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EXISTENCE OF THE SCATTERING OPERATOR FOR DISSIPATIVE HYPERBOLIC SYSTEMS WITH VARIABLE MULTIPLICITIES

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0. INTRODUCTION

The abstract theory of scattering developed by Lax and Phillips in [10], [11] enables one to prove the existence of the scattering operator for a large class of hyperbolic systems and equations. The verification of the basic axioms of the abstract scheme is rather complicated when the space dimension is even. The difficulties increase when one tries to check the axiom concerning the local energy decay for first order dissipative hyperbolic systems.

Another idea to study the existence of the scattering operator was exploited by Strauss [23] for moving obstacles. This idea is based on the strong Huygens principle, which holds only for odd space dimensions. Petkov [15] and Rangelov [16] used a similar approach for first order hyperbolic systems and odd space dimensions.

In this work we obtain the existence of the scattering operator for a large class first order dissipative hyperbolic systems in exterior domains. The approach used in the paper is different from those in [10], [11], [14] and enables us to prove directly the existence of the scattering operator. Our idea is to apply the time-dependent Enss' method (see [1], [2], [3], [19], [21], [22], [25], [26]) proposing a suitable adaptation for dissipative hyperbolic systems in exterior domains. We deal with systems which are not strictly hyperbolic and we do not make any assumptions on the multiplicities of the characteristics. This leads to some difficulties when we introduce the incoming and outgoing parts of Enss' decomposition. Our approach enables us to study mixed problems in unbounded domains $\Omega \subset \mathbf{R}^m$ with a smooth compact boundary $\partial\Omega$. More precisely, we consider the following mixed problem

$$(0.1)\qquad \begin{cases} E(x)\partial_t u = Gu \text{ on } (0,\infty)\times\Omega \\ \Lambda(x)u = 0 \text{ on } (0,\infty)\times\partial\Omega \\ u(0, x) = f(x), \end{cases}$$

where

$$G = \sum_{j=1}^{m} A_j(x)\, \partial_{x_j} + B(x), \tag{0.2}$$

and $E(x)$, $A_j(x)$, $B(x)$, $A(x)$ are matrices with properties described below. The assumptions (H_1)—(H_3) of the next section guarantee that $E(x) = E_0$, $A_j(x) = A_j^0$, $B(x) = 0$ for $|x|$ sufficiently large and the solution $u(t, x)$ to (0.1) can be represented as $u = V(t)f$, where $\{V(t),\ t \geqslant 0\}$ is a contraction semigroup in the energy Hilbert space H_E. The unperturbed system is connected with the Cauchy problem

$$E_0 \partial_t u = G_0 u \quad \text{on} \quad \mathbf{R} \times \mathbf{R}^m, \qquad u(0, x) = f(x), \tag{0.3}$$

where $G_0 = \sum_{j=1}^{m} A_j^0 \partial_{x_j}$. The operator $\mathrm{i} E_0^{-1} G_0$ can be defined as a selfadjoint one and the solution to the Cauchy problem (0.3) is given by $u = U_0(t)f$ with $U_0(t) = \exp(tE_0^{-1}G_0)$ being a unitary group on the Hilbert space H_{E_0}.

Since the perturbed semigroup is defined only for $t \geqslant 0$, the basic objects of the scattering theory — the wave operators, can be defined by the equalities

$$\Omega_+ f = \lim_{t \to +\infty} V(t) J^* U_0(-t) P_{\mathrm{ac}}(E_0^{-1} G_0) f,$$

$$Wg = \lim_{t \to +\infty} U_0(-t) J V(t) g \quad \text{for } g \in H_{E,\mathrm{b}}^{\perp}.$$

The operators $P_{\mathrm{ac}}(E_0^{-1}G_0)$, J, J^* and the space $H_{E,\mathrm{b}}^{\perp}$ are described in Section 1.

The main goal of this paper is to prove the existence of the wave operators Ω_+, W and to characterize the image of Ω_+. This enables one to prove the existence of the scattering operator

$$S = W\Omega_+ .$$

The present work is a generalization of [4], where we assume $E(x) = I$ and the principle symbol of the system has nonzero eigenvalues of constant multiplicity. On the other hand, there are many important examples in the mathematical physics as the Maxwell equations in an inhomogeneous media, when $E(x) \neq I$ and the eigenvalues of the matrix $E_0^{-1} \sum_{j=1}^{m} A_j^0 \xi_j$ have variable multiplicity (see Appendix 2).

The Enss' method is a typical tool for the analysis of Schrödinger type equations. Since we study mixed problems for general hyperbolic systems, some new ideas in the scheme proposed by Enss occur. First, we note that it is convenient to work with the following analogue of the usual Enss' condition (compare with [1]–[4]):

$$\lim_{\substack{R \to +\infty \\ R > R_0}} \|[(G-1)^{-k} - (G_0-1)^{-k}]\chi(|x| \geqslant R)\| = 0. \tag{0.4}$$

Here $\chi(A)$ denotes the multiplication by the characteristic function of the set A. The condition (0.4) can be proved by using finite dependence argument for first order hyperbolic systems.

The novelty of our approach is the suitable choice of the incoming and outgoing part in the Enss' decomposition. The main difficulty is connected with the fact that the non-zero roots $\lambda = a_j(\xi)$ of the characteristic equation $\det\left(\lambda E_0 + \sum_{j=1}^{m} A_j^0 \xi_j\right) = 0$ may have variable multiplicities and they could be non-differentiable functions.

Following the approach in [21], [4] and taking into account (0.4), we reduce the proof of the existence of the wave operators to a suitable estimate for the action of the unperturbed group $U_0(t)$ on the outgoing and incoming parts of the Enss' decomposition.

By using the Fourier transform one can connect the action of the unperturbed group with the following integrals

$$\int T_{jk}(\xi) T_{jl}(\xi) h_\alpha(\xi) \exp[\mathrm{i}(\langle x-\alpha, \xi\rangle - t a_j(\xi))]\, \mathrm{d}\xi, \tag{0.5}$$

where the functions $a_j(\xi)$, $T_{jk}(\xi)$ in general are non-differentiable, hence we can not exploit the usual integration by parts argument. A similar difficulty was overcomed by Yafaev in [26], where polar coordinates are used together with a suitable modification of the incoming and outgoing parts in the Enss' decomposition.

In this work we use a suitable modification of the cutt-off functions connected with the definition of outgoing and incoming parts. The crucial role in the analysis of the action of the unperturbed group is played by the following estimate

$$\partial_{|\xi|}[\langle x-\alpha, \xi\rangle - t a_j(\xi)] \geqslant Ct$$

of the phase function in (0.5). The above estimate holds for ξ in the "outgoing" cone and $t \geqslant t(\alpha)$ sufficiently large. For small $t \geqslant 0$ we apply another argument based on the finite speed of propagations.

It seems that this strategy to treat the case of dissipative hyperbolic systems of variable multiplicity is used for the first time. In this direction, we mention the paper [24] of Tamura, where an example of hyperbolic system with variable multiplicity is studied by using the explicit form of the characteristic roots, together with the recent works of Iwashita ([7], [8]).

Finally, we shall sketch the plan of the work. The assumptions and main results are formulated in Section 1. Some preliminary results are given in Section 2. The incoming and outgoing parts of Enss' decomposition and the action of the unperturbed group on them are described in Sections 3, 4. We prove the existence of the wave operator W in Section 5. In Section 6 we analyse the images of the wave operators and prove the existence of the scattering operator. Appendix 1 is devoted to the proof of the existence of the contraction semigroup $V(t)$, while Appendix 2 discusses an important application of the results obtained in the work.

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1. ASSUMPTIONS AND MAIN RESULTS

Our assumptions are close to those given in [17]. Let Ω be an open domain in $\mathbf{R}^m$ with a bounded complement and boundary of class C^1. Let G be the operator defined in (0.2) and $E(x)$, $A_j(x)$, be $(r\times r)$ matrices with elements in $C^1(\bar{\Omega})$, while $B(x)$ is a matrix of the same type with elements in $C(\bar{\Omega})$. We impose the conditions:

(H_1) a) $\begin{cases} A_j^*(x) = A_j(x),\ E(x) \text{ is a positively defined Hermitian matrix} \\ \text{for } x \in \bar{\Omega}, \end{cases}$

(H_1) b) $\begin{cases} \text{there exist constant matrices } E_0,\ A_j^0 \text{ and a number } R_0 > 0 \text{ such} \\ \text{that } A_j(x) = A_j^0,\ E(x) = E_0,\ B(x) = 0 \text{ for } |x| \geqslant R_0, \end{cases}$

(H_1) c) $\begin{cases} \text{the matrix } A(\xi) = -\sum_{j=1}^{m} A_j^0\xi_j \text{ has constant rank for } \xi = (\xi_1, \dots \\ \dots, \xi_m) \neq 0. \end{cases}$

According to the results in [17], the problem (0.1) is well-posed in the case $E(x) = E_0 = I$ provided the following conditions are fulfilled:

(H_2) a) $\operatorname{rank} A(x, \nu(x))$ is constant on each component of $\partial\Omega$,

(H_2) b) $B^*(x) + B(x) - \sum_{j=1}^{m} \partial_{x_j} A_j(x) \leqslant 0$ for $x \in \Omega$,

(H_2) c) $\langle u, A(x, \nu(x))u\rangle \leqslant 0$ for $u \in \operatorname{Ker} A'(x)$, $x \in \partial\Omega$,

(H_2) d) $\begin{cases} \operatorname{Ker} A(x) \text{ is the maximal linear subspace of } \mathbf{C}^r \text{ satisfying the pro-} \\ \text{perty } (\mathrm{H}_2)\text{ c)}. \end{cases}$

Here $A(x, \xi) = \sum_{j=1}^{m} A_j(x)\xi_j$ and $\nu(x) = (\nu_1, \dots, \nu_m)$ is the unit normal at $x \in \partial\Omega$ pointed into $\mathbf{R}^m \setminus \Omega$. The inner products in $\mathbf{C}^r$ and $\mathbf{R}^m$ will be denoted by $\langle\cdot,\cdot\rangle$.

Let H_E be the Hilbert space $L^2(\Omega; \mathbf{C}^r)$ of vector-valued functions with the inner product

$$(f, g)_E = \int_{\Omega} \langle E(x)f(x), g(x)\rangle \, dx.$$

The norm in H_E will be denoted by $\|\cdot\|_E$. Consider the operator $E^{-1}G$ with the domain $D(E^{-1}G)$ equal to the closure in the graph-norm $(\|f\|_E + \|E^{-1}Gf\|_E)$ of the set of functions $f \in C^1(\bar{\Omega}; \mathbf{C}^r) \cap H_E$ satisfying the boundary condition $\Lambda(x)f(x) = 0$ on $\partial\Omega$. Under the assumptions (H_1) and (H_2) we prove that $E^{-1}G$ generates a contraction semigroup $\{V(t),\, t \geqslant 0\}$ on H_E (see Lemma A.1). The solution to the problem (0.1) is given by $u(t, x) = V(t)f$.

Let $\mathrm{Ker}(E^{-1}G)^\perp$ be the orthogonal complement in the space H_E to the null-space of $E^{-1}G$. Our last assumption is (see [13]):

$$(\mathrm{H}_3)\quad \begin{cases} \text{for } f \in D(E^{-1}G) \cap \mathrm{Ker}(E^{-1}G)^\perp \text{ we have the coercive estimate } \|\partial_x f\| \leqslant \\ \leqslant C(\|Gf\| + \|f\|). \end{cases}$$

Here $\|\cdot\|$ is the norm in the Hilbert space $L^2(\Omega; \mathbf{C}^r)$ which is equivalent to the energy norm $\|\cdot\|_E$. We shall also denote by $\|\cdot\|$ the norm in $L^2(\mathbf{R}^m; \mathbf{C}^r)$ and by $(\cdot,\cdot)$ the inner product in these two spaces.

The dissipative operator $E^{-1}G$ is a perturbation of the operator $E_0^{-1}G_0 = E_0^{-1} \sum_{j=1}^{m} A_j^0 \partial_{x_j}$, which is the generator of a unitary group $U_0(t)$ on the Hilbert space $H_{E_0} = L^2(\mathbf{R}^m; \mathbf{C}^r)$ with inner product $(f, g)_{E_0} = (E_0 f, g)$. The solution to the Cauchy problem (0.3) is given by $u(t, x) = U_0(t)f$.

Denote by $J_0\colon H = L^2(\Omega; \mathbf{C}^r) \to H_0 = L^2(\mathbf{R}^m; \mathbf{C}^r)$ the operator of extention as zero in $\mathbf{R}^m \setminus \Omega$ and introduce the operators $E^{1/2}\colon H_E \to H$, $E_0^{1/2}\colon H_{E_0} \to H_0$, defined by the action of the corresponding matrices $E^{1/2}(x)$, $E_0^{1/2}$. Since the operators $E^{1/2}$ and $E_0^{1/2}$ are unitary ones, the operator $J = E_0^{-1/2} J_0 E^{1/2}$ maps isometrically H_E onto H_{E_0}. Similarly, the operator $J^* = E^{-1/2} J_0^* E_0^{1/2}$ maps H_{E_0} onto H_E.

Following [21], [4] denote by $H_{E,\mathrm{b}}^\perp$ the orthogonal complement to the linear space $H_{E,\mathrm{b}}$ spanned by the eigenvectors of $E^{-1}G$ with eigenvalues on the imaginary axis.

Our key result is

THEOREM 1.1. *Suppose the assumptions* (H_1)—(H_3) *fulfilled and* $\eta \in H_{E,\mathrm{b}}^\perp$. *Then there exists a sequence* t_n *tending to* $+\infty$, *such that*

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \|[JV(t) - U_0(t)J]V(t_n)\eta\|_{E_0} = 0.$$

This result enables one to prove the existence of the wave operator $Wf = \lim_{t\to+\infty} U_0(-t)JV(t)f$.

COROLLARY 1.2. *Suppose the assumptions* (H_1)—(H_3) *fulfilled and* $f \in H_{E,\mathrm{b}}^\perp$. *Then the following limit*

$$Wf = \lim_{t\to+\infty} U_0(-t)JV(t)f$$

exists.

The Enss' decomposition principle, used in the proof of Theorem 1.1, can be exploited to obtain some information about the spectrum of the perturbed generator.

COROLLARY 1.3. *The only possible finite limit point of the pure imaginary eigenvalues of $E^{-1}G$ is zero and any non-zero eigenvalue on the imginary axis has finite multiplicity.*

The second problem discussed in this paper is related to the images of the wave operators

$$\Omega_+ = \text{s-}\lim_{t\to+\infty} V(t)J^* U_0(-t)P_{ac}(E_0^{-1}G_0),$$

$$\Omega_- = \text{s-}\lim_{t\to+\infty} V^*(t)J^* U_0(t)P_{ac}(E_0^{-1}G_0).$$

Here $P_{ac}(E_0^{-1}G_0)$ denotes the projection on the absolutely continuous subspace of the selfadjoint operator $iE_0^{-1}G_0$. Proposition 2.3 guarantees that this space coincides with the orthogonal complement of the kernel of $E_0^{-1}G_0$.

First, we prove the existence of the wave operators $\Omega_\pm$ by means of Cook's method (see [19]). Consider the spaces

$$H_{E,\infty}^- = \{f \perp H_{E,b}\ ;\ \lim_{t\to+\infty} \|V(t)f\| = 0\},$$

$$H_{E,\infty}^+ = \{f \perp H_{E,b}\ ;\ \lim_{t\to+\infty} \|V^*(t)f\| = 0\}.$$

As it was mentioned in [4], the above spaces are closely connected with the images of the wave operators. These spaces are non-empty when disappearing solutions exist, i.e. solutions with $V(t)f = 0$ for large t (see [5]).

THEOREM 1.4. *Suppose the assumptions* (H_1)—(H_3) *fulfilled. Then the wave operators* $\Omega_\pm$ *exist and*

$$\overline{\operatorname{Ran}\Omega_-} = H_{E,b}^\perp \ominus H_{E,\infty}^- . \tag{1.1}$$

REMARK 1. If we replace the assumption (H_3) by the dual estimate

(H_3^*) $\|\partial_x f\| \leqslant C(\|G^*f\| + \|f\|)$ for $f \in \operatorname{Ker}(E^{-1}G^*)^\perp \cap D(E^{-1}G^*)$,

it is possible to prove that

$$\overline{\operatorname{Ran}\Omega_+} = H_{E,b}^\perp \ominus H_{E,\infty}^+. \tag{1.1$'$}$$

In the case of unitary group $V(t)$, Theorem 1.4 and Remark 1 yield the completeness of the wave operators. The equalities (1.1) and (1.1)′ show that in general

$$\operatorname{Ran}\Omega_- \neq \operatorname{Ran}\Omega_+ .$$

THEOREM 1.5. *Under the assumptions* (H_1)—(H_3) *the scattering operator* $S = W\Omega_+$ *exists.*

REMARK 2. In Appendix 2 we give an important example of equations satisfying the hypotheses (H_1)—(H_3). That is the Maxwell equations in inhomogeneous media with strictly dissipative boundary conditions ([12], [13]).

REMARK 3. In order to avoid the difficulties connected with the appearence of two energy inner products $(\cdot,\cdot)_{E_0}$ and $(\cdot,\cdot)_E$, we shall reduce our problem to the usual L^2 spaces. Using the unitary maps $E^{1/2}$ and $E_0^{1/2}$ we obtain the commutative diagram

$$\begin{array}{ccc} H_E & \xrightarrow{\quad J \quad} & H_{E_0} \\ E^{-1/2}\Big\uparrow\Big\downarrow E^{1/2} & & E_0^{-1/2}\Big\downarrow\Big\uparrow E_0^{1/2} \\ H & \xrightarrow{\quad J_0 \quad} & H_0\,. \end{array}$$

Now let us formulate Theorems 1.1, 1.4 and 1.5 and Corollaries 1.2 and 1.3 in the spaces H and H_0. We have to replace the operators $E^{-1}G$ and $E_0^{-1}G_0$ by the operators $\hat{G} = E^{1/2}(E^{-1}G)E^{-1/2}$ and $\hat{G}_0 = E_0^{1/2}(E_0^{-1}G)E_0^{-1/2}$, the group $U_0(t)$ and the semigroup $V(t)$ by the group $\hat{U}_0(t) = E^{1/2}U_0(t)E_0^{-1/2}$ and the semigroup $\hat{V}(t) = E^{1/2}V(t)E^{-1/2}$ etc. The subspaces $H_{E,\mathrm{b}}$ and $H_{E,\mathrm{b}}^{-}$ must be replaced by subspaces H_b and H_∞^- having the same definitions, formally assuming $E(x) = I$. The operators $\hat{G}_0$ and $\hat{G}$ satisfy the hypotheses (H_1)—(H_3) with new matrices $\hat{A}_j$, $\hat{A}_j^0$, $\hat{B}$ and $\hat{E}(x) = I$. Our advantage of the described reduction is that we can omit the hats and without loss of generality we can assume $E(x) = I$.

REMARK 4. Later we shall work essentially in the space $H_0 = L^2(\mathbf{R}^m, \mathbf{C}^r)$. In order to simplify the notations, given any $\psi \in H$ we shall denote by $\tilde{\psi} = J_0\psi$ the function $\tilde{\psi} \in H_0$, which is continued as 0 on $\mathbf{R}^m \setminus \Omega$. For any operator A in H we shall denote by $\tilde{A}$ the operator $J_0AJ_0^*$ on H_0.

REMARK 5. We deal with finite range perturbations (the condition b) in (H_1)) only to simplify the proofs. The results of the paper can be obtained for general short-range perturbations.

2. PRELIMINARIES

We need the following form of RAGE theorem for contraction semigroups, obtained by B. Simon in [21].

THEOREM 2.1 ([21]). *Let G be the generator of a contraction semigroup* $V(t) = \exp(tG)$, $t \geqslant 0$, *on the Hilbert space H and $H_\mathrm{b}^\perp$ be the subspace of H introduced in Section* 1. *Suppose that L is a bounded operator and $L(G - 1)^{-1}$ is a compact* *one. Then we have*

$$\lim_{T\to+\infty} T^{-1}\int_0^T \|LV(t)f\|^2\,dt = 0 \quad \textit{for } f \in H_b^{\perp}.$$

COROLLARY 2.2. *For any $f \perp H_b$ there exists a sequence $t_n \to +\infty$, such that*

$$\lim_{n\to+\infty} \|\chi(|x| < n)V(t_n)f\| = 0.$$

Proof. Theorem 2.1 implies that $\|LV(t_n)f\|$ tends to 0 as n tends to ∞ for suitably chosen sequence $t_n \to +\infty$. The coercive estimate (H_3) combined with the Rellich compactness theorem shows that the operator $L = \chi(|x| < R)(G-1)^{-1}$ is compact on $H_b^{\perp}$. Thus, the assertion of the corollary is true for any $f \in H_b^{\perp} \cap D(G)$. Applying a density argument we complete the proof of the corollary.

Next, we are going to study the unperturbed operator G_0. Let $\xi \neq 0$ and $a_1(\xi) \geqslant a_2(\xi) \geqslant \dots \geqslant a_{r-d_0}(\xi)$ be the non-zero eigenvalues of the matrix $A(\xi) = -\sum \xi_j A_j^0$, where $d_0 = \dim \operatorname{Ker} A(\xi)$ (see assumption (H_1)). The functions $a_j(\xi)$ are continuous and homogeneous of degree 1. Since we consider the case when the multiplicities of these eigenvalues may be variable, the functions $a_j(\xi)$ may have singularities in $\mathbf{R}^m\setminus\{0\}$. It is not hard to see that the integer $r-d_0$ is even and $a_j(\xi)>0$ for $\xi = 0$, $j = 1, 2, \dots, (r-d_0)/2$, while $a_j(\xi) < 0$ for $j > (r-d_0)/2$. Set

$$v_{\min} = \min\{|a_j(\omega)|\,;\, 1 \leqslant j \leqslant r-d_0,\ \omega \in S^{m-1}\},$$
$$v_{\max} = \max\{|a_j(\omega)|\,;\, 1 \leqslant j \leqslant r-d_0,\ \omega \in S^{m-1}\}.$$

Our assumption (H_1) guarantees that $v_{\min} > 0$.

PROPOSITION 2.3. *The spectrum of the self-adjoint operator* iG_0 *on the space* $(\operatorname{Ker} G_0)^{\perp}$ *is absolutely continuous.*

Proof. It is sufficient to prove that given any set $U \subset \mathbf{R}$ with Lebesgue measure $\mu(U) = 0$ we have $\mu\{\xi : a_j(\xi) \in U\} = 0$ for $j = 1, \dots, r-d_0$. Without loss of generality we can assume that $U \subset [A, B]$, where $0 < A < B < \infty$. Fix $\varepsilon > 0$ and let $I_k = [M_k, M_k + \varepsilon_k]$ be a family of intervals covering U, such that $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$. For $j \leqslant (r-d_0)/2$ we have

$$\mu\{\xi\,;\, a_j(\xi) \in I_k\} = \int_{S^{m-1}} d\omega \int_{M_k/a_j(\omega)}^{(M_k+\varepsilon_k)/a_j(\omega)} \lambda^{m-1}\,d\lambda =$$

$$= \int m^{-1}\{[(M_k+\varepsilon_k)/a_j(\omega)]^m - [M_k/a_j(\omega)]^m\}\,d\omega \leqslant$$

$$\leqslant \mu(S^{m-1})(m \cdot v_{\min})^{-m}[(M_k+\varepsilon_k)^m - M_k^m] \leqslant C(A, B)\varepsilon_k.$$

The last inequality follows from the fact the function $s \to s^m$ has a bounded derivative in the set $[A, B]$. Hence

$$\mu\{\xi;\ a_j(\xi) \in U\} \leqslant C(A, B) \sum_{k=1}^{\infty} \varepsilon_k \leqslant C(A, B)\varepsilon.$$

This completes the proof of the lemma.

The next equality, introduced in [4], plays an important role when we have to compare the perturbed and unperturbed systems.

PROPOSITION 2.4. Let $\theta \in C_0^\infty(\mathbf{R}^m)$ *and* $\theta(x) = 1$ *for* $|x| \leqslant R_0$. *Denote by* Q *the matrix* $Q(x) = \sum_{j=1}^{m} A_j^0 \partial_{x_j} \theta(x)$. *Then given any* $h \in H_0$ *we have the equality*

$$[U_0(t) - \tilde{V}(t)]h = -\tilde{V}(t)\theta h + \theta U_0(t)h +$$

$$+ \int_0^t \tilde{V}(t-s) Q U_0(s) h \, \mathrm{d}s. \tag{2.1}$$

Next we need the following variant of the principle of the causality (see [10]).

PROPOSITION 2.5. *Suppose* $\varphi \in H_0$, $\varphi(x) = 0$ *for* $|x| \leqslant R$. *Then* $(U_0(t)\varphi)(x) = 0$ *for* $|x| \leqslant R - t \cdot v_{\max}$.

This assertion leads to

PROPOSITION 2.6. ([4]). *For any integer* $k \geqslant 1$ *we have*

$$\lim_{R \to \infty} \|[J(G-1)^{-m}J^* - (G_0 - 1)^{-m}]\chi(|x| \geqslant R)\| = 0.$$

3. THE ENSS' DESCOMPOSITION

In this section we shall introduce the form of the Enss' decomposition in the phase space, which is essential in our considerations. Given any function $\varphi \in L^2(\mathbf{R}^m; \mathbf{C}^r)$ we denote by

$$\hat{\varphi}(\xi) = (\mathscr{F}\varphi)(\xi) = \lim_{n \to +\infty} \int_{|x|<n} \varphi(x) \exp(-\mathrm{i}\langle x, \xi \rangle) \, \mathrm{d}x$$

the Fourier transform of φ. Choose a function $f \in S(\mathbf{R}^m)$ with the properties:

a) $f(x) \geqslant 0$;

b) $\int f(x)\,dx = 1$;

c) $\operatorname{supp}_{\xi} \hat{f} \subset \{\xi\,;\,|\xi| \leqslant \varepsilon_M\}$, where the constant $\varepsilon_M > 0$ is fixed and will be specified later.

Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{Z}^m$, consider the function

$$f_\alpha(x) = (f * \chi_\alpha)(x) = \int f(x-y)\chi_\alpha(y)\,dy, \tag{3.1}$$

where χ_α is the characteristic function of the unit cube in $\mathbf{R}^m$ centered at α. Thus we obtain the following partition of the unity in the coordinate space $1 = \sum_{\alpha \in \mathbf{Z}^m} f_\alpha(x)$. Further we make a suitable decomposition in the dual space of the variables ξ. As it was mentioned in the introduction, we can not exploit the velocities $v_j(\xi) = \nabla a_j$, used in [4]. In [24] Yafaev suggests to take into account only the sign of $a_j(\xi)$. Following this idea, we replace the velocities $v_j(\xi)$ by the vector ξ if $a_j(\xi)>0$ and by $-\xi$, if $a_j(\xi) < 0$. More precisely, consider the functions $g_\pm \in C^\infty(\mathbf{R}^1)$ with the properties

$$g_+(s) = 0 \quad \text{if } s \leqslant -c_0, \qquad g_+(s) = 1 \quad \text{if } s \geqslant c_0, \tag{3.2}$$

$c_0 = v_{\min}/(4v_{\max})$ and $g_- = 1 - g_+$. Denote by $\Pi_0(\xi)$, $\Pi_+(\xi)$, $\Pi_-(\xi)$ the projections for nonzero ξ on the null-space, the space spanned by the eigenvectors corresponding to the positive and negative eigenvalues of the matrix $A(\xi)$, respectively. These projections are smooth homogeneous of degree 0 functions for nonzero ξ.

Define the matrix-valued functions

$$G^{\text{out}}_{\alpha,\sigma}(\xi) = g_\sigma(\langle \xi/|\xi|,\ \alpha/|\alpha|\rangle)\Pi_\sigma(\xi)\psi_M(A(\xi))F(A(\xi)), \tag{3.3 a)}$$

$$G^{\text{in}}_{\alpha,\sigma}(\xi) = g_{-\sigma}(\langle \xi/|\xi|,\ \alpha/|\alpha|\rangle)\Pi_\sigma(\xi)\psi_M(A(\xi))F(A(\xi)), \tag{3.3 b)}$$

where $\sigma = \pm$, ψ_M is a smooth function depending on a positive parameter M, such that $\psi_M(s) = 0$, if $|s| \leqslant 1/M$ or $|s| \geqslant M$, $\psi_M(s) = 1$, if $2/M < |s| < M/2$ and

$$F(s) = (s-\mathrm{i})^{-2}s = \mathrm{i}(s-\mathrm{i})^{-2} + (s-\mathrm{i})^{-1}.$$

It is clear that the elements of the matrix $\psi_M(A(\xi))$ are smooth for nonzero ξ. Hence the same is true for $G^{\text{out}}_{\alpha,\sigma}(\xi)$, $G^{\text{in}}_{\alpha,\sigma}(\xi)$. Since $(\psi_M F)(0) = 0$, we have $\Pi_0(\xi)(\psi_M F)(A(\xi)) = (\psi_M F)(A(\xi))\Pi_0(\xi) = 0$. Thus, we have the equalities

$$\sum_{\sigma=\pm}[G^{\text{in}}_{\alpha,\sigma}(\xi) + G^{\text{out}}_{\alpha,\sigma}(\xi)] = \sum_{\sigma=\pm}\Pi_\sigma(\xi)\psi_M(A(\xi))F(A(\xi)) = \psi_M(A(\xi))F(A(\xi)). \tag{3.4}$$

D
Consider the following pseudodifferential operators in H_0

$$P_n^{\mathrm{out}} = \sum_{|\alpha|>n/2} \sum_{\sigma=\pm} G_{\alpha,\sigma}^{\mathrm{out}}(D) f_\alpha \Pi_\sigma(D),$$

$$P_n^{\mathrm{in}} = \sum_{|\alpha|>n/2} \sum_{\sigma=\pm} G_{\alpha,\sigma}^{\mathrm{in}}(D) f_\alpha \Pi_\sigma(D).$$

Here $D = (-i\partial_{x_1}, \ldots, -i\partial_{x_m})$, $G_{\alpha,\sigma}^{\mathrm{out}}(D)$, $G_{\alpha,\sigma}^{\mathrm{in}}(D)$, $\Pi_\sigma(D)$ are pseudodifferential operators with symbols $G_{\alpha,\sigma}^{\mathrm{out}}(\xi)$, $G_{\alpha,\sigma}^{\mathrm{in}}(\xi)$, $\Pi_\sigma(\xi)$ and f_α is the multiplication by the function $f_\alpha(x)$. Note that the operators $P_n^{\mathrm{out}}, P_n^{\mathrm{in}}$ depend on the parameter M. From (3.4) we deduce that

$$\sum_{|\alpha|\leqslant n/2} \sum_{\sigma=\pm} \psi_M(iG_0)F(iG_0)\Pi_\sigma(D) f_\alpha \Pi_\sigma(D) + P_n^{\mathrm{out}} + P_n^{\mathrm{in}} = \psi_M(iG_0)F(iG_0). \tag{3.5}$$

The decomposition we will use in Section 5 is based on the above equality. Following [19], [21] one can obtain

PROPOSITION 3.1. *The operators* P_n^{out}, P_n^{in} *are uniformly bounded in* $H_0 = L^2(\mathbf{R}^m; \mathbf{C}^r)$ *for* $n = 1, 2, \ldots$.

The proof of this proposition is based on the estimate

$$\|P_n^{\mathrm{out}} h\|^2 = \sum_{\sigma=\pm} \sum_{\alpha,\beta} (f_\beta G_{\beta,\sigma}^{\mathrm{out}*}(D) G_{\alpha,\sigma}^{\mathrm{out}}(D) f_\alpha \Pi_\sigma(D) h, \Pi_\sigma(D) h) \leqslant$$

$$\leqslant \sum_{\sigma=\pm} \sum_{\alpha,\beta} \int f_\beta(x)\, |H_{\alpha,\beta,\sigma}(x-y)(\Pi_\sigma(D)h)(y)(\Pi_\sigma(D)h)(x)|\, f_\alpha(y)\, \mathrm{d}x\mathrm{d}y,$$

where the sum for α, β is taken over $|\alpha|>n/2$, $|\beta|>n/2$, and the functions $H_{\alpha,\beta,\sigma}(x) = \mathscr{F}^{-1}[G_{\beta,\sigma}^{\mathrm{out}*}(\xi) G_{\alpha,\sigma}^{\mathrm{out}}(\xi)]$ satisfy the estimate $|H_{\alpha,\beta,\sigma}(x)| \leqslant C(1+|x|^2)^{-m}$ with a constant C independent of α, β (see Lemma 3 from XI.17 in [19]).

4. FREE TIME EVOLUTION OF THE INCOMING AND OUTGOING PARTS OF ENSS' DECOMPOSITION

The main result of this section is

THEOREM 4.1. *Let* $\rho > R_0$ *be fixed. There exist a constant* n_0 *depending only on* ρ *and the matrices* A_j^0, *such that for* $t > 0$, $n > n_0$ *and for any integer* $N > 0$ *we have the estimates*

$$\|\chi(|x| < \rho) U_0(t) P_n^{\mathrm{out}}\| \leqslant C_N(1+t+n)^{-N}, \tag{4.1}$$

$$\|\chi(|x| < \rho) U_0^*(t) P_n^{\mathrm{in}}\| \leqslant C_N(1+t+n)^{-N}, \tag{4.2}$$

$$\|\chi(|x| < \rho) U_0^*(t) P_n^{\mathrm{in}*}\| \leqslant C_N(1+t+n)^{-N}, \tag{4.3}$$

with some positive constant C_N *depending only on* N, ρ, A_j^0.

REMARK. The above estimates are fulfilled if

$$n_0 \geqslant 4\rho(8v_{\max}/v_{\min} + 1).$$

To prove this theorem we need the following

LEMMA 4.2. *Let $\psi \in S(\mathbf{R}^m)$ and $0 < r < R$. For each integer N and $f \in H_0$ we have*

$$\|\chi(|x| < r)\psi(D)\chi(|x| > R)\| \leqslant C\|(1 - \Delta)^{N/2+m}\psi\|_{L^1}(R - r)^{-N}.$$

Proof. For $|x| < r$ and $|y| > R$ we have $|x - y| > R - r$. Hence

$$|[\psi(D)\chi(|x| > R)f](x)| = \left| \int_{|y|\geqslant R} (\mathscr{F}^{-1}\psi)(x - y)f(y)\,\mathrm{d}y \right| \leqslant$$

$$\leqslant (R - r)^{-m}(\sup_{z\in\mathbf{R}^m} |(1 + |z|^2)^{N/2+m}(\mathscr{F}^{-1}\psi)(z)|) \int (1 + |x - y|^2)^{-m} |f(y)|\,\mathrm{d}y$$

for $|x| < r$. Using Young's inequality, we obtain the needed estimate. This completes the proof of the lemma.

Proof of Theorem 4.1. Set $P^{\text{out}}_{\alpha,\sigma} = G^{\text{out}}_{\alpha,\sigma}(D)f_\alpha \Pi_\sigma(D)$, $\sigma = \pm$. To prove (4.1) it is sufficient to obtain the estimate

$$\|\chi(|x| < \rho)U_0(t)P^{\text{out}}_{\alpha,\sigma}\| \leqslant C_N(1 + t + |\alpha|)^{-N} \tag{4.4}$$

for $t > 0$ and $|\alpha|$ sufficiently large, more precisely for

$$|\alpha| > 2\rho(8v_{\max}/v_{\min} + 1). \tag{4.5}$$

Indeed, the equality $P^{\text{out}}_n = \sum_{|\alpha|>n/2} [P^{\text{out}}_{\alpha,+} + P^{\text{out}}_{\alpha,-}]$ and the estimate (4.4) lead to the needed inequality (4.1).

To prove (4.4) we consider two cases

(A) $\rho + tv_{\max} \leqslant |\alpha|/2$,

(B) $\rho + tv_{\max} > |\alpha|/2$

provided (4.5) fulfilled.

In the case (A) we use Lemma 4.2 together with the principle of causality. More precisely, the principle of causality formulated in Proposition 2.5 yields $\chi(|x| < \rho)U_0(t) = \chi(|x| < \rho)U_0(t)\chi(|x| < \rho + tv_{\max})$. Thus we obtain

$$\|\chi(|x| < \rho)U_0(t)P^{\text{out}}_{\alpha,\sigma}f\| \leqslant$$

$$\leqslant \|\chi(|x| < \rho + tv_{\max})G^{\text{out}}_{\alpha,\sigma}(D)\|\,\|\chi(|x| \leqslant 3|\alpha|/4)f_\alpha(x)\|_\infty\|\Pi_\sigma(D)f\| + \tag{4.6}$$

$$+ \|\chi(|x| < \rho + tv_{\max})G^{\text{out}}_{\alpha,\sigma}(D)\chi(|x| > 3|\alpha|/4)\|\,\|f_\alpha(x)\Pi_\sigma(D)f\|.$$

Consider the first term in the right hand side of (4.6). Since the norm $\|\chi(|x| \leqslant \leqslant \rho + tv_{\max})G^{\text{out}}_{\alpha,\sigma}(D)\|$ is uniformly bounded (see (3.3)), it is sufficient to estimate the term $\|\chi(|x| \leqslant 3|\alpha|/4)f_\alpha(x)\|_\infty$. For this purpose we shall use the definition of $f_\alpha = \int f(x-y)\chi_\alpha(y)\,\mathrm{d}y$, where χ_α is the characteristic function of the unit cube centered at α. We have $|x-y| \geqslant |\alpha| - |y-\alpha| - |x| \geqslant |\alpha| - m^{1/2} - 3|\alpha|/4 = |\alpha|/4 - m^{1/2}$ provided $y \in \operatorname{supp}\chi_\alpha$ and $|x| \leqslant 3|\alpha|/4$. Since $f \in S$, we find

$$\text{(4.7)}\qquad \|\chi(|x| \leqslant 3|\alpha|/4)f_\alpha\|_\infty \leqslant \int\limits_{|z| \geqslant |\alpha|/4 - m^{1/2}} f(z)\,\mathrm{d}z \leqslant C_N(1+|\alpha|)^{-N}.$$

To estimate the second term in the right hand side of (4.6) we shall use Lemma 4.2. Setting $r = \rho + tv_{\max}$ and $R = 3|\alpha|/4$, we obtain $R - r \geqslant |\alpha|/4$, when (A) holds. Hence,

$$\text{(4.8)}\qquad \begin{aligned} &\|\chi(|x| < \rho + tv_{\max})G^{\text{out}}_{\alpha,\sigma}(D)\chi(|x| > 3|\alpha|/4)\| \leqslant \\ &\leqslant C_N\|(1-\Delta)^{N/2+m}G^{\text{out}}_{\alpha,\sigma}(\xi)\|_{L^1}|\alpha|^{-N}. \end{aligned}$$

From the definition (3.3) of the function $G^{\text{out}}_{\alpha,\sigma}$ it is clear that the norm $\|(1 - \Delta)^{N/2+m}G^{\text{out}}_{\alpha,\sigma}\|_{L^1}$ is uniformly bounded with respect to α. Moreover we have

$$\text{(4.9)}\qquad C_N|\alpha|^{-N} \leqslant C_N(|\alpha|/2 + \rho + tv_{\max})^{-N} \leqslant C'_N(1 + t + |\alpha|)^{-N}.$$

From the estimates (4.6)—(4.9) we get the needed inequality (4.4) in the case (A). Next, we turn to the case

(B) $\rho + tv_{\max} > |\alpha|/2$. Here we are going to use a suitable modification of the stationary phase techniques. The action of the unperturbed group $U_0(t)$ has the following representation

$$U_0(t)P^{\text{out}}_{\alpha,\sigma}\varphi = (2\pi)^{-m}\int \mathscr{F}(P^{\text{out}}_{\alpha,\sigma}\varphi)(\xi)\exp(\mathrm{i}\langle x, \xi\rangle - \mathrm{i}tA(\xi))\,\mathrm{d}\xi.$$

We shall investigate only the case $\sigma = +$, since the case $\sigma = -$ can be treated in a similar manner. We mentioned in Section 2 that the positive eigenvalues of the matrix $A(\xi)$ are $a_1(\xi), \ldots, a_s(\xi)$, where $s = (r - d_0)/2$. Let $T(\xi)$ be a unitary matrix whose elements $T_{jk}(\xi)$ are homogeneous of degree 0 functions such that

$$T(\xi)A(\xi)T^{-1}(\xi) = \operatorname{diag}(a_1(\xi), \ldots a_{r-d_0}(\xi), 0, \ldots, 0).$$

Therefore, denoting by $(v)_k$, $k = 1, \ldots, r$, the components of any vector $v \in \mathbf{C}^r$, we obtain

$$(4.10) \qquad (U_0(t)P^{\text{out}}_{\alpha,+}\varphi)_k = \sum_{j=1}^{s} \int_K D^{\alpha}_{kj}(\xi) \exp(\mathrm{i}\langle x - \alpha, \xi\rangle - \mathrm{i}ta_j(\xi))\,\mathrm{d}\xi,$$

where $D^{\alpha}_{kj}(\xi) = (2\pi)^{-m} \sum_{l=1}^{r} \overline{T_{jk}(\xi)} T_{jl}(\xi) g_+(\langle \xi/|\xi|, \alpha/|\alpha|\rangle)(E_\alpha(\xi))_l$ and $E_\alpha(\xi) = [(\psi_M F)(A(\xi))\mathscr{F}(f_\alpha \Pi_+(D)\varphi)(\xi)]\exp(\mathrm{i}\langle\alpha, \xi\rangle)$. The integration domain K in (4.10) is defined by

$$K = \{\xi \,;\, c_1 \leqslant |\xi| \leqslant c_2\}, \quad c_1 = (Mv_{\max})^{-1}, \quad c_2 = M/v_{\min}.$$

Introduce polar coordinates $\lambda > 0$, $\omega \in S^{m-1}$, such that $\xi = \lambda\omega$. Then the right hand side in (4.10) takes the form

$$(4.11) \qquad \sum_{j=1}^{s} \int_{S^{m-1}} \int_{c_1}^{c_2} D^{\alpha}_{kj}(\lambda\omega)\exp(\lambda[\mathrm{i}\langle x - \alpha, \omega\rangle - \mathrm{i}ta_j(\omega)])\lambda^{m-1}\,\mathrm{d}\lambda\mathrm{d}\omega.$$

In order to integrate by parts with respect to λ in (4.11) consider the operators $L_j(x, \alpha, \omega, t, \partial_\lambda) = -\mathrm{i}[\langle x - \alpha, \omega\rangle - ta_j(\omega)]^{-1}\partial_\lambda$. It is clear that the exponential function in (4.11) can be represented as $L_j^N\{\exp(\lambda[\mathrm{i}\langle x - \alpha, \omega\rangle - \mathrm{i}ta_j(\omega)])\}$. The definition (3.2) of the function g_+ guarantees that $\langle \omega, \alpha/|\alpha|\rangle \geqslant -v_{\min}/(4v_{\max})$, when $\omega \in \operatorname{supp} g_+(\langle\omega, \alpha/|\alpha|\rangle)$. On the other hand, we have $a_j(\omega) \geqslant v_{\min} > 0$ for $j = 1, \ldots, s$ and we obtain

$$\langle x - \alpha, \omega\rangle - ta_j(\omega) \leqslant |x| + |\alpha|v_{\min}/(4v_{\max}) - tv_{\min}.$$

Since we consider the case (B), for $|x| < \rho$ we find

$$\langle x - \alpha, \omega\rangle - ta_j(\omega) \leqslant \rho + \rho v_{\min}/2v_{\max} + tv_{\min}/2 - tv_{\min} \leqslant 2\rho - tv_{\min}/2 =$$
$$= 2\rho - tv_{\min}/4 - tv_{\min}/4.$$

Comparing (4.5) and the condition (B), we derive $\rho + tv_{\max} \geqslant |\alpha|/2 \geqslant \rho + 8\rho v_{\max}/v_{\min}$. Hence, $2\rho - tv_{\min}/4 \leqslant 0$ and $|\langle x - \alpha, \omega\rangle - ta_j(\omega)| \geqslant tv_{\min}/4$. The last inequality shows that the differential operators L_j are defined correctly and integrating by parts in (4.11) we obtain the estimate

$$(4.12) \qquad |(U_0(t)P^{\text{out}}_{\alpha,+}\varphi)_k| \leqslant C_N t^{-N} \int_{S^{m-1}} \mathrm{d}\omega \int_{c_1}^{c_2} |(\partial_\lambda)^N(\lambda^{m-1}E_\alpha(\lambda\omega))|\,\mathrm{d}\lambda.$$

Now we can use the equalities $\partial_\lambda = \sum_{j=1}^{m} \xi_j/|\xi|\, \partial_{\xi_j}$ and $(\partial_\lambda)^N = \sum_{|\alpha|\leqslant N} c_\alpha(\xi)\, \partial_\xi^\alpha$, where $c_\alpha(\xi)$ are functions smooth in $\mathbf{R}^m\setminus\{0\}$ and bounded in K. This argument allows us to estimate the right hand side of (4.12) by

$$C_N t^{-N} \sum_{|\alpha|\leqslant N} \int_K |\xi|^{1-m} |\partial_\xi^\alpha[|\xi|^{m-1} E_\alpha(\xi)]|\, d\xi \leqslant$$

$$\leqslant C'_N t^{-N} \|(1-\Delta_\xi)^{N/2}[|\xi|^{m-1}E_\alpha(\xi)]\|.$$

Since $E_\alpha(\xi) = [(\psi_M F)(A(\xi))\mathscr{F}(f_\alpha \Pi_+(D)\varphi)(\xi)]\exp(\mathrm{i}\langle\alpha, \xi\rangle)$ and K is a compact set disjoint from 0, by using the Leibniz rule in the right hand side of the last estimate, we obtain

$$
\begin{aligned}
&|\chi(|x| \leqslant \rho) U_0(t) P^{\text{out}}_{\alpha,+}\varphi| \leqslant \\
&\leqslant C_N t^{-N} \|(1-\Delta)^{N/2}(\mathscr{F}(f_\alpha \Pi_+(D)\varphi)\exp(\mathrm{i}\langle\alpha,\xi\rangle))\| = \\
(4.13)\qquad &= C'_N t^{-N}\|(1+|x|^2)^{N/2}(f_\alpha\Pi_+(D)\varphi)(x+\alpha)\| \leqslant \\
&\leqslant C'_N t^{-N}\|(1+|x|^2)^{N/2} f_0(x)\|_\infty \|\Pi_+(D)\varphi\| \leqslant \\
&\leqslant C''_N t^{-N}\|\varphi\|,
\end{aligned}
$$

since $f_\alpha(x+\alpha) = f_0(x)$. Hence, after taking the norms in the ball $\{|x| \leqslant \rho\}$ of the both sides of the inequality (4.13) we obtain (4.4) in the case (B).

The proof of (4.2) is similar. Consider (4.3). The adjoint operator $P^{\text{in}*}_{\alpha,\sigma}$ has the form $\Pi_\sigma(D) f_\alpha G^{\text{in}*}_{\alpha,\sigma}$. Given any $\varphi \in H_0$, we see that the Fourier transform of $P^{\text{in}*}_{\alpha,\sigma}\varphi$ is a smooth vector valued function with values in $\operatorname{Ran}\Pi_\sigma(\xi)$. Then we have the inclusion

$$
\begin{aligned}
&\operatorname{supp}\mathscr{F}(P^{\text{in}*}_{\alpha,\sigma}\varphi) \subset \operatorname{supp}\hat{f}_\alpha + \operatorname{supp} G^{\text{in}}_{\alpha,\sigma}(\xi) \subset \\
(4.14)\qquad &\subset \operatorname{supp}\hat{f}_\alpha + \operatorname{supp} g_\sigma(\langle\alpha/|\alpha|, \xi/|\xi|\rangle) \cap K \subset \\
&\subset \{\xi : |\xi| \leqslant \varepsilon_M\} + \operatorname{supp} g_\sigma(\langle\alpha/|\alpha|, \xi/|\xi|\rangle) \cap K.
\end{aligned}
$$

Choosing $\varepsilon_M > 0$ sufficiently small we can arrange the property

$$
(4.15)\qquad \begin{cases} \text{if } \xi \in \operatorname{supp}\mathscr{F}(P^{\text{in}*}_{\alpha,\sigma}\varphi) \text{ then } \langle\alpha/|\alpha|, \xi/|\xi|\rangle \geqslant -3v_{\min}/(8v_{\max}) \text{ and} \\ \xi \in K' = \{\xi\,;\, c_1/2 \leqslant |\xi| \leqslant 2c_2\}. \end{cases}
$$

Therefore, we can apply the above used technique to prove the estimate (4.3).

This completes the proof of the theorem.

5. PROOF OF THEOREM 1.1

In this section we are going to prove Theorem 1.1, following closely the approach in [21], [4]. According to Remarks 3 and 4 we have to show that

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \|[\tilde{V}(t) - U_0(t)]\tilde{V}(t_n)\tilde{\psi}\| = 0 \tag{5.1}$$

for any $\psi \perp H_b$ and for suitably chosen sequence $t_n \to \infty$, depending on ψ. Lemma 2 from Section 9 in [21] asserts that the set

$$\{(G-1)^{-2}G\varphi \ ; \ \varphi \in D(G) \cap H_b^{\perp}\}$$

is dense in $H_b^{\perp}$. So, it sufficies to prove (5.1) for $\psi = F(\mathrm{i}G)\varphi$, where $\varphi \perp H_b$ and $\varphi \in \mathrm{D}(G)$. Corollary 2.2 enables one to find a sequence t_n tending to $+\infty$, such that

$$\|\chi(|x| \leqslant n)\,\varphi_n\| \to 0 \quad \text{as } n \to +\infty. \tag{5.2}$$

Here $\varphi_n = V(t_n)\varphi$, $\psi_n = V(t_n)\psi$. We write down the following decomposition, introduced by B. Simon in [21]

$$\tilde{\psi}_n = \psi_{n,M}^{(1)} + \psi_{n,M}^{(2)} + \psi_{n,M}^{\mathrm{out}} + \psi_{n,M}^{\mathrm{in}},$$

where (see (3.5))

$$\begin{aligned} \psi_{n,M}^{(1)} &= (F(\mathrm{i}G) - F(\mathrm{i}G_0))\tilde{\varphi}_n + \\ &+ \sum_{|\alpha|\leqslant n/2} \sum_{\sigma=\pm} (\psi_M F)(\mathrm{i}G_0)\Pi_\sigma(D)f_\alpha\Pi_\sigma(D)\tilde{\varphi}_n, \end{aligned} \tag{5.3}$$

$$\psi_{n,M}^{(2)} = (I - \psi_M(\mathrm{i}G_0))F(\mathrm{i}G_0)\tilde{\varphi}_n, \tag{5.4}$$

$$\psi_{n,M}^{\mathrm{out}} = P_n^{\mathrm{out}}\tilde{\varphi}_n, \quad \psi_{n,M}^{\mathrm{in}} = P_n^{\mathrm{in}}\tilde{\varphi}_n. \tag{5.5}$$

Let $M > 2$ be fixed. First we shall verify the property

$$\|\psi_{n,M}^{(1)}\| \text{ tends to 0 as } n \text{ tends to } \infty. \tag{5.6}$$

Consider the inequality

$$\begin{aligned} \|[F(\mathrm{i}G) - F(\mathrm{i}G_0)]\tilde{\varphi}_n\| &\leqslant C\|\,\chi(|x| \leqslant n)\tilde{\varphi}_n\| + \\ &+ C\|[F(\mathrm{i}G) - F(\mathrm{i}G_0)]\,\chi(|x| \geqslant n)\|\,\|\varphi_n\|. \end{aligned}$$

This inequality together with (5.2) and Proposition 2.6 guarantees that the first term in the right hand side of (5.3) tends to 0. In order to estimate the second term in the right hand side of (5.3), we shall use the equality

$$(\psi_M F)(\mathrm{i}G_0)\Pi_\sigma(D)f_\alpha\Pi_\sigma(D)\tilde{\varphi}_n = (\psi_M F)(\mathrm{i}G_0)\Pi_\sigma(D)f_\alpha\Pi_\sigma(D)\gamma(D)\tilde{\varphi}_n .$$

The function $\gamma(\xi) \in C_0^\infty(\mathbf{R}^m\setminus\{0\})$ is chosen so that $\gamma=1$ on K' (see (4.14) and (4.15)). Using the inequalities $\|\varphi_n\| \leqslant \|\varphi\|$ and $\sum f_\alpha \leqslant 1$, we obtain

$$\|\sum_{|\alpha|\leqslant n/2}\sum_{\sigma=\pm}(\psi_M F)(\mathrm{i}G_0)\Pi_\sigma f_\alpha \Pi_\sigma \tilde{\varphi}_n\| \leqslant$$

$$\leqslant C\sum_{\sigma=\pm}\|\sum_{|\alpha|\leqslant n/2} f_\alpha(\Pi_\sigma\gamma)(D)\tilde{\varphi}_n\| \leqslant$$

$$\leqslant C_1\|\chi(|x| \leqslant n)\tilde{\varphi}_n\| + C_1\|\sum_{|\alpha|\leqslant n/2} f_\alpha(x)\chi(|x| > 3n/4)\|_\infty +$$

$$+ C_1\sum_{\sigma=\pm}\|\chi(|x| \leqslant 3n/4)(\Pi_\sigma\gamma)(D)\chi(|x| > n)\|.$$

The property (5.2), the definition (3.1) of the function f_α, and Lemma 4.2 lead to the property (5.6).

The function $\varphi_{n,M}^{(2)}$ can be estimated as follows

$$\|\varphi_{n,M}^{(2)}\| \leqslant \|(1-\psi_M)F\|_\infty\|\varphi\| = \varepsilon(M) \to 0 \quad \text{as } M \to +\infty. \tag{5.7}$$

Let us turn now to the analysis of the outgoing part of Enss' decomposition. Applying Proposition 2.4, we get

$$\|[U_0(t) - \tilde{V}(t)]\,\psi_{n,M}^{\mathrm{out}}\| \leqslant \|\theta\psi_{n,M}^{\mathrm{out}}\| +$$

$$+ \|\theta U_0(t)\,\psi_{n,M}^{\mathrm{out}}\| + \int_0^t \|QU_0(s)\psi_{n,M}^{\mathrm{out}}\|\,\mathrm{d}s.$$

Theorem 4.1 and the above estimate yield

$$\lim_{n\to+\infty}\sup_{t\geqslant 0}\|[U_0(t) - \tilde{V}(t)]\,\psi_{n,M}^{\mathrm{out}}\| = 0. \tag{5.8}$$

Finally, we shall prove that

$$\lim_{n\to+\infty}\|\psi_{n,M}^{\mathrm{in}}\| = 0, \tag{5.9}$$

following the approach in Section 9 of [21]. Since $\|\theta(x)\varphi_n\|\to 0$ and since the operators P_n^{in} are uniformly bounded according to Proposition 3.1, it is sufficient to prove that

$$\|P_n^{\text{in}}(1-\theta)\tilde{V}(t_n)\tilde{\varphi}\| \to 0 \quad \text{as } n \to +\infty. \tag{5.10}$$

The property (5.10) follows from

$$\|P_n^{\text{in}}[(1-\theta)\tilde{V}(t_n) - U_0(t_n)]\| \to 0, \tag{5.11}$$

$$\|P_n^{\text{in}} U_0(t_n) g\| \to 0 \quad \text{for } g \in H_0. \tag{5.12}$$

The first property (5.11) is equivalent to

$$\|[\tilde{V}(t_n)^*(1-\theta) - U_0^*(t_n)]P_n^{\text{in}*}\| \to 0. \tag{5.13}$$

To obtain (5.13) we use the following variant of (2.1)

$$\begin{aligned}[U_0^*(t) - \tilde{V}^*(t)]h &= -\tilde{V}^*(t)\theta h + \theta U_0^*(t)h + \\ &+ \int_0^t \tilde{V}^*(t-s)Q_1 U_0^*(s)\,ds,\end{aligned} \tag{5.14}$$

where $Q_1(x)$ is a matrix-valued function with elements in $C_0^\infty(\mathbf{R}^m)$. From (5.14) we derive the estimate

$$\begin{aligned}\|[\tilde{V}^*(t_n)(1-\theta) - U_0^*(t_n)]P_n^{\text{in}*}\| &\leqslant \|\theta U_0^*(t_n)P_n^{\text{in}*}\| + \\ &+ \int_0^{t_n} \|Q_1 U_0^*(t)P_n^{\text{in}*}\|\,dt.\end{aligned}$$

Applying the inequality (4.3) from Theorem 4.1, we prove (5.11). In order to check (5.12) choose a function $g \in C_0^\infty(\mathbf{R}^m; \mathbf{C}^r)$ with support in $\{x\,;\,|x| \leqslant R\}$. Then the equality

$$\|P_n^{\text{in}} U_0(t)\,\chi(|x| \leqslant R)\| = \|\chi(|x| \leqslant R)U_0^*(t)P_n^{\text{in}*}\|$$

and Theorem 4.1 show that (5.12) holds for smooth functions g with compact support. Since these functions form a dense subset in H_0 and the operators $P_n^{\text{in}}U_0(t)$ are uniformly bounded, we conclude that (5.12) holds for any $g \in H_0$. This completes the proof of (5.9).

From (5.6)—(5.9) we obtain

$$\lim_{n\to+\infty} \sup_{t\geqslant 0}\|(U_0(t) - \tilde{V}(t))\tilde{V}(t_n)\tilde{\psi}\| \leqslant 2\varepsilon(M).$$

Taking $M \to +\infty$, we complete the proof of the theorem.

Proof of Corollary 1.2. We have to show that the limit

$$\lim_{t \to +\infty} U_0(-t)\tilde{V}(t)\tilde{f} \tag{5.15}$$

exists for $f \perp H_b$. Given any $f \perp H_b$, choose the sequence $\{t_n\}$ according to Theorem 1.1. For $t > t_n$, $s > t_n$ we have

$$\|U_0(-s)\tilde{V}(s)\tilde{f} - U_0(-t)\tilde{V}(t)\tilde{f}\| \leqslant \|[U_0(s-t_n) - \tilde{V}(s-t_n)]\tilde{V}(t_n)\tilde{f}\| + \\ + \|[U_0(t-t_n) - \tilde{V}(t-t_n)]\tilde{V}(t_n)\tilde{f}\|.$$

Applying Theorem 1.1 and the above estimate, we prove that the limit (5.15) exists. This proves the corollary.

Proof of Corollary 1.3. Suppose the contrary. Then we can find a sequence φ_n, such that $G\varphi_n = \mathrm{i}\mu_n\varphi_n$, μ_n being real numbers tending to the fixed number $\mu \neq 0$ and $\|\varphi_n\| = 1$. According to the results of B. Simon [21] we have $G^*\varphi_n = -\mathrm{i}\mu_n\varphi_n$ and $\varphi_k \perp \varphi_n$ for $k \neq n$. Hence, the sequence φ_n tends weakly to 0. Moreover the coercive estimate (H_3) implies that $\|\chi(|x| \leqslant n)\, \varphi_{k_n}\|$ tends to 0 for suitably chosen subsequence φ_{k_n}. Without loss of generality we can consider that this subsequence is φ_n. Therefore, setting $\psi_n = (\mu_n - \mathrm{i})^{-2}\mu_n\varphi_n$, we can use the decomposition (5.3)—(5.5). Then the properties (5.6)—(5.8) are fulfilled. Following the proof of (5.8) and using (4.2), we get

$$\lim_{n\to\infty} \sup_{t \geqslant 0} \|[U_0^*(t) - \tilde{V}^*(t)]\psi_{n,M}^{\mathrm{in}}\| = 0. \tag{5.16}$$

Now we shall use the equality

$$(\tilde{\psi}_n, \psi_{n,M}^{(1)}) + (\tilde{\psi}_n, \psi_{n,M}^{(2)}) + (\tilde{\psi}_n, \psi_{n,M}^{\mathrm{out}}) + (\tilde{\psi}_n, \psi_{n,M}^{\mathrm{in}}) = (\psi_n, \psi_n). \tag{5.17}$$

Since $\|\psi_n\| \leqslant C = \sup|\mu_n(\mu_n - \mathrm{i})^{-2}| < \infty$, the first term in (5.17) tends to 0, according to (5.6). The second term can be estimated above by $C\varepsilon(M)$. For the third term we have

$$|(\tilde{\psi}_n, \psi_{n,M}^{\mathrm{out}})| = |(\tilde{V}^*(t)\tilde{\psi}_n, \psi_{n,M}^{\mathrm{out}})| = |(\tilde{\psi}_n, \tilde{V}(t)\psi_{u,M}^{\mathrm{out}})| = \\ = \lim_{t\to+\infty} |(\tilde{\psi}_n, (\tilde{V}(t) - U_0(t))\psi_{n,M}^{\mathrm{out}})| \leqslant C \sup_{t>0} \|(\tilde{V}(t) - U_0(t))\psi_{n,M}^{\mathrm{out}}\|.$$

The property (5.8) implies that $(\tilde{\psi}_n, \psi_{n,M}^{\mathrm{out}})$ tends to 0 as n tends to $+\infty$. Similarly one can prove that $(\tilde{\psi}_n, \psi_{n,M}^{\mathrm{in}})$ tends to 0, using (5.16). Thus, we conclude that ψ_n tends to 0. This convergence contradicts the fact that

$$\|\psi_n\| = |(\mu_n - \mathrm{i})^{-2}\mu_n| \to |(\mu - \mathrm{i})^{-2}\mu| \neq 0.$$

This completes the proof of the corollary.

6. PROOF OF THEOREM 1.4 AND THEOREM 1.5

First, we shall prove the existence of the limits

$$\Omega_{\theta,+} = \text{s-}\lim_{t\to+\infty} \tilde{V}(t)\theta U_0(-t)\, P_{ac}(G_0),$$

$$\Omega_{\theta,-} = \text{s-}\lim_{t\to+\infty} \tilde{V}^*(t)\theta U_0(t)\, P_{ac}(G_0).$$

Here $\theta = \theta(x)$ is a smooth function, such that $\theta(x) = 0$ for $|x| \leqslant R_0 + 1$ and $\vartheta(x) = 1$ for $|x| \geqslant R_0 + 2$. To prove the existence of the operator $\Omega_{\theta,+}$ we use the equality

$$\Omega_{\theta,+}(t)f - \Omega_{\theta,+}(s)f = \int_s^t \tilde{V}(\sigma)Q_2U_0(-\sigma)P_{ac}(G_0)f\,d\sigma,$$

where $\Omega_{\theta,+}(t) = \tilde{V}(t)\theta U_0(-t)\, P_{ac}(G_0)$ and $Q_2 = G\theta - \theta G_0$ is a matrix-valued function with elements in $C_0^\infty(\mathbf{R}^m)$. On the other hand, integrating by parts, it is not difficult to obtain the estimate

$$\|Q_2U_0(t)\, P_{ac}(G_0)f\| \leqslant C_{N,f}(1 + |t|)^{-N} \tag{6.1}$$

for $f \in S(\mathbf{R}^m; \mathbf{C}^r)$. Indeed, Proposition 2.3 implies that $P_{ac}(G_0) = \Pi(D)$, where $\Pi = \Pi_+ + \Pi_-$. Hence, we have the equality

$$\begin{aligned} Q_2U_0(t)P_{ac}(G_0)f &= (2\pi)^{-m}Q_2 \times \\ &\times \int_{S^{m-1}} d\omega \int_0^{+\infty} \lambda^{m-1}\Pi(\omega)\hat{f}(\lambda\omega)\exp[i\lambda(\langle x, \omega\rangle - tA(\omega))]\,d\lambda. \end{aligned} \tag{6.2}$$

Consider the first order differential operator

$$L = -[\langle x, \omega\rangle - tA(\omega)]^{-1}\Pi(\omega)\, i\partial_\lambda.$$

If $t > 2\max\{x\,;\, x \in \operatorname{supp} Q_2(x)\}/v_{\min}$, we have the estimate

$$|[\langle x, \omega\rangle - tA(\omega)]^{-1}\Pi(\omega)| \leqslant Ct^{-1}.$$

Integrating by parts in (6.2), we obtain (6.1). This proves the existence of the limits $\Omega_{\theta,+}$. It is well known (see Proposition 5, XI.3 in [19]) that this fact implies the existence of the wave operator Ω_+. The existence of the wave operator Ω_- can be established by the same manner.

Finally, we turn to the point b) of Theorem 1.4. Here we follow essentially [4]. First, we shall verify the inclusion

$$\operatorname{Ran} \Omega_- \subset H_b^\perp \ominus H_\infty^- . \tag{6.3}$$

Let $\psi \in D(G)$ be an eigenvector of the operator G with eigenvalue on the imaginary axis, i.e. $G\psi = i\mu\psi$ for some real number μ. Then $V(t)\psi = \psi \exp(it\mu)$. From the equality

$$(\Omega_- f, \psi) = \lim_{t \to +\infty} (U_0(t) P_{ac}(G_0) f, \tilde{V}(t)\tilde{\psi}), \quad f \in H_0, \tag{6.4}$$

we conclude that

$$(\Omega_- f, \psi) = \lim_{t \to +\infty} (U_0(t) P_{ac}(G_0) f, \tilde{\psi}) \exp(-it\mu).$$

It is well known that $U_0(t)P_{ac}(G_0)f$ tends weakly to 0 as t tends to infinity. Hence $(\Omega_- f, \psi) = 0$ for any $\psi \in H_b$ and $f \in H_0$. The last property leads directly to the inclusion

$$\operatorname{Ran} \Omega_- \subset H_b^\perp .$$

Given any $\psi \in H_\infty^-$ we have $\lim_{t \to +\infty} \|V(t)\psi\| = 0$. Then the equality (6.4) shows that $\Omega_- f$ is orthogonal to the space H_∞^-.

Finally, we shall prove that

$$\overline{\operatorname{Ran} \Omega_-} = H_b^\perp \ominus H_\infty^- , \tag{6.5}$$

which is the conclusion b) of Theorem 1.4. Consider the restriction of the operator

$$\Omega_- W = \text{s-}\lim_{t \to \infty} V^*(t) J_0^* P_{ac}(G_0) J_0 V(t) \tag{6.6}$$

on the Hilbert space $H_- = H_b^\perp \ominus H_\infty^-$. The inclusion (6.3) implies that $\operatorname{Ran} \Omega_- W \subseteq H_-$. On the other hand the operator $\Omega_- W$ is a selfadjoint contraction operator. To prove that the closure of its image is H_- it sufficies to check the equality

$$\operatorname{Ker}(\Omega_- W) \cap H_- = \{0\}.$$

Indeed, assuming $\Omega_- W f = 0$ for some $f \in H_b^\perp \ominus H_\infty^-$, we obtain from (6.6) the relation

$$0 = (\Omega_- W f, f) = \lim_{t \to +\infty} \|P_{ac}(G_0) \tilde{V}(t) \tilde{f}\|^2. \tag{6.7}$$

Denote by $P_0 = I - P_{ac}(G_0)$ the orthogonal projection onto $\operatorname{Ker} G_0$. Then one can prove the equality

$$\lim_{n\to\infty} \|P_0\tilde{V}(t_n)\tilde{f}\| = 0 \tag{6.8}$$

for some suitable sequence $t_n \to +\infty$. More precisely, there exists a sequence t_n tending to ∞, such that $P_0\tilde{V}(t_n)\tilde{f}$ tends weakly to 0, according to Corollary 2.2. On the other hand,

$$P_0\tilde{V}(t_n)\tilde{f} = P_0U_0(-t_n)\tilde{V}(t_n)\tilde{f} \to P_0W\tilde{f}, \quad \text{as } n \to \infty.$$

Hence $P_0\tilde{V}(t_n)\tilde{f}$ tends strongly to 0.

From (6.7), (6.8) we derive that $V(t_n)f$ tends strongly to 0. Since $V(t)$ are contraction operators, we obtain

$$\lim_{t\to+\infty} \|V(t)f\| = 0$$

and hence $f \in H_\infty^-$. Our choice of the element f implies $f = 0$.

This proves (6.5) and completes the proof of Theorem 1.4.

Proof of Theorem 1.5. One can obtain the following analogue of the inclusion (6.5)

$$\operatorname{Ran}\Omega_+ \subset H_b^\perp \ominus H_\infty^+.$$

In its proof an essential role plays Theorem 9.1 in [21], which asserts that if $G\varphi = i\mu\varphi$, $\mu \in \mathbf{R}$, then $G^*\varphi = -i\mu\varphi$ and $V^*(t)\varphi = \varphi\exp(-it\mu)$. Hence the scattering operator $S = W\Omega_+$ is well-defined.

APPENDIX A1

LEMMA A.1. *Suppose the hypotheses* (H_1)—(H_3) *fulfilled. Then the operator* $E^{-1}G$ *is a generator of a contraction semigroup in* H_E.

Proof. The proof is based on the results of Rauch [17]. According to Remark 3, we can assume $E(x) = E^0 = I$. To prove the lemma it is sufficient to show that (see [18])

$$\operatorname{Re}(Gu, u) \leqslant 0 \quad \text{for each } u \in D(G), \tag{1}$$

$$\operatorname{Ran}(G - I) = H. \tag{2}$$

The property (1) follows immediately from the definition of D(G). Consider (2): Theorem 5 in [17] shows that for each $f \in H$ the equation $(G - I)u = f$ has a unique solution $u \in H$ satisfying the boundary conditions in a generalized sense [17]. This solution belongs to $D(G)$ according to Theorem 4 in [17]. Thus (2) is established. This completes the proof of the lemma.

APPENDIX A2

Next we shall consider the important example of the Maxwell equations in homogeneous media

$$\varepsilon(x)\partial_t E = \nabla \times H, \quad \mu(x)\partial_t H = -\nabla \times E, \quad \text{on } (0,\infty) \times \Omega, \tag{1}$$
$$\langle \nabla, \varepsilon(x)E \rangle = 0, \quad \langle \nabla, \mu(x)H \rangle = 0 \quad \text{on } (0,\infty) \times \Omega,$$

where E and H are the electric and magnetic fields respectively, while $\varepsilon(x)$, $\mu(x)$ are (3×3) symmetric positive matrices, connected with the anisotropy of the media. The unperturbed system is

$$\varepsilon_0 \partial_t E = \nabla \times H, \quad \mu_0 \partial_t H = -\nabla \times E, \quad \text{on } (0,\infty) \times \mathbf{R}^3, \tag{2}$$
$$\langle \nabla, \varepsilon_0 E \rangle = 0, \quad \langle \nabla, \mu_0 H \rangle = 0 \quad \text{on } (0,\infty) \times \mathbf{R}^3.$$

We assume that for some $R_0 > 0$ we have $\varepsilon(x) = \varepsilon_0$, $\mu(x) = \mu_0$ provided $|x| \geqslant R_0$. According to the results of Majda [13] any strictly dissipative boundary condition for Equation (1) satisfies the coercive estimate (H_3). There is a simple example of such boundary conditions [13]

$$n \times [E + \gamma(n \times H)] = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $\gamma(x)$ is a smooth positive function on the boundary $\partial\Omega$. Equations (1), (3) can be written in the form of the mixed problem (0.1), discussed in the previous sections. For the purpose it is sufficient to set

$$A(\xi) = \begin{pmatrix} 0 & -\xi\times \\ +\xi\times & 0 \end{pmatrix}, \quad E(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}, \quad u = \begin{pmatrix} E \\ H \end{pmatrix}.$$

Note that the last two equations in (2) and (1) are equivalent to the properties $f \perp \operatorname{Ker}(E_0^{-1}G_0)$ and $f \perp \operatorname{Ker}(E^{-1}G)$ respectively. The results in [13] guarantee that the hypotheses (H_1)—(H_3) are fulfilled. Moreover, choosing $\varepsilon_0^{-1} = \operatorname{diag}(1, 2, 3)$, $\mu_0 = I$, it is not difficult to see that for $\xi = (\pm \lambda, 0, \lambda)$ the eigenvalues of the matrix $E_0^{-1}A(\xi)$ are 0, 2λ, -2λ and each eigenvalue has multiplicity 2. On the other hand, for any other choice of $\xi \neq 0$ we have four pure nonzero eigenvalues.

So the treatment of the systems with characteristics of variable multiplicities is well-motivated.

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