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A TALE OF THREE C^*-ALGEBRAS

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1. INTRODUCTION

It was the best of C^*-algebras, it was the worst of C^*-algebras, it contained both. The three C^*-algebras to be discussed here are AF_θ, a particular approximately finite C^*-algebra introduced by E. G. Effros and C. L. Shen in [5], Irr_θ the well known irrational rotation C^*-algebra, and a third C^*-algebra C_θ introduced by J. Cuntz [3]. M. Pimsner and D. Voiculescu showed in [14] that Irr_θ could be embedded into AF_θ. Probably the main use of this embedding has been that allied with the work of M. Rieffel [15] it enabled one to show that the range of the unique trace on $\mathrm{K}_0(\mathrm{Irr}_\theta)$ is $\mathbf{Z} + \mathbf{Z}\theta$. A. Kumjian has also shown that AF_θ can be embedded in Irr_θ [9]. The third C^*-algebra C_θ, which contains Irr_θ, was used with the six term exact sequence for the K-groups [13] to give an alternative computation of the trace on $\mathrm{K}_0(\mathrm{Irr}_\theta)$. In this paper we detail what we feel is the most natural relationship between these three C^*-algebras. Firstly C_θ also naturally contains AF_θ. Actually it is gotten from it by adjoining the rotation operator U_0, while C_θ is gotten from Irr_θ by adjoining certain periodic maps U_{θ_k} which naturally approximate the rotation operator. And the closeness of the C^*-algebras, is seen in the, fact that their spectra are almost the same.

In [12], M. Pimsner gives necessary and sufficient conditions for a covariance algebra to be embeddable in an AF algebra. Unfortunately, to my knowledge no one has yet succeeded in using this for C^*-algebras that are somehow naturally related. For this reason we have chosen a route to our results that is not the shortest. There are other examples of AF algebras and covariance algebras that are naturally associated [11], but one will in general be given only one of these, and will want to find the other. So we first start with AF_θ and show how Irr_θ can be naturally associated to it. Since the range of the trace on $\mathrm{K}_0(\mathrm{Irr}_\theta)$ aroused such interest, we then show how this relationship is sufficient to allow its computation. Then we proceed from Irr_θ and find AF_θ. At this stage it is even easier to calculate the range of the trace, and in fact one does not actually even explictly need AF_θ. In the course of this we arrive at a dynamical way of describing the continued fraction expansion for an irrational number. This is presumably well known to people in number theory, but we know of no explicit reference. It does immediately give the known results on partitioning the interval, see [15] for example. But the main thing that it makes absolutely clear is why Irr_θ must have associated with it an AF algebra whose K_0 group is dicyclic. Finally this approach leads to Denjoy's theorem on the conjugacy of minimal homeomorphisms of the unit circle [4], and an alternative definition of rotation number. Since this method of approaching the theorem is "folklore", we omit the details. However it is worth noting that since something more than rotation numbers is necessary in higher dimensions this sort of approach via AF algebras might be useful there.

One way of interpreting our results is as follows. A UHF algebra may be considered as a crossed product algebra [18]. An arbitrary AF algebra may be considered as a representation of a cross product algebra, but clearly the choice of groups is critical. This is where going from AF_θ to Irr_θ (or to C_θ) is useful. Essentially a complicated non abelian group is replaced by **Z**.

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2. THE APPROXIMATELY FINITE ALGEBRA

As mentioned previously, the particular C^*-algebra, AF_θ, in which we are interested is the one described by E. G. Effros and C. L. Shen [5]. Briefly we recall the construction. If θ has $[c_0, c_1, \dots]$ as its continued fraction expansion, hen θ is approximated by a sequence of rationals p_n/q_n where $p_{n+1} = c_n p_n + p_{n-1}$, $q_{n+1} = c_n q_n + q_{n-1}$, and the sequences start with $p_0 = c_0$, $p_{-1} = 1$, $q_0 = 1$, $q_{-1} = 0$. We consider an embedding φ_n of $M_{q_n} \oplus M_{q_{n-1}}$ into $M_{q_{n+1}} \oplus M_{q_n}$ as follows

$$a \oplus b \to \underbrace{a \oplus \dots \oplus a}_{c_n \text{ times}} \oplus b \oplus a.$$

We then define AF_θ to be the direct limit of this sequence of C^*-algebras, [2], $AF_\theta = \varinjlim \mathscr{A}_n$ where $\mathscr{A}_n = M_{q_n} \oplus M_{q_{n-1}}$.

We intend to look for a copy of Irr_θ in AF_θ in the most naive way possible. Now, if S^1 is the unit circle, and if μ is Haar measure, then Irr_θ is just the C^*-algebra of operators on $L^2(S^1, \mu)$, generated by the multiplication operators M_f, for f in $C(S^1)$, and the unitary operator U_θ which translates the functions. Now as Strătilă and Voiculescu point out in [17], inside any AF algebra there is a natural abelian C^*-algebra, and a group of unitaries acting on it. So we will look for a copy of $C(S^1)$ inside the abelian algebra, and look for U_θ in the group of unitaries.

Using a maximal abelian self-adjoint subalgebra (m.a.s.a.) to break down the structure of an AF algebra was successfully employed in [6], to study the CAR algebra, and was used in [17] to study general AF algebras. Let us recall how the structures in [17] apply in the case of AF_θ. We have in Bratteli diagram form [2],

$$\begin{array}{ccccc} \mathscr{A}_n & = & M_{q_n} & \oplus & M_{q_{n-1}} \\ \Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle c_n} & \swarrow\!\!\searrow & \\ \mathscr{A}_{n+1} & = & M_{q_{n+1}} & \oplus & M_{q_n}. \end{array}$$

We can inductively choose a m.a.s.a. $\mathscr{D}_n$ in each $\mathscr{A}_n$, such that $\mathscr{D}_n$ contains $\varphi_{n-1}(\mathscr{D}_{n-1})$. Again schematically, if D_n is a diagonal subalgebra in M_n, we have,

$$\begin{array}{ccccc} \mathscr{D}_n & = & D_{q_n} & \oplus & D_{q_{n-1}} \\ \Big\downarrow{\scriptstyle \varphi_n} & & \Big\downarrow{\scriptstyle c_n} & \swarrow\!\!\searrow & \\ \mathscr{D}_{n+1} & = & D_{q_{n+1}} & \oplus & D_{q_n}. \end{array}$$

Then $\mathscr{A} = \varinjlim \mathscr{A}_n$ has a diagonal subalgebra $\mathscr{D} = \varinjlim \mathscr{D}_n$. The usefulness of this is in the following:

THEOREM [17]. *Given any* AF *algebra* $\mathscr{A}$, *then as above one has*

(a) *a m.a.s.a.* $\mathscr{D}$ *in* $\mathscr{A}$,

(b) *a conditional expectation* P *of* $\mathscr{A}$ *with range* $\mathscr{D}$ *and*,

(c) *a subgroup* $\mathscr{U}$ *of the unitary group of* $\mathscr{A}$, *such that*

  (i) $u^*\mathscr{D}u = \mathscr{D}$ *for all* u *in* $\mathscr{U}$, *and*

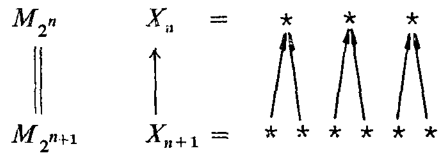
  (ii) $P(u^*xu) = u^*P(x)u$ *for all* x *in* $\mathscr{A}$.

Further, if $\mathscr{X}$ *is the Gelfand spectrum of* $\mathscr{D}$, *and* $\mathscr{G}$ *is the group of homeomorphisms of* $\mathscr{X}$ *induced by* $\mathscr{U}$, *then* $\mathscr{A} \cong A(\mathscr{X}, \mathscr{G})$, *a* C^**-algebra which is a particular representation of the cross product* C^**-algebra* $C^*(C(\mathscr{X}), \mathscr{G})$.

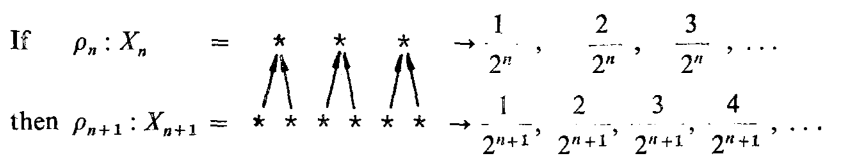
Now $\mathscr{X}$ can be identified as follows: If X_n denotes the spectrum of $\mathscr{D}_n$ and Y_n is the spectrum of D_{q_n}, then φ_n induces a map from X_{n+1} to X_n, which we will also denote by φ_n. We have

$$\begin{array}{ccccc} X_n & = & Y_n & \cup & Y_{n-1} \\ \Big\uparrow{\scriptstyle \varphi_n} & & \Big\uparrow{\scriptstyle c_n} & \nwarrow\!\!\nearrow & \\ X_{n+1} & = & Y_{n+1} & \cup & Y_n. \end{array}$$

The spectrum of $\mathscr{D}$ is easily seen to be $\mathscr{X} = \varprojlim X_n$. We recall that the projective limit may be thought of as sequences $\{x_n\}$ with x_n in X_n and $\varphi_n(x_{n+1}) = x_n$. In general this is not a very transparent description of $\mathscr{X}$ so we want to try and map it onto some simpler space. For illustration let us first consider the usual well known treatment of the CAR algebra. Here $\mathscr{A} = \varinjlim M_{2^n}$, and the Bratteli diagrams are just,



So $\mathscr{X} = \prod_1^\infty \{0, 1\}$ and we have $\rho: \mathscr{X} \to [0, 1]$, by $\rho(\{x_n\}) = \sum_1^\infty \frac{x_n}{2^n}$ [6]. Another way to look at this map is to define $\rho_n: X_n \to [0, 1]$ inductively as follows:

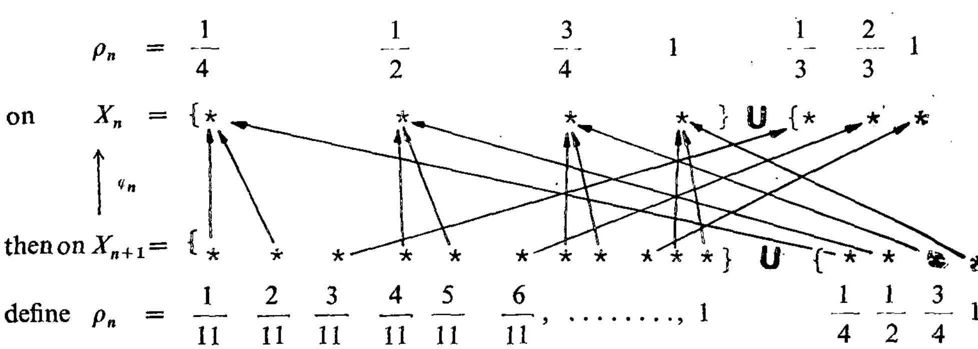


Now define $\rho(\{x_n\}) = \lim \rho_n(x_n)$. One has $\rho(\mathscr{X}) = [0, 1]$, and one can easily see that this gives a representation of $\mathscr{X}$ as $[0, 1]$, cut at each dyadic rational point.

For AF_θ, we proceed in an analogous way. We will construct our maps so that,

$$\rho_n : X_n = Y_n \cup Y_{n-1} \to \left\{\frac{1}{q_n}, \frac{2}{q_n}, \ldots, 1\right\} \cup \left\{\frac{1}{q_{n-1}}, \ldots, 1\right\}.$$

We define the ρ_n's inductively, and the procedure is best illustrated by example. Suppose ρ_n has been defined, taking the values below



The key feature here is that the right hand side of X_n is distributed among the left hand side according to order

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < 1 = 1,$$

and this determines the value of ρ_{n+1} on the preimages in X_{n+1}. Since $aq_{n-1} \neq bq_n$ for any integers a and b, $0 < b < q_{n-1}$, and $n > 3$, the only ambiguity arises at 1. The choice of order here is not important at the moment.

The idea of the above is that if $\varphi_n(x_{n+1}) = x_n$ then the value of $\rho_{n+1}(x_{n+1})$ should stay as close as possible to $\rho_n(x_n)$. Then if $x = \{x_n\}$ is in $\mathscr{X}$, $\rho(x) = \lim \rho_n(x_n)$ should exist. This seems like it should be true for general reasons, yet in proving it we found ourselves drawn into the details of the continued fraction expansion for θ. This gives more than the desired convergence, and it would be nice if it could be dispensed with at this juncture. We need the following facts from the theory of continued fractions [7],

(i) $\left| \dfrac{p_n}{q_n} - \dfrac{p_{n+1}}{q_{n+1}} \right| = \dfrac{1}{q_n q_{n+1}}$, and

(ii) θ lies between $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n+1}}{q_{n+1}}$.

Now let $X_n = Y_n \cup Y_{n-1} = \{a_i\}_{i=1}^{q_n} \cup \{b_i\}_{i=1}^{q_{n-1}}$ and suppose $x = \{x_n\}$ is such that $x_n = a_i$, and $\rho_n(a_i) = i/q_n$. In X_{n+1}, there are c_n elements in Y_{n+1}, and one element in Y_n, which are mapped to a_i. Consider those in Y_{n+1} first. On the last of these, ρ_{n+1} takes the value

$$\frac{c_n i + l_n}{q_{n+1}} \quad \text{where} \quad \frac{l_n}{q_{n-1}} < \frac{i}{q_n} < \frac{l_n + 1}{q_{n-1}}.$$

In X_{n+2}, on the last element in Y_{n+2} that gets mapped to a_i, ρ_{n+2} takes the value

$$\frac{c_{n+1}(c_n i + l_n) + l_{n+1}}{q_{n+2}} \quad \text{where} \quad \frac{l_{n+1}}{q_n} < \frac{c_n i + l_n}{q_{n+1}} < \frac{l_{n+1} + 1}{q_n}.$$

We claim that $l_{n+1} = i - 1$. First

$$\frac{l_{n+1}}{q_n} < \frac{c_n i + l_n}{q_{n+1}}.$$

$$l_{n+1} q_{n+1} < i c_n q_n + q_n l_n = i(q_{n+1} - q_{n-1}) + q_n l_n = i q_{n+1} + q_n l_n - i q_{n-1}.$$

Thus $l_{n+1} < i$ since $q_n l_n - i q_{n-1} < 0$. Also

$$\frac{l_{n+1} + 1}{q_n} > \frac{c_n i + l_n}{q_{n+1}}$$

$$q_{n+1}(l_{n+1} + 1) > i c_n q_n + l_n q_n =$$

$$= i(q_{n+1} - q_{n-1}) + l_n q_n = i q_{n+1} + l_n q_n - i q_{n-1} >$$

$$> i q_{n+1} - q_n > \qquad \text{since } \frac{i}{q_n} < \frac{l_{n+1}}{q_{n-2}}$$

$$> (i - 1) q_{n+1}.$$

Thus $l_{n+1} > i - 2$, giving the claim.

Now taking $l_{n+1} = i - 1$, we see that the numerator of the appropriate ρ_n on each last element referred to above, appears on the left hand side of the pairs

r_n	r_{n-1}
i	$l_n + 1$
$c_n i + l_n$	i
$c_{n+1}(c_n i + l_n) + i - 1$	$c_n i + l_n$.

The pattern is $r_{n+1} = c_n r_n + r_{n-1} - 1$.

Now consider $\{x_n\}$ such that $x_n = a_{i-1}$. Here we obtain the pairs

r'_n	r'_{n-1}
$i - 1$	$l_n + 1$ or l_n
$c_n(i - l) + l_n$ (or $l_n - 1$)	$i - 1$.

The pattern is $r'_{n+1} = c_n r'_n + r'_{n-1} - 1$. Thus

$$r_{n+1} - r'_{n+1} = c_n(r_n - r'_n) + r_{n-1} - r'_{n-1}.$$

Let

$$d_n = r_n - r'_n.$$

The pairs (d_n, d_{n-1}) start as

$$1,\ 0 \quad \text{or} \quad 1,\ 1.$$

So solve

$$\begin{cases} 1 = \alpha p_n + \beta q_n \\ 0 = \alpha p_{n-1} + \beta q_{n-1} \end{cases} \quad \text{or} \quad \begin{cases} 1 = \alpha p_n + \beta q_n \\ 1 = \alpha p_{n-1} + \beta q_{n-1} \end{cases}$$

noting that the first of the continued fraction properties guarantees an integer solution, then

$$d_{n+1} = c_n(\alpha p_n + \beta q_n) + \alpha p_{n-1} + \beta q_{n-1} = \alpha p_{n+1} + \beta q_{n+1}$$

and

$$d_{n+k} = \alpha p_{n+k} + \beta q_{n+k}.$$

Thus

$$\lim_{k\to\infty} \frac{d_{n+k}}{q_{n+k}} = \alpha\theta + \beta.$$

Starting at any Y_n, only two lengths are possible. The number of each is $q_n - q_{n-1}$ or q_{n-1}. Hence the lengths of these intervals goes to zero.

Now if x_{n+1} is in Y_n, then $\rho_{n+1}(x_{n+1}) = \rho_n(x_n) = i/q_n$. And on the låst of the elements in Y_{n+1} that get mapped to a_i, we saw that the value of ρ_{n+1} was

$$= \frac{c_n i + l_n}{q_{n+1}}, \quad \text{where } \frac{l_n}{q_{n-1}} < \frac{i}{q_n} < \frac{l_n + 1}{q_{n-1}}$$

$$< \frac{c_n i + \dfrac{i q_{n-1}}{q_n}}{q_{n+1}} = \frac{i}{q_n}.$$

On the last of the c_n elements in Y_{n+1} that get mapped to a_{i+1}, the value of ρ_{n+1} is

$$= \frac{c_n(i + 1) + h_n}{q_{n+1}}, \quad \text{where } \frac{h_n}{q_{n-1}} < \frac{i+1}{q_n} < \frac{h_n + 1}{q_{n-1}}$$

$$> \frac{c_n(i + 1) + (i + 1)\dfrac{q_{n-1}}{q_n} - 1}{q_{n+1}} > \frac{i}{q_n}.$$

So ρ_{n+2} of the point that we are interested in, will lie between ρ_{n+2} of these points, and consequently within one of the intervals above. Thus $\rho(x) = \lim \rho_n(x_n)$ exists for all x. The construction ensures that ρ is continuous on $\mathscr{X}$.

The second part of the structure of an AF algebra is the group of unitaries $\mathscr{U}$, or the group of homeomorphisms $\mathscr{G}$. So it is natural to look here for U_θ. Since $\frac{p_n}{q_n} \to \theta$ we do the following:

First note that the functions ρ_n have put an order on each of the bases that determine $\mathscr{D}_n$ inside $M_{q_n} \oplus M_{q_{n-1}}$. In effect a system of matrix units has been chosen.

Let U_{θ_n} be that element in $M_{q_n} \oplus M_{q_{n-1}}$ that shifts these bases forward through p_n and p_{n-1} elements respectively. Then U_{θ_n} acts on $\mathscr{D}_n$ by $D \to U_{\theta_n} D U_{\theta_n}^*$ and $\varphi_m \ldots \varphi_n(U_{\theta_n})$ acts similarly on $\mathscr{D}_{n+m}$. In this way we get an action on $\mathscr{D}$ given by an operator in $\mathscr{A}$, call it U_{θ_n} also, $\mathscr{D} \to U_{\theta_n} \mathscr{D} U_{\theta_n}^*$. This in turn defines a homeomorphism of $\mathscr{X}$ which will be denoted by θ_n.

Next we want to calculate $\rho(\theta_n(x))$. For simplicity of notation we may consider the functions ρ_n defined on $\mathscr{X}$, by $\rho_n(\{x_n\}) = \rho_n(x_n)$. Now $\rho_n(\theta_n(x))$ is directly seen to be $\rho_n(\theta_n(x)) + p_n/q_n$ (p_{n-1}/q_{n-1}) if $\pi_n(x)$ is in Y_n (Y_{n-1}) respectively. To calculate $\rho_{n+1}(\theta_n(x))$ we need:

LEMMA. *If $1 \neq i/q_n$, then between any i/q_n, and $i/q_n + p_n/q_n$ there lie exactly p_{n+1} (p_{n-1}) terms of the form j/q_{n+1} (j/q_{n-1}) respectively.*

NOTE. Here $i/q_n + p_n/q_n$ is to be taken mod 1. And this applies to all that follows also.

Proof. The proof is identical, mutatis mutandi, for $n+1$ or $n-1$. We do $n+1$. Suppose

$$\frac{d-1}{q_{n+1}} < \frac{i}{q_n} < \frac{d}{q_{n+1}}$$

then

$$\frac{d}{q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} = \left(\frac{d}{q_{n+1}} - \frac{i}{q_n}\right) + \left(\frac{i}{q_n} + \frac{p_n}{q_n}\right) + \left(\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right).$$

We know

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{\pm 1}{q_n q_{n+1}}$$

and clearly

$$\frac{d}{q_{n+1}} - \frac{i}{q_n} \geqslant \frac{1}{q_n q_{n+1}}$$

so

$$\frac{d}{q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} \geqslant \frac{i}{q_n} + \frac{p_n}{q_n}.$$

Similarly,

$$\frac{d}{q_{n+1}} + \frac{p_{n+1}-1}{q_{n+1}} = \frac{d-1}{q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} \leqslant$$

$$\leqslant \frac{i}{q_n} - \frac{1}{q_n q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} \leqslant \frac{i}{q_n} + \frac{p_n}{q_n}.$$

Thus p_{n+1} terms of the form j/q_{n+1} lie between $i/q_n + p_n/q_n$.

Since the end point $i/q_n + p_n/q_n$ may be equal to 1 and thus to one of the j/q_{n+1}, we cannot in this case say that this number of terms lies strictly between. This is important because of the ambiguity that remained in the definition of the ρ_n's. We can remove this by completing the order. For this if $p_{n-1}/q_{n-1} > p_n/q_n$, let ρ_n take the values 1^- on Y_{n-1}, and 1^+ on Y_n, with $1^- < 1^+$ of course. If $p_{n-1}/q_{n-1} < p_n/q_n$, then we let ρ_n take 1^+ on Y_{n-1} and 1^- on Y_n. This alternation may seem strange, but it is exactly the fact that $\frac{p_n}{q_n}$ is alternately greater than and less than $\frac{p_{n+1}}{q_{n+1}}$, that causes the lemma to fail at the end point. Notice that once we have distinguished, in any way, between the 1's, we have for any x_1, x_2, in X,

$$x_1 = x_2 \Leftrightarrow \rho_n(x_1) = \rho_n(x_2), \quad \forall n.$$

Returning to the calculation of $\rho_{n+1}(\theta_n(x))$, we must consider various cases. Suppose first

$$\frac{p_{n-1}}{q_{n-1}} > \frac{p_n}{q_n}.$$

Case 1.

$$\rho_n(x) = i/q_n \neq 1^+$$

and

$$\rho_{n+1}(x) = k/q_{n+1}.$$

Then

$$\rho_n(\theta_n(x)) = \rho_n(x) + p_n/q_n.$$

Schematically, using the lemma, we have

$$\begin{array}{ccccccc}
\rho_n & = & \dfrac{i}{q_n} & \cdots\cdots\cdots\cdots & & \dfrac{i}{q_n} + \dfrac{p_n}{q_n} & \\
 & & \cdot & \left(\dfrac{j}{q_{n-1}} \leftarrow \quad \rightarrow \dfrac{j}{q_{n-1}} + \dfrac{p_{n-1}}{q_{n-1}}\right) & & & \\
\Big\uparrow \varphi_n & \nearrow & \uparrow \quad \nwarrow & & \nearrow & \uparrow & \nwarrow \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\
\rho_{n+1} & = & \dfrac{k}{q_{n+1}} & & & ? &
\end{array}$$

Thus

$$\rho_{n+1}(\theta_n(x)) = \frac{k}{q_{n+1}} + \frac{c_n p_n + p_{n-1}}{q_{n+1}} =$$

$$= \rho_{n+1}(x) + \frac{p_{n+1}}{q_{n+1}}.$$

Case 2.

$$\rho_n(x) = i/q_n$$

and

$$\rho_{n+1}(x) = i/q_n.$$

Then

$$\rho_{n+1}(\theta_n(x)) = \rho_n(x) + p_n/q_n$$

is immediate.

Case 3.

$$\rho_n(x) = i/q_{n-1} \neq 1^-$$

and

$$\rho_{n+1}(x) = k/q_{n+1}.$$

Then

$$\rho_{n+1}(\theta_n(x)) = \rho_{n+1}(x) + \frac{p_{n+1}}{q_{n+1}}$$

follows as in Case 1, using the lemma.

The effect of ordering 1^- and 1^+ above was to make $\rho_n(x) = 1 - p_n/q_n$ in Case 1 and $\rho_n(x) = 1 - p_{n-1}/q_{n-1}$ in Case 2, compute correctly. If $p_{n-1}/q_{n-1} < < p_n/q_n$, as happens on alternate terms, an exactly similar analysis holds. Let $E_n = = \{x : \rho_n(x) = 1^+, 1^-\}$. Then the three cases show

$$\rho_{n+1}(\theta_n(x)) = \rho_{n+1}(\theta_{n+1}(x)) \quad \text{if } x \notin E_n.$$

The same argument shows, since $E_n \supset E_{n+1} \supset \ldots$, that

$$\rho_{n+k}(\theta_n(x)) = \rho_{n+k}(\theta_{n+k}(x)) \quad \text{if } x \notin E_n.$$

It follows hence that for all m,

$$\rho_m(\theta_n(x)) = \rho_m(\theta_{n+1}(x)) \quad \text{if } x \notin E_n$$

so by the previous remark, about the ρ_n's separating points,

$$\theta_n(x) = \theta_{n+1}(x) \quad \text{if } x \notin E_n.$$

It is easy to check that $\bigcap E_n = \{x_0\}$ where $\rho_n(x_0) = 1^+$, all n. For x not in E_n we know $\theta_{n+p}(x) = \theta_n(x)$ for all p. Thus for $x \neq x_0$, we can define $\theta(x) = \theta_n(x)$ if $x \notin E_n$. It is clear that

$$\theta_n(E_n) = \left\{ x : \rho_n(x) = 1 + \frac{p_n}{q_n} \text{ or } = 1 + \frac{p_{n-1}}{q_{n-1}} \right\}$$

so that the intersection of all these sets consists of just the two points x_1, and x_2, where $\rho_n(x_1) = 1 + \frac{p_n}{q_n}$ and $\rho_n(x_2) = 1 + \frac{p_{n-1}}{q_{n-1}}$. We have a similar pair of points for all powers of the θ_n applied to E_n. To complete the definition of θ we have a choice of $\theta(x_0) = x_1$ or $= x_2$. In either case, from the above we see that

$$\rho(\theta(x)) = \rho(x) + \theta \quad \text{all } x.$$

If we had chosen a different system of ordering 1^+ and 1^- we could have arrived at worse E_n's but unfortunately not at $\bigcap E_n = \emptyset$. If the exact behaviour of the θ_n's, and of θ itself on $\mathscr{X}$ is not transparent at the moment, we can assure the reader that the next section will clarify things.

Actually the cases considered above show more than was stated so far. Recall that φ_n denoted the embedding of $M_{q_n} \oplus M_{q_{n-1}}$ into $M_{q_{n+1}} \oplus M_{q_n}$. Now a part of the diagonal of $M_{q_{n+1}} \oplus M_{q_n}$ corresponds to E_n, and it follows from the above that $\varphi_n(U_{\theta_n}) = U_{\theta_{n+1}}$ except on those columns. In the abstract setting there is no operator U_θ corresponding to θ for the U_{θ_n} to converge to. They certainly do not converge in norm, for, among other reasons if they did converge to a unitary operator, this would induce a homeomorphism of $\mathscr{X}$, and we know that θ is not one. But we have succeeded in associating two different covariance algebras with AF_θ. It seems worthwhile to consider what happens in the concrete setting of a representation of AF_θ, but we shall postpone this until the final section.

One place to see the interplay between the θ_k's and θ is to consider traces. If τ is any faithful finite trace on AF_θ, then for any A in $\mathscr{A}$, one has $\tau(A) = \tau P(A)$, where P is the conditional expectation referred to earlier [17]. Identifying $\mathscr{D}$ and $C(\mathscr{X})$, we will write D_f for f in $C(\mathscr{X})$. Now any trace τ is a positive linear functional on $C(\mathscr{X})$, and so there is a finite measure μ on $\mathscr{X}$, such that if A is in $\mathscr{A}$, then $\tau(A) = \int P(A)\,d\mu$. Now $\tau(U^*_{\theta_k} D_f U_{\theta_k}) = \tau(D_f)$ for all f, means that μ

is invariant for all θ_k. This implies that μ has no atoms since μ is finite. Now μ is also invariant for θ since if $A \subset \mathscr{X}$, write $A = \bigcup_i A_i \cup \{x_0\}$, where $A_i = A \cap \cap (E_i \setminus E_{i+1})$. Then

$$\mu(\theta(A)) = \sum_i \mu(\theta(A_i)) = \sum_i \mu(\theta_i(A_i)) = \sum_i \mu(A_i) = \mu(A).$$

The converse is also true. Suppose μ is invariant for θ. For any k we may write $\mathscr{X} = \sum_{i=1}^{q_k} B_i \cup \sum_{i=1}^{q_{k-1}} C_i$, where $\sum$ denotes disjoint union and B_i (C_i) are those points $x = \{x_n\}$ for which x_k is a particular point in Y_k (Y_{k-1}) respectively. On $B_1, \ldots, B_{q_k-1}$, $C_1, \ldots, C_{q_{k-1}-1}$ we know $\theta = \theta_k$, so if A is contained in these $\mu(\theta_k(A)) = \mu(A)$. If $A \subset B_{q_k}$, then $\mu(\theta_k(A)) = \mu(\theta^{q_k-1}\theta_k(A)) = \mu(\theta_k^{q_k}(A)) = \mu(A)$. Similarly if $A \subset C_{q_{k-1}}$. So part of the relationship between the $\{\theta_k\}$ and θ, is that they have the same invariant measures.

So any trace τ on AF_θ, gives a measure μ, on $\mathscr{X}$, and this can be used to get a trace on Irr_θ. We had $\rho : \mathscr{X} \to [0, 1]$, and identifying 0 and 1 we may take ρ as mapping onto S^1. Any element T of Irr_θ may be represented by its Fourier series $\sum_{-\infty}^{\infty} f_n U_\theta^n$. Defining $\tau(T) = \int f_0 \circ \rho \, d\mu$ gives a faithful trace on Irr_θ. It is well known that there is a unique one, namely $\tau(T) = \int f_0 dm$ where m is Haar measure on S^1. We want to consider the range of this trace. Suppose, for the moment, that there is a trace τ on AF_θ.

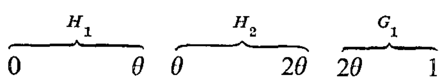
Let $E = \sum_n f_n U_\theta^n$ be in Irr_θ. Writing out the condition that $E^2 = E$, we see that E is an idempotent if and only if certain conditions hold among $\{f_n \circ \theta^m\}$. Now $\rho(\theta_k(x)) = \rho(\theta(x))$ except on E_k, and $\mathrm{diam}(E_k) \to 0$. For any n, since f_n is continuous, if k is chosen large enough then f_n is nearly constant on $\rho(\theta_k(E_k)) = \rho(\theta(E_k))$ and $\|f_n \circ \rho \circ \theta_k - f_n \circ \rho \circ \theta\|$ is small. Notice that the actual effect of θ_k and θ within E_k is irrelevant. Let $A_k = \sum_{-k}^{k} D_{f_n \circ \rho} U_{\theta_k}^n$, an element in AF_θ. We may approximate E by a finite sum of its terms, and let A_k just involve those terms. If k is large enough then $\|A_k^2 - A_k\|$ will be small. Thus if $B_k = (1/2)(A_k + A_k^*)$, for large k, we certainly have $\|B_k^2 - B_k\| < 1/4$. This is enough to ensure that the spectrum is disconnected at 1/2. Let p be the characteristic function of $[1/2, 2]$, then p is continuous on the spectrum of B_k, so $p(B_k)$ is an indempotent in $C^*(B_k)$. We had $\|B_k^2 - B_k\|$ small so $\|B_k - p(B_k)\|$ is small also. Then, $\tau(p(B_k))$ is close to $\tau(B_k) = \int P(B_k) d\mu = \int f_0 \circ \rho \, d\mu = \tau(E)$. Hence $\tau(p(B_k)) \to \tau(E)$. Recall that U_{θ_k} agrees with $U_{\theta_{k+1}}$ except on certain columns corresponding to E_k. On finite dimensional spaces the unitary group is connected, thus we can connect U_{θ_k} and $U_{\theta_{k+1}}$ via a path of unitaries U_t which all agree with U_{θ_k} and $U_{\theta_{k+1}}$ except on those pieces. But as noted previously then $\|U_t^* D_{f_n} \circ \rho U_t - D_{f_n} \circ \rho \circ \theta\|$ is small. Thus if $A_t = \sum_{-k}^{k} D_{f_n} \circ \rho U_t^n$ and $B_t = (1/2)(A_t + A_t^*)$ we have $\|B_t^2 - B_t\| < 1/4$. Now $B_0 = B_k$ and $B_1 - B_{k+1} = D_{f_{k+1}} \circ \rho U_{\theta_{k+1}}^{k+1} + D_{f_{-k-1}} \circ \rho U_{\theta_{k+1}}^{-k-1}$ which goes to zero as $k \to \infty$. Thus, if k is large enough, $p(B_t)$ gives a path of idempotents connecting $p(B_k)$ and an idempotent equivalent to $p(B_{k+1})$. Thus $\tau(p(B_k)) = \tau(p(B_{k+1})) = \tau(E)$.

This shows that the range of the trace on the idempotents in Irr_θ is contained in the range of any trace on AF_θ. Pimsner and Voiculescu [14] give a direct proof of the existence and uniqueness of a trace on AF_θ whose value on idempotents is contained in $\mathbf{Z} + \mathbf{Z}\theta$. With Rieffel's work [15], this shows that $\tau(\mathrm{K}_0(\mathrm{Irr}_\theta)) = \mathbf{Z} + \mathbf{Z}\theta$. We shall see an even more direct approach in the next section.

3. THE COVARIANCE ALGEBRA

In this section we want to see how starting with the covariance algebra Irr_θ, one is naturally led to AF_θ. In the last section it was expressing AF_θ in terms of the diagonal algebra $\mathscr{D}$ and the U_{θ_k}'s that led to θ. So given θ on $[0, 1)$ we want to approximate it by θ_k's. Recall here that the θ_k's were homeomorphisms on $\mathscr{X}$, but not on $[0, 1]$, and so we should not look for that.

Since the irrational translation is ergodic, it is not possible to approximate it by periodic transformations that only disagree on diminishing pieces of $[0, 1)$. But we can use pairs of such transformations. In what follows each θ_i is to be a rigid translation. Define $H_1 = [0, \theta)$, $H_2 = \theta(H_1)$, and suppose $\theta(H_2)$ is not disjoint from $H_1 \cup H_2$, then put $G_1 = [0, 1)\setminus \bigcup H_i$, as below:

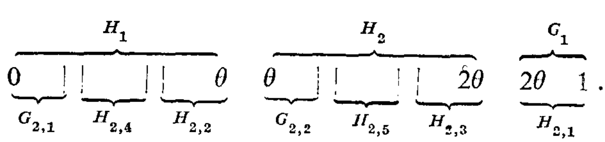


Define

$$\theta_1(H_1) = H_2, \quad \theta_1(H_2) = H_1, \quad \text{and} \quad \theta_1(G_1) = G_1.$$

Then $\theta_1 = \theta$ except on H_2 and G_1.

For the next step, define $H_{2,1} = G_1$, and $H_{2,j+1} = \theta(H_{2,j})$, as long as they are disjoint. Continuing the above example, this works for $j = 1, 4$. Then put $G_{2,1} = H_1 \setminus \bigcup H_{2,j}$ and $G_{2,2} = H_2 \setminus \bigcup H_{2,j}$, as below:



Define

$$\theta_2(H_{2,j}) = H_{2,j+1 \bmod 5} \quad \text{and} \quad \theta_2(G_{2,j}) = G_{2,j+1 \bmod 2}.$$

Then $\theta_2 = \theta$ except on $H_{2,5}$ and on $G_{2,2}$.

For the next step, let $H_{3,1} = G_{2,1}$. Define $H_{3,i+1} = \theta(H_{3,i})$, again until one can go no further. Now one will have a piece left over in each of the $H_{2,i}$, $1 \leqslant i \leqslant 5$. θ_3 will be a cycle on these and $\theta_3 = \theta$ except on the last $H_{3,\cdot}$ and $G_{3,5}$. No tice that we have pairs of cycles of periods

$$\begin{array}{cc} 2 & 1 \\ 2.2+1 & 2 \\ ?.5+2 & 5. \end{array}$$

This looks like a continued fraction expansion and in fact the numbers 2, 2, ?, are the expansion for θ. This can be established by induction. The first step can be seen by inspection. At stage n, we have q_n pieces of length α_{n-1} and q_{n-1} pieces of length α_n say. This is one of the partitioning results [16]. (Notice that the new smaller length was always for the right hand smaller cycle.) Then $q_n\alpha_{n-1} + q_{n-1}\alpha_n = 1$. Assume that $\alpha_{n-1}/\alpha_n = r_n$, then the continued fraction expansion is generated by putting $r_n = \text{integer} + \dfrac{1}{r_{n+1}}$. Clearly the same c_n is obtained at this stage. Write

$$\frac{\alpha_{n-1}}{\alpha_n} = c_n + \frac{1}{r_{n+1}}$$

$$\alpha_{n-1} = \alpha_n c_n + \frac{\alpha_n}{r_{n+1}}.$$

So

$$1 = q_n\alpha_{n-1} + q_{n-1}\alpha_n = q_n\alpha_n c_n + q_n\alpha_n/r_{n+1} + q_{n-1}\alpha_n = q_{n+1}\alpha_n + q_n\alpha_n/r_{n+1}.$$

But $1 = q_{n+1}\alpha_n + q_n\alpha_{n+1}$. So $\alpha_{n+1} = \alpha_n/r_{n+1}$ or $r_{n+1} = \alpha_n/\alpha_{n+1}$ taking us to the next stage.

Notice that this is a very dynamic method of generating the continued fraction expansion for θ. Take any point (here 0) and follow its orbit under translation, counting as we go. This decomposition of [0,1) will work for any ergodic homeomorphism and this can be used as an alternative definition of the rotation number [8].

The fundamental role of these θ_k's is underlined by the way in which we can use them to calculate the range of the trace on $K_0(\mathrm{Irr}_\theta)$ without explicitly mentioning AF_θ.

We may suppose Irr_θ faithfully represented on $L^2(S^1, \mu)$, where μ is arc length by the translation operator U_θ and the multiplication operators M_f, $f \in C(S^1)$. The unique faithful trace is easily seen to be $\tau(\sum M_{f_n} U_\theta^n) = \int f_0 d\mu$. If we allow $f \in L^\infty(S^1)$ we still obtain a trace on the larger C^*-algebra by the same formula. Corresponding to θ_k we have unitary translation operators U_{θ_k}. We can write

$$U_{\theta_k} = U_\theta M_{\chi_{S^1 \setminus (H_{k,q_k} \cup G_{k,q_{k-1}})}} + U_\theta^{*q_k} M_{\chi_{H_{k,q_k}}} + U_\theta^{*q_{k+1}} M_{\chi_{G_{k,q_{k-1}}}} .$$

So the U_{θ_k}'s are all in the larger C^*-algebra. As in the last section given any idempotent P in Irr_θ we can find $P_k = \sum M_{f_n} U_{\theta_k}^n$ with the same trace. Now θ_k is not a homeorphism on S^1, but it is continuous except at the end points of the intervals. Thus P_k is seen to lie in a copy of $M_{q_k}(H_{k,1}) \oplus M_{q_{k-1}}(G_{k,1})$. ($M_n(X)$ denotes $M_n \otimes C(X)$.) $H_{k,1}$ ($G_{k,1}$) have lengths α_{k-1} (α_k) respectively, which are in $\mathbf{Z} + \mathbf{Z}\theta$ by construction. But any trace on the above matrix algebra must give $\tau(P_k) = a\alpha_{k-1} + b\alpha_k$ for some integers a, b. Hence the desired result.

Let us return to the θ_k. These are not homeomorphisms on $[0,1)$, but we can make them into homeomorphisms by simply adjoining the end points of all the intervals, and extending the definition of θ_k in the obvious way. Now θ can be defined on this by mapping the point $-\theta$ to either the point 0 or to the point 1, and then mapping left hand end points to left hand end points, and similarly for the right hand end points. It is now clear what was happening in the last section. The point x_0 there, is now the point $-\theta$, while the points x_1, and x_2 there, are now 0 and 1. As we saw there, θ is not a homeomorphism on this cut interval. To make it into one we simply have to further cut the interval at the backward translates of θ, and extend θ in the obvious way. This brings us to the third of the C^*-algebras.

4. CUNTZ'S C^*-ALGEBRA

The third C^*-algebra, C_θ was introduced in [3]. It is the covariance algebra $C^*(C(\tilde{\mathscr{X}}), \mathbf{Z})$ gotten from $\mathrm{Irr}_\theta = C^*(C(S^1), \mathbf{Z})$ by cutting S^1 at all the translates of a given point. Let $\mathscr{X}'$ be $S^1 = [0,1)$ cut at just the forward translates. We saw in the last section how $\mathscr{X}'$ decomposed into successive collections of pieces $H_{k,i}$ and $G_{k,i}$. It is easy to see that we can inductively define ρ_k, continuous, sending each $H_{k,i}$ and $G_{k,i}$ to a point in X_k (X_k as in Section 2), so as to make the following diagram commute

$$\begin{array}{ccc} X_k & \xleftarrow{\ \rho_k\ } & \mathscr{X}' \\ \uparrow{\scriptstyle \varphi_k} & & \| \\ X_{k+1} & \xleftarrow{\ \rho_{k+1}\ } & \mathscr{X}' . \end{array}$$

Every point in $\mathscr{X}'$, is completely determined by its images $\rho_k(x)$, so it is easy to see that this space has the universal attracting property in the correct cateogory. Thus by the uniqueness of projective limits we have that $\mathscr{X}' = \mathscr{X}$ the spectrum of the diagonal C^*-algebra in AF_θ. Also, from the last section we know that translation by θ has a matrix decomposition with respect to the $H_{k,i}$'s and the $G_{k,i}$'s of the form

$$\begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & * \\ 1 & 0 & \cdot & \cdot & \cdot & * \\ 0 & 1 & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & 0 & * \\ 0 & 0 & \cdot & \cdot & 1 & * \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & * \\ 1 & 0 & \cdot & \cdot & \cdot & * \\ 0 & 1 & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * \\ 0 & 0 & \cdot & \cdot & 1 & * \end{pmatrix}$$

except for extra entries in the columns with the *'s in them. Applying the correct idempotent we find that

$$\begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{pmatrix}$$

is in C_θ. But this operator together with all diagonals generates $M_{q_k} \oplus M_{q_{k-1}}$ as a C^*-algebra. In this way we see that Cuntz's algebra contains AF_θ.

Thus the three C^*-algebras are $C^*(C(S^1), \theta)$, a representation of $C^*(C(\mathscr{X}), \{\theta_k\})$, and $C^*(C(\tilde{\mathscr{X}}), \theta)$. Let us briefly consider representations of these algebras. Standard arguments [10], [17], tell us that any irreducible representation of AF_θ, is on a space $L^2(\mathscr{X}, \mu, H)$, where μ is quasiinvariant with respect to the θ_k. If the measure is non transitive, or is atomic but none of the points $-\theta$, 0 or 1, are in the orbit of atoms, then the operators U_{θ_k} converge weakly to an operator U_θ. In this way we get irreducible representations of the other two algebras. If we start with an irreducible representation of Irr_θ, which is not based on the orbit of 0, then we can define operators U_{θ_k}, and get an irreducible representation of AF_θ. In all of these cases, we may express the relationship between the (representations of the) three C^*-algebras as

$$\begin{array}{ccccc} \mathrm{AF}_\theta & & & & \mathrm{Irr}_\theta \\ & \searrow & & \swarrow & \\ & \text{adjoin } U_\theta & & \text{adjoin } \{U_{\theta_k}\} & \\ & \searrow & & \swarrow & \\ & & C_\theta. & & \end{array}$$

If the representation of AF_θ is on the orbit of $-\theta$, then the weak limit of the U_{θ_k} is a unilateral backward shift. From the orbit of 0, and that of 1, one gets a forward shift. The representation of Irr_θ, on the orbit of 0, corresponds to the direct sum of the former and the first of the latter pair. C_θ also has representations based on the orbit of 1, and these correspond to a direct sum of the former and the second of the latter. In all of the cases, the correspondence between irreducible representations preserves unitary equivalence. So we see that the spectra of these three C^*-algebras are the same, except for a couple of points, and it is no wonder that one can use one of them to deduce properties of the others.

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