \ldots, K_N$ satisfying

$$K_j = F_j + G_j, \quad F_j = (S_j - Q_j)K_j(S_j - Q_j), \quad G_j = K_j - F_j,$$

$$\|F_j\| < 5\varepsilon, \quad \|G_j\| < \varepsilon/2^j \quad (j = 1, 2, \ldots, N),$$

such that, if

$$B_N = \left[R_1 - \sum_{j=2}^{N} (j-1)\varepsilon(R_j - R_{j-1}) \right] B + \sum_{j=1}^{N} K_j,$$

then

$$\sigma_0(T - B_N) = \emptyset.$$

It is completely apparent that B_N is a compact operator, and

$$\|B_N\| \leqslant \|B\| + \max_j \|F_j\| + \sum_{j=1}^{N} \|G_j\| <$$

$$< \|B\| + 5\varepsilon + \sum_{j=1}^{N} \varepsilon/2^j < \|B\| + 6\varepsilon < 1.$$

Thus, the operator $K = B_N$ satisfies all our requirements.

We conclude that

$$\delta_0(T) = \inf\{\|K\| : K \in \mathscr{K}(\mathscr{H}),\ \sigma_0(T - K) = \emptyset\}. \qquad ▨$$

In certain interesting cases, the distance formula from T to a similarity-invariant class of operators involves, not the removal of all the normal eigenvalues, but only those in a certain region of the plane (with respect to the essential spectrum of T). For instance, it may be necessary to remove normal eigenvalues in $\sigma_0(T) \setminus \sigma_e(T)^\wedge$, where $\sigma_e(T)^\wedge$ is the *polynomial hull* of $\sigma_e(T)$ (= the complement of the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$ = the union of $\sigma_e(T)$ and all its holes; see [10], [11], [16]).

Ad hoc modification of the proof of Theorem 2.1 produce analogous results for these special problems. For instance, we have

COROLLARY 2.3. *Let $T \in \mathscr{L}(\mathscr{H})$; then*

$$\inf\{\|B\| : B \in \mathscr{L}(\mathscr{H}),\ \sigma_0(T-B)\setminus\sigma_e(T-B)^\wedge = \emptyset\} =$$

$$= \inf\{\|K\| : K \in \mathscr{K}(\mathscr{H}),\ \sigma_0(T-K)\setminus\sigma_e(T)^\wedge = \emptyset\}.$$

3. FILLING IN THE HOLES

THEOREM 3.1. *Let $T \in \mathscr{L}(\mathscr{H})$ and let Φ be the union of a collection of bounded components of $\sigma(T)\setminus\sigma_0(T)$. Given $\varepsilon > 0$ there exists $K \in \mathscr{K}(\mathscr{H})$, with*

$$\|K\| < \max\{m_e(\lambda - T) : \lambda \in \Phi\} + \varepsilon$$

such that

$$\sigma(T-K) = \sigma(T)\cup\Phi, \quad \sigma_0(T-K) = \sigma_0(T)\setminus\Phi,$$

$$\mathrm{nul}(\lambda - [T-K]) = \mathrm{nul}(\lambda - [T-K])^* = 1 \quad \textit{for all } \lambda \in \Phi,$$

$$\mathrm{min.ind}(\lambda - [T-K])^k = \mathrm{min.ind}(\lambda - T)^k \quad \textit{for all } \lambda \in \rho_{s\text{-}F}(T)\setminus\Phi$$

and all $k = 1, 2, \ldots,$ and

$$T-K \mid \bigvee\{\mathscr{H}(T-K;\lambda) : \lambda \in \sigma_0(T-K)\}$$

is unitarily equivalent to

$$T \mid \bigvee\{\mathscr{H}(T;\lambda) : \lambda \in \sigma_0(T-K)\}.$$

(Here $\bigvee$ denotes "the closed linear span of".)

The proof follows by a combination of the arguments of [12] and the result of Lemma 3.2 below. Lemma 3.2 appears "intuitively true", and it would be highly desirable to have a more elementary proof, but the author was unable to find it.

LEMMA 3.2. *Let $\Omega = \mathrm{interior}\,\Omega^-$ be a bounded open subset of* **C**. *Given $\varepsilon > 0$ there exist $A, N \in \mathscr{L}(\mathscr{H})$ such that $[A^*, A] := A^*A - AA^*$ is compact, N is normal, $\sigma(A) = \Omega^-$, $\sigma(N) \subset \Omega^-$, $\sigma_e(A) = \sigma_e(N) = \partial\Omega$, $\mathrm{nul}(\lambda - A) = \mathrm{nul}(\lambda - A)^* = 1$ for all $\lambda \in \Omega$, $A - N$ is compact, and $\|A - N\| < \varepsilon$.*

Proof. First we shall consider the case when Ω is connected. Given $\varepsilon > 0$, there exists a subnormal operator $S = S(\Omega)$ with $\sigma(S) = \Omega^-$ and $\sigma_e(S) = \partial\Omega$ such that $[S^*, S]$ is compact, $\|[S^*, S]\| < \eta$ and $\mathrm{ind}(\lambda - S) = -\mathrm{nul}(\lambda - S)^* = -1$ for all $\lambda \in \Omega$. (This is Lemma 5.2 of [7].)

If $\Omega^* = \{\bar{\lambda} : \lambda \in \Omega\}$, $S(\Omega^*)$ is the operator constructed in [7, Lemma 5.2] with Ω replaced by Ω^*, and $B = S(\Omega) \oplus S(\Omega^*)^*$, then $\sigma(B) = \Omega^-$, $\sigma_e(B) = \partial\Omega$, $[B^*, B]$ is compact, $\|[B^*, B]\| < \eta$ and $\mathrm{nul}(\lambda - B) = \mathrm{nul}(\lambda - B)^* = 1$ for all $\lambda \in \Omega$. It follows from the Brown-Douglas-Fillmore theorem that $B = M + K$, for some normal M and some K compact [4], [5]; moreover, M can be chosen equal to a diagonal normal operator of uniform infinite multiplicity such that $\sigma(M) = \sigma_e(M) = \partial\Omega$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$ such that $Me_n = \lambda_n e_n$ $(n = 1, 2, \ldots)$ for a suitable sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\{\lambda_n\}^- = \partial\Omega$, and let P_n denote the orthogonal projection of $\mathscr{H}$ onto $\bigvee \{e_j\}_{j=1}^{n}$. Since K is compact, $\|K - P_nKP_n\| \to 0$ $(n \to \infty)$, and therefore $B_n = M + P_nKP_n$ satisfies $\|[B^*, B]\| < 2\eta$, provided n is large enough.

Observe that $B_n = (P_n(M + K)|\mathrm{ran}\, P_n) \oplus M|\ker P_n$ ($M|\ker P_n$ is normal). It follows that $C_n = P_n(M + K)|\mathrm{ran}\, P_n$ acts on a finite dimensional space and satisfies $\|[C_n^*, C_n]\| < 2\eta$. Furthermore, since $S(\Omega)$ is subnormal, we see that $\|(\lambda - B)^{-1}\| = (\mathrm{dist}[\lambda, \Omega])^{-1}$ for all $\lambda \notin \Omega^-$, whence it readily follows that $\sigma(C_n) \subset \Omega_\eta := \{\lambda \in \mathbf{C} : \mathrm{dist}[\lambda, \Omega] \leqslant \eta\}$ and $\|(\lambda - C_n)^{-1}\| \leqslant (\mathrm{dist}[\lambda, \Omega_\eta])^{-1}$ for all $\lambda \notin \Omega_\eta$, for all n large enough.

Fix n so that all these conditions are satisfied. It follows from [4] (see especially the beginning of Section 5.7) that there exist normal operators N_n and M_n, acting on finite dimensional spaces, such that $\sigma(N_n) \cup \sigma(M_n) \subset \Omega^-$ and

$$\|C_n \oplus N_n - M_n\| < f(\eta) \to 0 \quad (\eta \to 0).$$

Thus, if η is small enough, then $\|C_n \oplus N_n - M_n\| < \varepsilon/2$. If $A_\eta = B_n \oplus N_n$, then $A_\eta = S(\Omega) \oplus [S(\Omega^*)^* \oplus N_n] + G_n$, where G_n is a compact operator such that $\|G_n\| \to 0$ $(n \to \infty)$. By using [1], (or [9, Chapter 3]), we can find a finite rank operator F_η of arbitrarily small norm ($\|F_\eta\| < \min[\varepsilon/2 - \|G_n\|, \eta]$) such that F_η is reduced, and equal to 0, on the subspace correspondin g to the direct summand $S(\Omega)$, $F_\eta = 0 \oplus F'_\eta$, and

$$A = S(\Omega) \oplus [S(\Omega^*)^* \oplus N_n + F'_\eta]$$

satisfies $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$, $\mathrm{nul}(\lambda - A) = \mathrm{nul}(\lambda - A)^* = 1$ for all $\lambda \in \Omega$, and $\|(\lambda - A)^{-1}\| \leqslant (\mathrm{dist}[\lambda, \Omega_\varepsilon])^{-1}$ for all $\lambda \notin \Omega_\varepsilon$.

If N is the normal operator $M_n \oplus M|\ker P_n$, then

$$\|A - N\| \leqslant \|F_\eta\| + \|G_n\| + \|C_n \oplus N_n - M_n\| < \varepsilon.$$

This completes the proof for the case when Ω is connected. If $\Omega = \bigcup_k \Omega_k$ (disjoint union of the components of Ω), then for each Ω_k we can construct A_k and

N_k as above, with $\|(\lambda - A_k)^{-1}\| \leqslant (\mathrm{dist}[\lambda, (\Omega_k)_{\varepsilon_k}])^{-1}$ for all $\lambda \notin \Omega_{\varepsilon_k}$ $(k = 1, 2, \ldots)$. It easily follows that $A = \underset{k}{\oplus} A_k$ and $N = \underset{k}{\oplus} N_k$ satisfy all our requirements. ▨

REMARK. Let A and N be as in Lemma 3.2. Since N is normal and $\sigma(N) \subset \Omega^-$, the inequality $\|A - N\| < \varepsilon$ automatically implies that $\sigma(A) \subset \Omega_\varepsilon$. Since $\sigma(A) = \Omega^-$, the same inequality implies that $\sigma(N)_\varepsilon \supset \Omega^-$. (Use the fact that $\|(\lambda - N)^{-1}\| = (\mathrm{dist}[\lambda, \sigma(N)])^{-1}$ for all $\lambda \notin \sigma(N)$.) Thus, the normal eigenvalues of N "flood" Ω (within ε).

Proof of Theorem 3.1. First Perturbation. Let $R = \rho(\tilde{T}) \in \mathscr{L}(\mathscr{H}_\rho)$, where ρ is a faithful unital $*$-representation of the C^*-algebra $C^*(\tilde{T})$ generated by $\tilde{T}$ and $\tilde{1}$ on a separable Hilbert space $\mathscr{H}_\rho$. By replacing, if necessary, ρ by $\rho^{(\infty)}$, we can directly assume that R is unitarily equivalent to $R^{(\infty)}$. By Voiculescu's theorem, there exists $K_1 \in \mathscr{L}(\mathscr{H})$, with $\|K_1\| < \varepsilon/12$, such that $T - K_1 = U(T \oplus R_0 \oplus R_1 \oplus R_2 \oplus \ldots)U^*$, where R_n is unitarily equivalent to R for all $n = 0, 1, 2, \ldots$, and U is a unitary mapping from $\mathscr{H} \oplus \mathscr{H}_\rho^{(\infty)}$ onto $\mathscr{H}$ [19]. Moreover, R can be chosen so that

$$R = \begin{pmatrix} M & C \\ 0 & B \end{pmatrix},$$

where M is a diagonal normal operator of uniform infinite multiplicity, $\sigma(M) = \sigma_e(M) = \partial(\Phi^-)$ and $\sigma(B) = \sigma_e(B) = \sigma_e(T)$ (see, e.g., [9, Chapter 4]).

By Lemma 3.2 (and its proof), there exist a normal operator M_0, acting on a finite dimensional space, with $\sigma(M_0) \subset \Omega :=$ interior Φ^-, and an operator A such that $[A^*, A]$ and $A - M \oplus M_0$ are compact, $\sigma(A) = \Omega^- = \Phi^-$, $\sigma_e(A) = \sigma(M) = \partial\Omega$, $\mathrm{nul}(\lambda - A) = \mathrm{nul}(\lambda - A)^* = 1$ for all $\lambda \in \Omega$, and $\|A - M \oplus M_0\| < \varepsilon/12$.

Suppose $M_0 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ with respect to some orthonormal basis of its underlying space, and consider the operator

$$R_0 \oplus R_1 \oplus R_2 \oplus \ldots \oplus R_m = \begin{pmatrix} M & C \\ 0 & B \end{pmatrix} \oplus \left\{ \underset{j=1}{\overset{m}{\oplus}} R_j \right\}.$$

Let $\gamma_0 = \max\{m_e(\lambda - T) : \lambda \in \Phi\}$. Since $\lambda_j \in \Phi^-$, and $\|(\lambda - R)^{-1}\| = \|(\lambda - \tilde{R})^{-1}\| = m_e(\lambda - T)^{-1}$ for all $\lambda \notin \sigma_e(T)$, it readily follows that $m_e(\lambda_j - R_j) = m_e(\lambda_j - T) \leqslant \gamma_0$ for all $j = 1, 2, \ldots, m$. By proceeding as in [12, proof of Lemma 2.10], we can find compact perturbations C_j ($j = 1, 2, \ldots, m$; C_j acts on the space of R_j) such that

$$\|C_j\| < \gamma_0 + \varepsilon/12, \quad \lambda_j \in \sigma_0(R_j - C_j) \quad (j = 1, 2, \ldots, m).$$

Thus, if $K_2 = U^*\left(0 \oplus 0 \oplus \left\{\bigoplus_{j=1}^{m} C_j\right\} \oplus 0 \oplus 0 \oplus \ldots\right)U$, then $K_2 \in \mathscr{K}(\mathscr{H})$, $\|K_2\| = \max_j\|C_j\| < \gamma_0 + \varepsilon/12$, and

$$T - (K_1 + K_2) + U\left[T \oplus \begin{pmatrix} M & C \\ 0 & B \end{pmatrix} \oplus \left\{\bigoplus_{j=1}^{m} \begin{pmatrix} \lambda_j & G_j \\ 0 & H_j \end{pmatrix}\right\} \oplus R_{m+1} \oplus R_{m+2} \oplus \cdots\right]U^* =$$

$$= U\left[T \oplus \begin{pmatrix} M \oplus \left\{\bigoplus_{j=1}^{m} \lambda_j\right\} & C \oplus \left\{\bigoplus_{j=1}^{m} G_j\right\} \\ 0 & B \oplus \left\{\bigoplus_{j=1}^{m} H_j\right\} \end{pmatrix} \oplus R_{m+1} \oplus R_{m+2} \oplus \cdots\right]U^*,$$

where λ_j acts on a space of dimension one for all $j = 1, 2, \ldots, m$. Therefore, $M \oplus \oplus \left\{\bigoplus_{j=1}^{m} \lambda_j\right\}$ is unitarily equivalent to $M \oplus M_0$ and there exists a compact operator K_3, with $\|K_3\| < \varepsilon/12$, such that

$$T - (K_1 + K_2 + K_3) = U\left[T \oplus \begin{pmatrix} A' & C \oplus \left\{\bigoplus_{j=1}^{m} G_j\right\} \\ 0 & B \oplus \left\{\bigoplus_{j=1}^{m} H_j\right\} \end{pmatrix} \oplus R_{m+1} \oplus R_{m+2} \oplus \cdots\right]U^*,$$

where A' is unitarily equivalent to A. The compact operator K_3 is reduced by the image under U of the space of $M \oplus \left\{\bigoplus_{j=1}^{m} \lambda_j\right\}$, and $K_3 = 0$ on the orthogonal complement of this subspace. Thus, if $\mathscr{H}_1 = \bigvee \{\mathscr{H}(T;\lambda) : \lambda \in \sigma_0(T) \setminus \Phi^-\}$, then $T|\mathscr{H}_1$ is unitarily equivalent to $U^*[T - (K_1 + K_2 + K_3)]U|\mathscr{R}_1$ for a suitable subspace $\mathscr{R}_1$ of $\mathscr{H} \oplus \mathscr{H}_\rho^{(\infty)}$ invariant under $U^*[T - (K_1 + K_2 + K_3)]U$.

Second Perturbation. The operator $T|\mathscr{H}_1$ admits an upper triangular matrix with respect to some orthonormal basis of $\mathscr{H}_1$) of the form

$$T|\mathscr{H}_1 = \begin{pmatrix} \mu_1 & & & * \\ & \mu_2 & & \\ & & \mu_3 & \\ 0 & & & \ddots \end{pmatrix},$$

where $\operatorname{card}\{j : \mu_j = \lambda\} = \dim\mathscr{H}(T;\lambda)$ for each $\lambda \in \sigma_0(T) \setminus \Phi$; furthermore, if

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H} \ominus \mathscr{H}_1 \end{matrix},$$

then $\sigma(T_1)\cup\sigma(T_2)=\sigma(T)$ (because $\sigma(T_1)$ coincides with the left spectrum of T_1, $\sigma_l(T_1)$, and therefore $\sigma(T_1)=\sigma(T|\mathscr{H}_1)\subset\sigma(T)$), and $\sigma_0(T)\setminus\sigma_0(T_1)\subset\sigma_0(T_2)$ (see [9, Corollary 3.4] and [15, Proposition 4]). Thus, we have

$$T-(K_1+K_2+K_3)=U\left\{\begin{pmatrix}T_1 & T_{12}\\ 0 & T_2\end{pmatrix}\oplus\begin{pmatrix}A' & C\oplus\left\{\bigoplus_{j=1}^{m}G_j\right\}\\ 0 & B\oplus\left\{\bigoplus_{j=1}^{m}H_j\right\}\end{pmatrix}\oplus\left[\bigoplus_{j=m+1}^{\infty}R_j\right]\right\}U^*=$$

$$=U\left\{\begin{pmatrix}T_1 & 0 & T_{12} & 0\\ 0 & A' & 0 & C\oplus\left\{\bigoplus_{j=1}^{m}G_j\right\}\\ 0 & 0 & T_2 & 0\\ 0 & 0 & 0 & B\oplus\left\{\bigoplus_{j=1}^{m}H_j\right\}\end{pmatrix}\oplus\left[\bigoplus_{j=m+1}^{\infty}R_j\right]\right\}U^*=$$

$$=U\left\{\begin{pmatrix}T_1\oplus A' & T_{12}\oplus C\oplus\left\{\bigoplus_{j=1}^{m}G_j\right\}\\ 0\oplus 0 & T_2\oplus B\oplus\left\{\bigoplus_{j=1}^{m}H_j\right\}\end{pmatrix}\oplus\left[\bigoplus_{j=m+1}^{\infty}R_j\right]\right\}U^*.$$

By construction, $\sigma_0(T_2)\cap\sigma_0(T)=\sigma_0(T)\cap\Phi$, $\sigma_0(B)=\emptyset$, and $\sigma_0(R_j)=\emptyset$ (because $B\simeq B^{(\infty)}$ and $R_j\simeq R\simeq R^{(\infty)}$) and

$$\sigma_0\left(\bigoplus_{j=1}^{m}H_j\right)\subset\{\lambda\in\mathbf{C}:m(\lambda-R)\leqslant\max_j\|C_j\|\}\subset\Delta_{\gamma_0+\varepsilon/12}(T).$$

Indeed, if $\lambda\in\sigma_0(R_j-C_j)$ and x is a unit vector such that $(R_j-C_j)x=x$, then

$$m_e(\lambda-R)=m(\lambda-R_j)\leqslant\|C_jx\|\leqslant\|C_j\|<\gamma_0+\varepsilon/12,$$

o that $\lambda\in\Delta_{\gamma_0+\varepsilon/12}(R)=\Delta_{\gamma_0+\varepsilon/12}(T)$ for all $\lambda\in\sigma_0(H_j)$ $(j=1,2,\dots,m)$.

According to [1], [9, Chapter 3], there exists a compact operator F_j, with $\|F_j\|<\varepsilon/12$, such that $H_j'=H_j-F_j$ satisfies $\sigma_0(H_j')=\sigma_0(H_j)$ and $\min.\mathrm{ind}(\lambda-H_j')=$ $=0$ for $\lambda\in\rho_{s\text{-}F}(H_j)\setminus\sigma_0(H_j)$. Now we can write

$$T_2\oplus\left\{\bigoplus_{j=1}^{m}H_j'\right\}=\begin{pmatrix}T_3 & *\\ 0 & T_4\end{pmatrix}\begin{matrix}\mathscr{H}_2\\ \mathscr{H}_2'\end{matrix},$$

where $\mathscr{H}_2=\bigvee\left\{\mathscr{H}\left(T_2\oplus\left\{\bigoplus_{j=1}^{m}H_j\right\};\lambda\right):\lambda\in\sigma_0\left(T_2\oplus\left\{\bigoplus_{j=1}^{m}H_j'\right\}\right)\cap\Phi\right\}.$

The proof of Lemma 3.2 shows that $A' = A_+ \oplus A_-$, where $A_+ \in \mathcal{L}(\mathcal{H}_+)$, $A_- \in \mathcal{L}(\mathcal{H}_-)$, $\sigma(A_+) = \sigma(A_-) = \Omega^- = \Phi^-$, $\sigma_e(A_+) = \sigma_e(A_-) = \partial\Omega$, and $-\mathrm{ind}(\lambda - A_+) = \mathrm{nul}(\lambda - A_+)^* = \mathrm{ind}(\lambda - A_-) = \mathrm{nul}(\lambda - A_-) = 1$ for all $\lambda \in \Omega$. By using, once again, the results of [1], [9], we can find $F_0 \in \mathcal{K}(\mathcal{H}_- \oplus \mathcal{H}_2)$, with $\|F_0\| < \varepsilon/12$, such that

$$\mathrm{nul}(\lambda - [A_- \oplus T_3 - F_0]) = 1$$

for all $\lambda \in \Phi$, and

$$\mathrm{nul}(\lambda - [A_- \oplus T_3 - F_0])^* = 0$$

for all $\lambda \in \rho_{s\text{-}F}(A_- \oplus T_3)$. (To see this, use the fact that T_3 is a triangular operator and [9, Corollary 3.40] or [15, Proposition 4].)

Therefore, there exists $K_4 \in \mathcal{K}(\mathcal{H})$, with

$$\|K_4\| < (\max_j \|F_j\|) + \|F_0\| < 2\varepsilon/12,$$

such that $\mathcal{R}_1$ reduces K_4, $K_4|\mathcal{R}_1 = 0$, and

$$T - (K_1 + K_2 + K_3 + K_4) =$$

$$= U\left\{\begin{pmatrix} T_1 \oplus A_+ & * & * \\ 0 & (A_- \oplus T_3 - F_0) & * \\ 0 & 0 & T_4 \oplus B \end{pmatrix} \oplus [\bigoplus_{j=m+1}^{\infty} R_j]\right\} U^* =$$

$$= U\begin{pmatrix} L & * \\ 0 & T_4 \oplus [\bigoplus_{j=m+1}^{\infty} R_j] \end{pmatrix} U^*,$$

where

$$L = \begin{pmatrix} T_1 \oplus A_* & * & * \\ 0 & (A_- \oplus T_3 - F_0) & * \\ 0 & 0 & B \end{pmatrix}.$$

Third Perturbation. The operator $T_5 = T - (K_1 + K_2 + K_3 + K_4)$ is a compact perturbation of T, and therefore it satisfies $\rho_{s\text{-}F}(T_5) = \rho_{s\text{-}F}(T)$ and $\mathrm{ind}(\lambda - T_5) = \mathrm{ind}(\lambda - T)$ for all $\lambda \in \rho_{s\text{-}F}(T)$; moreover,

(a') $\sigma(T_5) = \sigma(L) \cup \sigma(T_4) = \sigma(T) \cup \Phi$, and $\sigma_0(T_5) = [\sigma_0(T) \setminus \Phi] \cup \sigma_0(T_4)$,

(b) if $\lambda \in \Phi$, then $\lambda \notin \sigma(T_1)$ and $\mathrm{nul}(\lambda - T_5) = \mathrm{nul}(\lambda - T_5)^* = 1$,

(c) $U\mathscr{R}_1$ is invariant under T_5 and $T_5|U\mathscr{R}_1$ is unitarily equivalent to $T|\mathscr{H}_1$.

(d′) if $\lambda \in \rho_{s\text{-}F}(T)\setminus\left[\Phi \cup \sigma_0\left(\bigoplus_{j=1}^{m} H_j'\right)\right]$, then $\text{min.ind}(\lambda - T_5)^k = \text{min.ind}(\lambda - T)^k$ for all $k = 1, 2, \ldots$, and

(e′) if $\lambda \in \sigma_0\left(\bigoplus_{j=1}^{m} H_j'\right) = \sigma_0\left(\bigoplus_{j=1}^{m} H_j\right)$, then $\text{min.ind}(\lambda - T_5)^k = \text{min.ind}(\lambda - T)^k + \text{nul}\left(\lambda - \bigoplus_{j=1}^{m} H_j'\right)^k$ for all $k = 1, 2, \ldots$.

On the other hand, if $\mathscr{H}_L$ denotes the space of L and P_L is the orthogonal projection onto $U\mathscr{H}_L$, then the compact operator $K_1 + K_2 + K_3 + K_4$ satisfies

$$\|K_1 + K_2 + K_3 + K_4\| < \gamma_0 + 5\varepsilon/12$$

and

$$\max\{\|(1 - P_L)(K_1 + K_2 + K_3 + K_4)\|, \quad \|(K_1 + K_2 + K_3 + K_4)(1 - P_L)\|\} < 5\varepsilon/12,$$

Since $\sigma_0\left(\bigoplus_{j=1}^{m} H_j'\right) \subset \Delta_{\gamma_0+\varepsilon/12}(R)$, by applying Corollary 2.2, we can find $K_5' \in \mathscr{K}(\mathscr{H}_L)$, with

$$\|K_5'\| < \gamma_0 + 2\varepsilon/12,$$

such that

$$\sigma_0\left(T_4 \oplus \left[\bigoplus_{j=m+1}^{\infty} R_j\right] - K_5'\right) = \emptyset$$

and

$$\text{min.ind}\left(\lambda - \left\{T_4 \oplus \left[\bigoplus_{j=m+1}^{\infty} R_j\right] - K_5'\right\}\right)^k = \text{min.ind}(\lambda - T_4)^k$$

for all $\lambda \in \rho_{s\text{-}F}(T) \setminus \sigma_0(T_4)$ and all $k = 1, 2, \ldots$.

Let $K_5 = U\{0(\text{on } \mathscr{H}_L) \oplus K_5'\}U^*$ and $K = K_1 + K_2 + K_3 + K_4 + K_5$; then K_5, K are compact operators,

$$K = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{matrix} U\mathscr{H}_L \\ U\mathscr{H}_L^{\perp} \end{matrix},$$

$$\|K\| \leqslant \left\|\begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix}\right\| = \max\{\|D_{11}\|, \|D_{22}\|\} + \max\{\|D_{12}\|, \|D_{21}\|\} <$$

$$< (\gamma_0 + 7\varepsilon/12) + 5\varepsilon/12 = \max\{m_e(\lambda - T) : \lambda \in \Phi\} + \varepsilon,$$

and $T - K$ satisfies

(a) $\sigma(T - K) = \sigma(T) \cup \Phi$, $\sigma_0(T - K) = \sigma_0(T) \setminus \Phi$,

(b) if $\lambda \in \Phi$, then $\mathrm{nul}(\lambda - [T - K]) = \mathrm{nul}(\lambda - [T - K])^* = 1$,

(c) $U\mathscr{R}_1 = \bigvee \{\mathscr{H}(T - K; \lambda) : \lambda \in \sigma_0(T - K)\}$ is invariant under $T - K$ and $T - K | U\mathscr{R}_1$ is unitarily equivalent to $T|\mathscr{H}_1$, and

(d) if $\lambda \in \rho_{s\text{-}F}(T) \setminus \Phi$, then

$$\text{min.ind}(\lambda - [T - K])^k = \text{min.ind}(\lambda - T)^k$$

for all $k = 1, 2, \ldots$.

The proof of Theorem 3.1 is now complete. ▨

The same argument can be applied to modify the behavior of T on components of $\rho_{s\text{-}F}(T)$ where the index is not zero. We can also combine these arguments with the results of [12] in order to prove results of the following kind :

THEOREM 3.3. *Let $T \in \mathscr{L}(\mathscr{H})$ and let Φ be a union of bounded components of $\sigma(T) \setminus \sigma_0(T)$. Given $p \geqslant 1$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$, with*

$$\|K\| < \max[\max\{m_e(T; \lambda) : \lambda \in \sigma_0(T) \setminus \Phi\}, \max\{m_e(\lambda - T) : \lambda \in \Phi\}] + \varepsilon$$

such that

$$\sigma(T - K) = [\sigma(T) \setminus \sigma_0(T)] \cup \Phi, \quad \sigma_0(T - K) = \emptyset,$$

and for each $\lambda \in \rho_{s\text{-}F}(T)$,

$$\text{min.ind}(\lambda - [T - K])^k = \begin{cases} 0, & \text{if } \lambda \in \sigma_0(T), \\ \text{min.ind}(\lambda - T)^k, & \text{if } \lambda \in \rho_{s\text{-}F}(T) \setminus [\sigma_0(T) \cup \Phi], \\ \text{min.ind}(\lambda - T)^k + kp, & \text{if } \lambda \in \Phi, \end{cases}$$

for all $k = 1, 2, \ldots$.

This research was partially supported by a Grant of N.S.F.

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Received November 4, 1986; revised February 13, 1987.