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L^p AND SPECTRAL THEORY FOR A CLASS OF GLOBAL ELLIPTIC OPERATORS

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The present paper deals with certain aspects of "global elliptic theory" on $\mathbf{R}^n$ and continues our research started in [13], [14], [15], [16]. To explain the problem and provide motivation we shall describe some essential features of "local theory" (compact manifolds and domains) of regular elliptic operators.

One such feature is the presence of a "single model", the Laplacian $-\Delta$ upon which the theory is based. Indeed, the scale of Sobolev spaces H_s is defined, in terms of "fractional Laplacians" $(1-\Delta)^s$. All operators can be compared by their order = degree of the principal symbols $a(x, \xi)$ in the ξ-variable. Precisely, $\deg b(x, \xi) < \deg a(x, \xi)$ implies "$B = b(x, D)$ is small (compact) relative to $A = a(x, D)$". Therefore, lower order terms can often be ignored in calculation.

An illustration of this principle is the celebrated "Weyl formula" for asymptotic distribution of large eigenvalues of a selfadjoint elliptic operator

$$N(\lambda) = \#\{\text{eigenvalues} \leqslant \lambda\} \sim \mathrm{Vol}\{(x, \xi) : \text{principal symbol } A \leqslant \lambda\} \quad \text{as } \lambda \to \infty.$$

Notice that in the local theory any regular elliptic operator can serve as a "model", as all of them are comparable to $(-\Delta)^s$, and hence to each other.

In the global setting $\mathbf{R}^n$ (non compact manifolds) or in the case of degenerate (singular) ellipticity no such single model exists. Depending on the particular type of degeneracy (singularity) of coefficients one can construct different classes of operators and appropriate "Sobolev scales". Degeneracy-singularity is often measured in terms of the distance function $d(X) = \mathrm{dist}(x; \partial\Omega)$, to the boundary of the region Ω. Precisely, let $a(x, \xi) = \sum a_{ij}(x)\, \xi_i \xi_j$ be the principal symbol of A. Denote by $\{\lambda_1(x), \ldots, \lambda_n(x)\}$ the eigenvalues of matrix $(a_{ij}(x))$ and by $\lambda(x)$ the isotropic modulus of ellipticity,

$$\lambda(x)|\xi|^2 \leqslant \sum a_{ij}(x)\xi_i\xi_j \leqslant \mathrm{Const}\, \lambda(x)|\xi|^2.$$

Typical conditions are given either in terms of $\lambda(x)$ or $\{\lambda_1, \ldots, \lambda_n\}$ as

(I) $\lambda(x) = \mathrm{O}(d^{-\alpha}(x))$ as $x \to \partial\Omega$ (singular)

(II) $\lambda(x) = \mathrm{O}(d^{\alpha}(x))$ as $x \to \partial\Omega$ (degenerate).

Degenerate elliptic operators of type II are usually called "Tricomi" (or "Legendre" in the ODE case). Both classes were studied by a number of authors (see [29] and references there).

The class of operators considered in the present paper is modelled after $-\nabla\cdot\rho\nabla + V$ (and its powers), where both parameters: "metric" ρ and "potential" V can be "large" (singular) or "small" (degenerate) on a "thin" subset $\Sigma\subset\mathbf{R}^n$.

An interesting case arises when "degeneracy" of ρ is combined with "singularity" of V,

$$\rho \approx d(x, \Sigma)^\alpha \quad \text{and} \quad V\approx d(x, \Sigma)^{-\beta}.$$

These are so called Laguerre-type operators modelled after the classical: $\partial x\partial + \dfrac{x^2+\alpha^2}{4x}$ on $(0, \infty)$.

More generally we consider m-th order elliptic operators of the form,

$$A_0 = \rho \sum_{|\alpha|=m} a_\alpha \mathrm{D}^\alpha + V = \rho a(x;\mathrm{D}) + V; \quad \mathrm{D} = \frac{1}{i}\partial_x\,;\ \alpha = (\alpha_1, \ldots, \alpha_n);$$

with "uniformly elliptic" symbol $a(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ and their perturbations $B = \sum_{|\alpha|\leqslant m} b_\alpha(x)\mathrm{D}^\alpha$ with possibly singular coefficients b_α.

Typical problems of operator theory that will be addressed in the present work are:

(I) Closedness and essential selfadjointness of operators A in L^2 and other L^p spaces, their L^p-domains.

(II) Existence and estimates of the resolvent $R_\varsigma = (\varsigma - A)^{-1}$, the semigroup $\{e^{-tA}\}$ and other related "kernels of A".

(III) Smoothing properties of $\{R_\varsigma\}$ and $\{e^{-tA}\}$ in L^p or Sobolev scales (so called "hypercontractivity problem").

(IV) Conditions for compactness of $(\varsigma - A)^{-1}$, and "relative compactness" of B with respect to A_0; structure of L^2 (L^p)-spectra of A.

(V) Estimates and asymptotics of eigenvalues and eigenfunctions of A.

The basic tool in studying problems, (I—V) is the resolvent kernel of A. We construct and estimate R_ς in Section 2. The method of Section 2 follows ideas of our earlier work [14], [15], [16], namely, using pseudodifferential calculus and perturbation expansions. However, the analysis becomes more involved due to a more complicated structure of operators.

After the main estimates (Lemma 1) we proceed to corollaries and applications (Theorems 1 — 4), which include

(i) L^p-closedness and essential selfadjointness of A;

(ii) bounds on L^p-spectra of A;

(iii) resolvent summability;

(iv) existence of a holomorphic semigroup e^{-tA};

(v) "L^p-smoothing" of kernels R_ς and e^{-tA} ("supercontractivity").

Section 2 is preceded by an introductory part (Section 1), which describes pairs of parameters (ρ, V) and introduces the corresponding classes of symbols and operators A_0 and $A = A_0 + B$. We establish basic properties of A_0 and A, such as product formula, adjoint, etc., and give a few examples both of second (Schrödinger-type) and higher order. Among others they include some classical orthogonal polynomial expansions, like Hermite and Laguerre, as well as their multivariable modifications.

The last section of the paper (Section 3) is devoted to spectral theory of operators A_0 and A. First we give conditions for compactness of R_ς and relative compactness of B with respect to A_0 and establish "L^p-stability" of the discrete spectra. Finally for self-adjoint operators A with purely discrete spectrum we derive an analogue of the "Weyl formula" (1) for asymptotic distribution of large eigenvalues (Theorem 6).

The latter problem has a long history (since the early work of H. Weyl in 1913) and an extensive literature.

Most of the known results fall into two large categories: "local results" on compact manifolds and domains (see for instance [1], [17]); "global results" for Schrödinger operators $-\Delta + V(x)$ on $\mathbf{R}^n$ ([23]).

There are fewer global results for operators of higher order and "singular coefficients". We shall mention the papers [29], [19], which established "Weyl formulas" for fairly general pseudodifferential operators of "Weyl-type"; the paper [5] which treats differential operators with singular leading coefficients, recent works [24] and [26], [27] on newly discovered "nonclassical asymptotics" for Schrödinger operator with "degenerate potential", and the monograph [29], which treats among others Tricomi operators. Our result is close to [10], but it differs from all of the above references due to its emphasis on "degenerate-singular" coefficients.

In conclusion let us remark that Theorem 6 along with other results of the paper extends to pseudodifferential operators of the form $\rho\alpha(x, D) + V$.

Another extension includes a wider class of "nonisotropically deformed" symbols $a(x, \xi) \to a(x, \rho\xi) + V$, where the scalar (conformally flat) metric, of the present work is replaced by the matrix-valued $\rho = (\rho_{ij}(x))_1^n$.

This extension will be discussed elsewhere.

We acknowledge the referee's comments that helped us to clarify the argument and improve the presentation of the paper.

1. CLASSES OF SYMBOLS AND OPERATORS

Throughout the paper we shall consider functions and differential operators, defined either on $\mathbf{R}^n$ or a complement $\Omega = \mathbf{R}^n \setminus \Sigma$ of a "thin" closed subset Σ in $\mathbf{R}^n$, e.g. a hypersurface or a closed submanifold.

The coefficients (symbols) of operators are assumed to be smooth, and in the case of $\mathbf{R}^n \setminus \Sigma$ they are allowed certain "degeneracy" or "singularity" on Σ. The singularity or degeneracy will by typically measured in terms of the distance function: $d(x) = \mathrm{dist}(x; \Sigma)$.

The following standard notations are used:

1) $L^p(w)$ denotes a weighted L^p-space on $\mathbf{R}^n$ (or Ω) with the norm

$$\|f\| = \left(\int |fw|^p \, \mathrm{d}x \right)^{1/p}.$$

2) $L_0^\infty(\Omega)$ is a subspace of $L^\infty(\Omega)$ which consists of functions that vanish at $\{\Sigma\}$ and $\{\infty\}$, i.e.

$$\sup\{|f(x)| : |x| \geqslant R \text{ and } d(x, \Sigma) < \varepsilon\} \to 0 \quad \text{as } R \to \infty,\ \varepsilon \to 0.$$

3) Given a symbol $p(x, \xi)$ on $\Omega \times \mathbf{R}^n$, $p_{(\beta)}^{(\alpha)}$ denotes its partial derivative $\mathrm{D}_\xi^\alpha \partial_x^\beta p$, of multiorders α in ξ and β in x where $\mathrm{D} = -\mathrm{i}\partial$.

4) If p depends on one set of variables (x or ξ) then $p^{(\alpha)}$ will denote its partial derivative in the appropriate variable.

5) The standard (left) convention will be mostly used for ΨDO's $P = p(x, \mathrm{D})$ with symbols $p(x, \xi)$

$$P[u] = \frac{1}{(2\pi)^n} \iint \mathrm{e}^{\mathrm{i}\xi\cdot(x-y)} p(x, \xi) u(y) \mathrm{d}y \mathrm{d}\xi,$$

but occasionally another (right) convention and more general symbols $p(x, y, \xi)$ will appear.

The class of operators A of the paper is described by two parameters: smooth functions $\rho \geqslant 0$, $V > 0$. Without loss of generality we can assume $V(x) \geqslant 1$.

Any such operator A will typically consist of the principal part $A_0 = \rho \sum_{|\alpha|=m} a_\alpha(x)D^\alpha + V$ and a perturbation $B(x) = \sum_{|\alpha|\leqslant m} b_\alpha(x)D^\alpha$.

Parameters (ρ, V) and coefficients $\{a_\alpha; b_\alpha\}$ are subject to certain constraints which we shall now describe.

These conditions are written in a somewhat technical form, convenient in the proof of resolvent estimates of Section 2, but their real meaning is to provide a control of degeneracy-singularity of (ρ, V) in terms of ratios $\rho^{(\alpha)}/\rho$; $V^{(\beta)}/V$ etc.

The basic hypothesis on $\{a_\alpha(x)\}$ is the standard uniform ellipticity

$$C_1|\xi|^m \leqslant \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \leqslant C_2|\xi|^m \quad \text{with } C_1, C_2 > 0, \text{ uniformly in } x.$$

A prototype of our basic hypothesis (H1) below is the "finite propagation speed" condition of [15] in case of one (leading order) parameter ρ,

$$\rho^{(\alpha)}/\rho = O(\rho^{-|\alpha|/m}), \quad |\alpha| \leqslant m, \tag{1.0}$$

which is close to the more conventional $\displaystyle\int^{+\infty} \frac{dx}{\sqrt[m]{\rho}} = \infty$ (cf. [6]). It essentially limits the rate of growth of ρ at ∞ by $O(|x|^m)$ (for m-th order operators!).

In the case of two parameters (ρ, V) we have more flexibility, and the proper analogue of (1.0) is given in terms of functions

$$F(x) = (\rho/V)^{N/m} \prod_1^k (\rho^{(\alpha^i)}/\rho) \prod_1^l (V^{(\beta^j)}/V) \tag{1.1}$$

where $\alpha^1, \ldots, \alpha^k$; $\beta^1, \ldots, \beta^l$ are tulpes of multiindices with combined norm $N = \sum |\alpha^i| + \sum |\beta^j|$.

We require

(H1) $F(x)$ to belong to L^∞ or L_0^∞ on $\mathbf{R}^n \setminus \Sigma$ for any tuple $\alpha^1, \ldots, \alpha^k$; $\beta^1, \ldots, \beta^l$, of norm $N \leqslant m$ = order of A_0.

An equivalent simpler form of (H1) can be stated in terms of $\log\rho$ and $\log V$,

(H2) $F(x) = (\rho/V)^{N/m}(\log\rho)^{(\alpha)}(\log V)^{(\beta)} \in L^\infty$ (or L_0^∞), for all α, β, whose combined norm $N = |\alpha| + |\beta| \leqslant m$.

The coefficients a_α of the leading part are assumed to satisfy

$$\rho^{|\beta|/m} a_\alpha(\beta) \in L^\infty, \quad |\beta| \leqslant m, \tag{H3}$$

in addition to the uniform ellipticity hypothesis.

Condition (H3) allows to replace the parameter ρ in the numerator of the function F (1.1) by products ρa_α, so that the hypothesis (H1) would still hold. This modification of (H1) will appear in the proof of the Lemma (Section 2).

The coefficients b_α of the perturbation are allowed local L^p-type singularities. Precisely, as in [14] we introduce for each term $b_\alpha D^\alpha$ with $b_\alpha \in L^{p_\alpha}$ its fractional order

$$\nu = \nu_\alpha = \frac{n}{p_\alpha} + |\alpha|, \tag{1.2}$$

and require

(H4) $\nu \leqslant m =$ order of A_0, for all terms $b_\alpha D^\alpha$ of B.

The fractional order ν in (1.2) combines the differential order $|\alpha|$ and local L^p-singularity of the coefficient. The extra term $\frac{n}{p_\alpha}$ corresponds to a shift in the L^p-scale caused by multiplication with an L^{p_α}-singular coefficient b_α: $\frac{1}{p} \to \frac{1}{p} + \frac{1}{p_\alpha}$. This shift can be balanced by a fractional derivative $(-\Delta)^{-s/2}$ of order $s = \frac{n}{p_\alpha}$, according to the Sobolev-Hardy-Littlewood inequality ([28]). So an L^{p_α}-singular coefficient has the same effect on the L^p-scale as a "fractional derivative" of order $\frac{n}{p_\alpha}$.

In addition to (H4) we need some control of b_α relative to parameters (ρ, V). The latter is conveniently expressed in terms of weights

$$w = \rho^{-\nu/m} V^{\nu/m-1} \tag{1.3}$$

where ν is the fractional order of $b_\alpha D^\alpha$. These weights interpolate between $w = 1/\rho$ (for $\nu = m$) and $w = 1/V$ (for $\nu = 0$).

We require b_α to be in the weighted L^p-space

(H5) $b_\alpha \in L^{p_\alpha}(w)$, where $w = \rho^{-\nu/m} V^{\nu/m-1}$ and $\nu = \frac{n}{p_\alpha} + |\alpha|$.

Let us remark that for top order coefficients $\{b_\alpha : |\alpha| = m\}$ one has $p_\alpha = \infty$, $w = 1/\rho$ and condition (H5) reduces to

$$b_\alpha(x)/\rho \in L^\infty. \tag{1.4}$$

In other words, top order b_α behaves like leading coefficients ρa_α of A_0, except no regularity of b_α is required in general.

Often instead of (1.4) a stronger "L_0^∞-hypothesis" will be used

(H6) $\qquad b_\alpha/\rho \in L_0^\infty(\mathbf{R}^n \setminus \Sigma)$ for top order coefficients $\{b_\alpha\}_{|\alpha|=m}$.

Let us comment on conditions (H1—6) and classes of operators A_0 and B.

1. An alternative pair of parameters can be used to describe our classes $A_0 = \rho a(x, D) + V$: functions $\varphi = \sqrt[m]{\rho/V}$ and V.

Then (H1) reduces to

(H7) $\qquad \varphi^{|\alpha|} \dfrac{\varphi^{(\alpha)}}{\varphi}$ and $\varphi^{|\alpha|} \dfrac{V^{(\alpha)}}{V}$ are in L^∞ or L_0^∞ for all $|\alpha| \leqslant m$.

The hypothesis (H7) is much easier to check. It has another advantage as one of parameters φ is now "uncoupled" from the other V.

Condition (H7) has implications both for "global" (∞) and "local" (Σ) behavior of (φ, V).

The first condition (H7) limits the rate of growth of φ at $\{\infty\}$ by $O(|x|)$, respectively $\rho/V = O(|x|^m)$ for m-th order operators (cf. [15]).

As for V, (H7) puts a very mild constraint on its possible growth rate at $\{\infty\}$ (see examples below).

Locally conditions (H7) allow φ to degenerate (vanish) on Σ to "first or higher" degree, namely $\varphi(x) = O(d(x))$, $d = \operatorname{dist}(x, \Sigma)$.

One interesting case arises when $\varphi(x)$ vanishes on Σ to degree $\mu > 0$, i.e. $\rho(x) = O(d^\mu(x))$. If $\mu < m$ we do not get sufficient "nullity" of φ unless the potential $V(x)$ is itself "singular" to degree $\nu \geqslant m - \mu$, i.e. $V(x) = O(d^{-\nu}(x))$. Thus locally we must have

(1.5) $$\mu + \nu = \text{"nullity of } \rho\text{"} + \text{"singularity of } V\text{"} \geqslant m.$$

As an illustration of this principle we shall discuss operators of the Laguerre--type (Examples 2.3).

Condition (1.5) provides only "L^∞-hypotheses" (H1—7) on functions F, φ etc., whereas in most results below a stronger "L_0^∞" is needed. The latter can be achieved by strengthening (1.5) to

(1.5) $$\varphi(x) = \begin{cases} O(d^s(x)) & \text{with } s > 1 \text{ (i.e. } \mu + \nu > m) \text{ "locally"} \\ O(|x|^\tau) & \text{with } \tau < 1 \text{ "globally".} \end{cases}$$

The most interesting critical case: $\mu + \nu = m$ and $\tau = 1$ (like Laguerre's) can still be treated by the method of Sections 2, 3 but requires more careful analysis.

2. Let us observe that our class of symbols $\sigma(x, \xi) = \rho a + V$, and powers $(\rho a + V)^s$, formally belong to generalized ΨD Calculi of Beals-Fefferman [4] and Hörmander [18]. Indeed, introducing "weight factors" $\varphi(x, \xi) = \sqrt[m]{\rho/V}$ and $\Phi(x, \xi) = |\xi|$ in the terminology of [4] hypothesis (H1) implies the following basic estimate of Beals' classes

$$|\sigma^{(\alpha)}_{(\beta)}/\sigma| \leqslant \text{Const}\ \Phi^{-|\alpha|}\varphi^{-|\beta|}.$$

However, other hypotheses of [4] when specialized to $\varphi = \sqrt[m]{\rho/V}$ and $\Phi = |\xi$ are different from our (H1—7). In particular, they do not allow local "degeneracies" and "singularities". Therefore the general theories of [3], [4] and [18] are not readily applicable.

For this and other reasons we prefer to work directly with symbols $(\rho a + V)^s$ and exploit their specific structure rather than involve the general machinery of Beals-Hörmander.

Let us also observe a certain similarity of our hypotheses (H1—7) and those of H. Triebel ([29], Chapters 6, 7).

Now we shall state some properties of the operators A and $A + B$.

PROPOSITION 1. *The product $A \cdot B$ and the adjoint B^* of two operators $A = a(x)D^\alpha$ and $B = b(x)D^\beta$ with coefficients $a \in L^p(w')$; $b \in L^q(w'')$, of fractiona orders $\nu(A) = \frac{n}{p} + |\alpha|$ and $\nu(B) = \frac{n}{q} + |\beta|$, belong to the above class of perturbations provided $b(x)$ is sufficiently smooth,* i.e. *$b \in L^q_{|\alpha|}(w'')$ for the product, and $L^q_{|\beta|}(w)$ for the adjoint. Moreover, the fractional order of the product, $\nu = \nu(A \cdot B) = \nu(A) + \nu(B)$ and the corresponding weight $w = w'w''$.*

It is assumed here that fractional orders $\nu(A)$, $\nu(B)$ are bounded by some integers m', m'' $(\nu(A) \leqslant m',\ \nu(B) \leqslant m'')$ and the corresponding weights w', w'', w are determined by the pairs: $\{\nu(A); m'\}$, $\{\nu(B); m''\}$, $\{\nu; m = m' + m''\}$.

The proof easily follows by expanding the product and adjoint according to the standard "Leibniz rule"

$$A \cdot B = a \sum_{0 \leqslant \gamma \leqslant \alpha} \binom{\alpha}{\gamma} b^{(\alpha-\gamma)} D^{\beta+\gamma}. \tag{1.6}$$

Then using Sobolev embedding

$$b^{(\alpha-\gamma)} \in L^q_{|\gamma|} \subseteq L^{p'}; \quad \frac{1}{p'} = \frac{1}{q} - \frac{|\gamma|}{n}$$

and Hölder inequality

$$\|ab^{(\alpha-\gamma)}\|_{L^r(w)} \leqslant \|a\|_{L^p(w')} \|b^{(\alpha-\gamma)}\|_{L^{p'}(w'')}: \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{p'}; \ w = w'w''$$

one can show that all terms of (1.6) belong to appropriate L^p-weighted classes and their order $\nu \leqslant \nu(A) + \nu(B)$. Similar argument applies to B^*.

From Proposition 1 immediately follows that any power $(A_0 + B)^k$ (k—integer) belongs to the same class of operators provided coefficients of A_0 and B are sufficiently smooth. The latter can be interpolated to fractional powers, when all definitions above are properly modified to include pseudodifferential operators of the form: $\rho a + V$ (cf. [29]).

Another property of operators $A_0 + B$ is given in

PROPOSITION 2. *The class of operators $A_0 + B$ is invariant under conjugation with any function $w = \rho^s V^t$ or, equivalently, $w = \varphi^s V^t$.*

Indeed, by the product formula the conjugate ΨDO $w^{-1}A_0 w$ has symbol

$$\rho a(x, \xi) + \rho \sum_{1 \leqslant |\alpha| \leqslant m} a^{(\alpha)}(x; \xi) \frac{w^{(\alpha)}}{w}. \tag{1.7}$$

But $\dfrac{w^{(\alpha)}}{w} \sim (\ln w)^{(\alpha)} = (\ln \varphi + \ln V)^{(\alpha)}$ is of order $O(\varphi^{-|\alpha|})$ by hypothesis (H7). So all terms $b(x, \xi) = \rho \dfrac{w^{(\alpha)}}{w} a^{(\alpha)}(x; \xi)$ of (1.7) represent perturbations of the above type.

An important consequence of Proposition 2 is that it allows to extend most of results stated below in L^p-spaces to weighted spaces $L^p(w)$ with any weight $w = \rho^s V^t$. This in turn can be used to extend our results to operators on certain manifolds diffeomorphic to $\mathbf{R}^n$.

We conclude this section with examples of pairs (ρ, V) and operators $A_0 + B$.

EXAMPLE 1. (Schrödinger-type operators).

$$A = -\nabla \cdot \rho \nabla + \vec{b} \cdot \nabla + V.$$

Hypotheses (H1—4) reduce to

$$\frac{\partial_j \rho}{\sqrt{\rho V}}\,;\ \frac{\sqrt{\rho}\,\partial_j V}{V^{3/2}}\,;\ \frac{\partial^2 \rho}{V}\,;\ \frac{\rho\,\partial^2 V}{V^2} \in L^\infty \quad (\text{or } L_0^\infty) \tag{1.8}$$

and

$$\vec{b} \in L^p(w), \quad \text{where } w = \rho^{-\nu/2} V^{\nu/2-1} \text{ and } \nu = \frac{n}{p} \leqslant 1. \tag{1.9}$$

Using the pair (φ, V) and functions $\log\varphi$, $\log V$ instead of (ρ, V) we can rewrite these conditions as

$$\partial\varphi;\ \varphi\partial^2\varphi;\ \varphi\frac{\partial V}{V}\ \text{and}\ \varphi^2\frac{\partial^2 V}{V}\in L^\infty\quad(\text{or } L_0^\infty).$$

One specific example is $\varphi(x)\approx|x|^s$, $V\approx\exp(|x|^\tau)$ for large x, where $0<s<1$, $s+\tau\leqslant 1$. These can be modified by any polynomial factor, e.g. $\dfrac{P(x)}{(1+x^2)^{l/2}}$ $(l=\deg P)$. The L_0^∞-hypotheses at ∞ is provided by $s+\tau<1$.

Other specific examples include differential operators associated to classical orthogonal polynomials.

EXAMPLE 2. Operators of the "Hermite" and "Laguerre" type: $H=-\Delta+(|x|^p+1)$ and $L=-\partial\cdot|x|^s\partial+\dfrac{|x|^p+1}{|x|^\tau}$ $(s+r\geqslant 2)$.

Conditions (1.8) that involve only the potential $V(x)$ become

$$\frac{\partial V}{V^{3/2}};\ \frac{\partial^2 V}{V^2}\in L^\infty\quad(\text{or } L_0^\infty).$$

They are obviously satisfied for $V=|x|^p+1$ as well as many other polynomial (and exponential) functions. Note that "L_0^∞" here refers only to the global behavior at ∞ (as $\rho=1$ has no "local degeneracies"!)

The classical Laguerre operator $L_\alpha=-\partial x\partial+\dfrac{x^2+\alpha^2}{4x}$ is usually considered on the half line $\mathbf{R}_+$, but the natural extension, $-\partial|x|\,\partial+\dfrac{x^2+\alpha^2}{4|x|}$, allows to translate all results from $\mathbf{R}$ to $\mathbf{R}_+$.

To verify hypotheses (1.8) for L_α we calculate: $\varphi=\sqrt{\rho/V}=(2x/\sqrt{\alpha^2+x^2})$ and perturbation coefficient $b=1\in L^\infty$, has "fractional" order $\nu=1$.

Hypotheses (H1—5) are easily seen to hold in the "strongest" L_0^∞-form at $\{\infty\}$ and this is also true for more general operators: $-\partial|x|^s\partial+\dfrac{|x|^p+1}{|x|^\tau}$ provided $p>\tau$. Locally (at the "degeneracy" $\Sigma=\{0\}$) "L_0^∞" holds iff $s+\tau>2$ (sub-critical case).

The Laguerre operator L_α itself represents the critical case $(s+\tau=2)$, so only a weaker "L^∞-hypotheses" holds near Σ. However, evaluating L^∞-norms of φ' and $\varphi\varphi''$ we get $\|\varphi'\|_\infty=\dfrac{1}{\alpha}$, $\|\varphi\varphi''\|_\infty=\dfrac{4}{9\alpha^2}$, both become small for large α. Therefore all results of Sections 2, 3 are still applicable to L_α with sufficiently large α.

In connection with the Hermite and Laguerre operators we shall mention paper [2], which proved L^p-convergence of the Hermite and Laguerre expansions in the range $4/3 < p < 4$. This implies, in particular, stability of L^p-spectra in this range: "L^p-spectrum of L" is equal to "L^2-spectrum" for all $4/3 < p < 4$.

The results of Sections 2, 3 (Theorems 2, 4 and 5) extend this corollary of [2] to the whole range of L^p, $1 < p < \infty$.

COROLLARY 1. *L^p-spectra of operators H and L_α with sufficiently large α are identical in all L^p-spaces, $1 < p < \infty$, and equal to their L^2-spectrum.*

EXAMPLE 3. A multivariable version of "Laguerre-type" operator can be defined as in Example 2 with $|x|$ being the norm of $x \in \mathbf{R}^n$.

More interesting versions arise when $|x|$ is replaced with some other functions, like $L = -\partial \cdot |xy|^s \partial + \dfrac{(x^2+y^2)^{p/2} + C}{|xy|^\tau}$ on $\mathbf{R}^2$.

Here the degeneracy set is $\Sigma = \{(x, y) : xy = 0\}$, all hypotheses (H1—5) are satisfied, if $s + \tau \geqslant 2$; $p \geqslant \tau$ for "L^∞"; or $s + \tau > 2$; $p > \tau$ for "L_0^∞". So all results of Sections 2—3 apply to such L.

2. CONSTRUCTION AND ESTIMATES OF THE RESOLVENT AND SEMIGROUP KERNELS

Our basic method follows [14], [15]. Namely, we first construct the "free" (unperturbed) resolvent $R_\varsigma^0 = (\varsigma - A_0)^{-1}$, using pseudodifferential calculus, then proceed to $R_\varsigma = (\varsigma - A)^{-1}$ via perturbation series expansions.

The free resolvent R_ς^0 is approximated by its parametrix $K = K_\varsigma(x; x-y)$ a ΨDO with symbol $\sigma_K = \dfrac{1}{\varsigma - (\rho a + V)}$. Here ς varies over the set $\mathbf{C} \setminus \mathbf{R}_+$, complement of the range of symbol $\rho a + V$. By the standard product formula we get

$$(\varsigma - A_0) K_\varsigma = I - L_\varsigma$$

where the symbol of the remainder, a ΨDO L_ς, is computed explicitly

$$\sigma_L = \sum_{1 \leqslant |\gamma| \leqslant m} \frac{1}{\gamma!} (\rho a + V)^{(\gamma)} \left(\frac{1}{\varsigma - \rho a - V} \right)_{(\gamma)}. \tag{2.1}$$

From (2.1) we get R_ς^0 in the form of the Neumann series

$$R_\varsigma^0 = K \sum_0^\infty L^k = K(I - L)^{-1}. \tag{2.2}$$

Given $R^0_\varsigma = (\varsigma - A_0)^{-1}$ we construct $R_\varsigma = (\varsigma - A)^{-1}$ via the perturbation series expansions

$$R_\varsigma = R^0_\varsigma \sum_{k=0}^{\infty} (BR^0_\varsigma)^k = R^0_\varsigma \sum_{0}^{\infty} [BK_\varsigma(I-L)^{-1}]^k. \tag{2.3}$$

To show convergence of both series (2.2)—(2.3) and establish resolvent identities for R^0 and R we need norm estimates of operators L_ς and BK_ς. These are given in the following lemma, which is similar to Lemma 1 of [14], [15].

LEMMA 1. *Operator L_ς is bounded in all L^p-spaces $1 < p < \infty$, while BK_ς is bounded in the range $1 < p \leqslant \min\{p_\alpha\}$, the minimum being taken over L^p-classes of all coefficients b_α.*

Both operators are estimated as follows

$$\|L_\varsigma\| \leqslant c_p(\varsigma); \quad \|BK_\varsigma\| \leqslant c_p(\varsigma) \sum \|b_\alpha\| \tag{2.4}$$

with $c_p(\varsigma)$ depending on the leading symbol, $a(x, \xi)$, and parameters (ρ, V).

We shall outline the proof of the lemma, and give explicit form of constant $c_p(\varsigma)$ in the RHS of (2.4).

It is convenient to introduce the following functions:

$$F = F_\varsigma(x) = \frac{\prod_1^k \rho^{(\alpha^i)} \prod_1^l V^{(\beta^j)}}{\rho^{k-N/m}|\varsigma - V|^{l+N/m}}; \quad N = \sum |\alpha^i| + \sum |\beta^j| \leqslant m \tag{2.5}$$

and a modified version of F_ς where derivatives of ρ in the numerator are replaced with derivatives of $a_\alpha \rho$, i.e. leading coefficients of A:

$$\text{weight: } w = w_\varsigma(x) = \rho^{\nu-/m}|\varsigma - V|^{\nu/m-1} \tag{2.6}$$

$$\text{argument (angle): } \theta = \theta(x) = \arg(\varsigma - V(x)); \tag{2.7}$$

and

$$\text{a ``uniformly elliptic'' symbol: } \sigma = \sigma_\theta(x, \xi) = \frac{\xi^\nu}{[e^{i\theta} - a(x, \xi)]^N} \tag{2.8}$$

with $|\nu| \leqslant Nm$.

The function F_ς and the weight w_ς are parameter dependent versions of F and w of hypotheses (H1—5) of Section 1 (the latter corresponds to $\varsigma = 0$).

By the product formula (2.1) the symbols of both operators L_ς and BK_ς consist of linear combinations of terms,

$$\rho\xi^\nu\left(\frac{1}{\varsigma-\rho a-V}\right)_{(\gamma)};\quad |\nu|=m=|\gamma|\quad\text{and}\quad b_\alpha\xi^\nu\left(\frac{1}{\varsigma-\rho a-V}\right)_{(\gamma)},\quad |\nu|=|\alpha|-|\gamma|.$$

Expanding each composite derivative $\partial_x^\gamma\left(\frac{1}{\varsigma-\rho a-V}\right)$ by the "iterated chain rule"

$$(f\circ a)^{(\gamma)}=\sum_{\substack{\alpha^1,\ldots,\alpha^k\\ 1\leqslant k\leqslant|\gamma|}} c_{\alpha^1,\ldots,\alpha^k}(f^{(k)}\circ a)\prod_1^k a^{(\alpha^j)};\qquad \sum_1^k\alpha^j=\gamma$$

we can rewrite the symbols of L_ς and BK_ς as finite sums of terms

$$\Psi(x,\xi)=\frac{\xi^\nu\prod_1^k(\rho a)^{(\alpha^i)}\prod_1^l V^{(\beta^j)}}{(\varsigma-\rho a-V)^{k+l+1}};\quad \text{with } |\nu|\leqslant(k+1)m-|\gamma|. \tag{2.9}$$

Next using homogeneity of $a(x,\xi)$ in ξ and "pulling out" the dilating factor

$$\delta=\delta(x)=\sqrt[m]{\rho/|\varsigma-V|}$$

from the numerator and denominator of (2.9) yields a representation of Ψ in terms of the functions and symbols (2.5—2.8), introduced above.

Namely,

$$\Psi=F_\varsigma(x)\sigma_0(x;\delta\xi).$$

The corresponding ΨDO $\Psi(x;\mathrm{D})$ is thus reduced to the product of two operators: a multiplication with a function $F_\varsigma(x)$, followed by a so-called δ-dilation ([15]), $\frac{1}{\delta^n}M\left(x;\frac{z}{\delta}\right)$, of a uniformly elliptic kernel

$$M(x;z)=\int\sigma_\theta(x;\xi)e^{i\xi\cdot z}d\xi;\qquad z=x-y.$$

If the symbol σ_θ has strictly negative order (as in the case of L_ς), then its kernel $M(x,\varsigma)$ admits by [14] a convolution type L^1-radial bound

$$|M(x;z)|\leqslant C(\theta)H(|z|)$$

with

$$H(|z|) = \begin{cases} 1 + |z|^{s-n}, & |z| \leqslant 1 \\ e^{-\gamma|z|}, & |z| \geqslant 1; \end{cases} \qquad \gamma = \gamma_0 \sin \theta/m \tag{2.10}$$

and with constant $C(\theta) = c_0|\sin \theta/2|^{-\mu}$ for a sufficiently large $\mu > 0$.

In [15] we observed that δ-dilations of an L^1-radial kernel $H(|x-y|)$, consequently M_δ, can be estimated in terms of the Hardy-Littlewood maximal function

$$u^*(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|y|\leqslant r} |u(x-y)|\mathrm{d}y.$$

Indeed,

$$|H_\delta u(x)| \leqslant \|H\|_1 u^*(x).$$

Combining the latter with L^∞-bound of F_ς (hypothesis H1) and the standard L^p-estimates of the maximal operator $u \to u^*$ we derive the following bound for operator L in all L^p-spaces $(1 < p < \infty)$

$$\|L_\varsigma\| \leqslant \sum \|F_\varsigma M_\delta\| = c_p \sum \|F_\varsigma \sin^{-\mu}(\theta)\|_\infty. \tag{2.11}$$

The summation in (2.11) extends to all terms in the expansion of $L = \sum F_\varsigma(x) M_\delta(x, z)$, and yields the required norm estimate of L_ς.

The argument for BK_ς goes along the same lines, but some modifications are needed to account for L^p-singular coefficients.

Precisely, by the iterated chain rule symbol of BK_ς is written as the sum of terms

$$b_\alpha \frac{\prod\limits_1^k (\rho a)^{(\alpha^i)} \prod\limits_1^l V^{(\beta^j)}}{(\varsigma - \rho a - V)^{k+l+1}} \xi^\nu; \qquad |\nu| = km + |\alpha| = |\gamma|$$

where each tuple of multiindices $\{\alpha^i; \beta^j\}$ represents a partition of $\gamma = \sum \alpha^i + \sum \beta^j$. As we already observed the L^{p_α}-singular coefficient b_α shifts the scale: $\frac{1}{p} \to \frac{1}{p} + \frac{1}{p_\alpha}$, and this shift is to be matched by a suitable negative fractional derivative. So we multiply and divide BK_ς by the fractional Laplacian $\Lambda^s = (-\Delta)^{s/2}$ and write the expansion of BK_ς as

$$BK_\varsigma = \sum \tilde{F}_\varsigma(x) M_\delta(x; z) \Lambda^{-s}(z). \tag{2.12}$$

Now the resulting ΨDO M_δ, the middle factor of (2.12), has symbol $\sigma_\theta = \frac{\xi^\nu|\xi|^s}{(e^{i\theta}-a)^N}$, which may be of order zero, if B has terms $b_\alpha D^\alpha$ of the highest fractional order $\frac{n}{p_\alpha} + |\alpha| = m$.

The corresponding ΨDO-kernel $M(x,z)$ becomes a singular integral of the Calderon-Zygmund type ([28]). It does not admit an L^1-convolution bound. We proceed as in [15], i.e. split σ_θ into the sum $\sigma^0 + \sigma^1$ of two symbols: σ^0, homogeneous of degree zero in ξ, and σ^1 of strictly negative order. The corresponding kernel M splits into the sum $M^0 + M^1$. Here the first (Calderon-Zygmund) part M^0 is not affected by δ-dilations (because of the "0-th degree" of homogeneity), while the second (L^1-radially bounded) M^1_δ can be estimated by the maximal function as above.

Next we write down explicitly the multiplication factor $\tilde{F}_\varsigma$ in (2.12)

$$\tilde{F}_\varsigma = \frac{b_\alpha}{\rho^{(|\alpha|+s)/m}|\varsigma - V|^{1-(|\alpha|+s)/m}} \frac{\prod_1^k (\rho a)^{(\alpha^i)} \prod_1^l V^{(\beta^j)}}{\rho^{k-|\gamma|}|\varsigma - V|^{l+|\gamma|}} = (w_\varsigma b_\alpha)F_\varsigma .$$

In other words $\tilde{F}_\varsigma$ is a product of an L^{p_α}-function $w_\varsigma b_\alpha$ and an L^∞-function F_ς (2.5).

Combining estimates of M_δ with the standard Hölder inequality for wb_α and Sobolev-Hardy-Littlewood for Λ^{-s} we get the final result

$$\|BK_\varsigma\| \leqslant c_p \sum \|b_\alpha w_\varsigma\| \; \|F_\varsigma \sin^{-\mu}\theta\|_\infty . \tag{2.13}$$

Norms of b_α in (2.13) are taken in appropriate weighted L^p-classes with weights w_ς given in (2.6).

The limitation on the scale $1 < p \leqslant \max\{p_\alpha\}$ results from the fact that the fractional Laplacian $\Lambda^{-s}\left(s = \frac{n}{p_\alpha}\right)$ in (2.12) is not allowed to "push" L^p outside the range $0 \leqslant \frac{1}{p} \leqslant 1$. This completes the proof.

In order to apply Lemma 1 to summation of series (2.2—2.3) we need to analyse the RHS of estimates (2.11) and (2.13).

Remembering that $V \geqslant 1$, choosing a specific value of ς, e.g. $\varsigma = 0$, and replacing $\varsigma - V(x)$ in the denominator of F_ς (2.5) by $V(x)$ we estimate

$$\|F_\varsigma \sin^{-\mu}\theta\|_\infty \leqslant \|F_0\|_\infty \sup_x \left|\frac{V(x)}{\varsigma - V(x)}\right|^{l+N/m} \sin^{-\mu}\pi/2. \tag{2.14}$$

Here F_0 is the original function (1.1) and $\theta_0(x) = \pi$.

Similar estimates hold for $\|wb\|_{p_\alpha}$. But these bounds are not enough to sum geometric series (2.2)—(2.3), unless norms $\|F\|_\infty$ and $\|wb\|_p$ are sufficiently small. So stronger hypotheses on ρ, V and b_α are needed.

One of them is the L_0^∞-condition (instead of L^∞) on the functions F and top order perturbation coefficients $\{b_\alpha\}_{|\alpha|=m}$ (L^{p_α}-condition on b_α with $p_\alpha < \infty$, does not require any change). Both conditions "L_0^∞" and "L^{p_α}" with $p_\alpha < \infty$ allow to divide $b = b_\alpha$ (respectively F) into two parts: compactly supported b' and small b'': $\operatorname{supp} b' \subseteq \{|x| \leqslant R, \operatorname{dist}(x, \Sigma) > \varepsilon\}$ and $\|b''\|_{p_\alpha} < \varepsilon$. Moreover, b' can be chosen in $L^{p'}$ so that the fractional order $\nu' = \dfrac{n}{p'} + |\alpha|$ is less than m. The order ν' will show up in the estimate of $\|bw_\varsigma\|$ below.

Such decomposition is obvious for coefficients b_α of order $|\alpha| < m$. As for the top-order b_α we simply require $\|b_\alpha/\rho\|_\infty$ to be sufficiently small.

To show that each term $\|bD^\alpha K_\varsigma\|$ becomes small for large ς we decompose $b = b' + b''$, as above, then use estimates (2.13) of the lemma for both terms. Let us start with b',

$$\|b'D^\alpha K_\varsigma\| \leqslant c \sum \|b'w'_\varsigma\|_{p'} \|F_\varsigma \sin^{-\mu}\|_\infty,$$

where w'_ς means the weight $\rho^{-\nu'/m}|\varsigma - V|^{\nu'/m-1}$ of order $\nu' < m$.

Note that as long as $\operatorname{supp} b'$ is separated from the boundary Σ any particular choice of the weight, w or w', in the L^p-norm of b' is irrelevant, as all spaces $L^{p'}(w)$ with weights $w = \rho^{-\tau}V^{\tau-1}$ $(0 \leqslant \tau \leqslant 1)$ are equivalent to the standard $L^{p'}$ on $\operatorname{supp} b'$.

The important point for us is ς-dependence of w' (or w), which allows to prove convergence $\|b'w'_\varsigma\| \to 0$ as $\varsigma \to \infty$. Indeed,

$$\|b'w'_\varsigma\|_{p'} \leqslant c\,\|b'\|_{p'} \max_{\operatorname{supp} b'} |\varsigma - V(x)|^{\nu'/m-1} \tag{2.15}$$

the exponential $\nu'/m - 1$ being negative and x varying over a compact region $\operatorname{supp} b'$, the RHS of (2.15) obviously goes to zero as $\varsigma \to \infty$.

As for $b''D^\alpha K_\varsigma$ it can be estimated as in (2.14)

$$\|b''D^\alpha K_\varsigma\| \leqslant C\|b''w_\varsigma\|_p \leqslant C\|b''w\| \sup \left|\frac{\varsigma - V(x)}{V}\right|^{\nu/m-L}.$$

So the norm $\|b''D^\alpha K_\varsigma\|$ is controlled by $\|b''w\|$ which is ε-small. A similar argument (partition into "compactly supported" and "ε-small" parts) applies to functions F_ς subject to the "L_0^∞-condition".

Notice for any fixed $\varepsilon > 0$ and small angle $\theta = \arg\varsigma$ the RHS of (2.15) can made arbitrarily small by choosing sufficiently large $r = |\varsigma| \geqslant r_0$.

Therefore both series (2.2) and (2.3) converge absolutely in the region

(2.16) $$\Omega = \{\varsigma : |\arg \varsigma| > \theta;\ r = |\varsigma| > r_0\}$$

for any $\theta > 0$ and a sufficiently large $r_0 = r_0(\theta)$.

REMARK 1. Sometimes it is convenient to replace "L_0^∞" with the stronger "L_ε^∞"-hypothesis: potential V dominates ρ in such a way that quotients F in (H1) remain bounded after the exponent of V is lowered by $\varepsilon > 0$,

(L_ε^∞) $$F^\varepsilon = \frac{\prod_1^k \rho^{(\alpha^i)} \prod_1^l V^{(\beta^j)}}{\rho^{k-N/m} V^{l-\varepsilon+N/m}} \in L^\infty .$$

Then one easily verifies that $\|F_\varsigma\|_\infty \leqslant \|F^\varepsilon\|_\infty \sup_{V \geqslant 0} \frac{V^{p-\varepsilon}}{|\varsigma - V|^r}$, $p = l + N/m$, so

$$\|F_\varsigma\|_\infty \leqslant r^{-\varepsilon} |\sin\theta|^{-\mu} \|F^\varepsilon\|_\infty ; \quad r = |\varsigma|,\ \theta = \arg\varsigma.$$

Therefore the first of two series, (2.2), converges in the complement of a parabolic-shaped region

(2.17) $$\Omega = \{r^\varepsilon |\sin\theta|^\mu \leqslant C\}$$

about $\mathbf{R}_+$. Two examples illustrate the "L_ε^∞"-hypothesis:

I) Bounded V, consequently "finitely propagating" ρ (studied in [15]).

II) Bounded function ρ (with all derivatives) and V satisfying $|V^{(\alpha)}/V| \leqslant$ $\leqslant$ const, i.e. operators of the form: "uniformly elliptic" + "large potential". In both cases one can show

$$\|F_\varsigma\|_\infty \leqslant C r^{-1/m}.$$

So under any of two assumptions "L_0^∞" or "L_ε^∞" on functions

$$F = \frac{\prod_1^k \rho^{(\alpha^i)} \prod_1^l V^{(\beta^j)}}{\rho^{1-N/m} V^{1+N/m}} ; \quad N = \sum |\alpha^i| + \sum |\beta^j|$$

and top-order coefficients $\{b_\alpha/\rho\}_{|\alpha|=m}$ norms of operators L_ς and BK_ς become small (less than 1) in the appropriate region $\Omega \subset \mathbf{C}$ and therefore both series (2.2), (2.3) converge absolutely in Ω.

REMARK 2. As we already mentioned in the critical "L^∞-case" (Laguerre-type operators of Examples 2, 3 with $s + \tau = 2$) the RHS of (2.11) and (2.13) does not decrease with $\varsigma \to \infty$. Therefore more careful analysis is required. To get RHS < < 1 for convergence of series (2.2) — (2.3) it suffices for instance, that functions $F_\varsigma(x)$ (2.5) with $\varsigma = 0$ (or some fixed ς_0) have sufficiently small L^∞-norm. This was shown to be the case for the Laguerre operator: $L_\alpha = \partial x \partial + \frac{x^2 + \alpha^2}{4x}$, with sufficiently large α. More generally, it holds for operators: $-(\partial \cdot \rho \partial)^{m/2} + V$ with $\rho \approx d^s(x)$; $V \approx \frac{C}{d(x)^\tau}$ in the critical case : $s + \tau = m$, provided $C > 0$ is sufficiently large.

Now we can state the first main result of the paper.

Let A_0 and $A = A_0 + B$ be operators of Section 1 satisfying "L_0^∞" (or "L_ε^∞") hypothesis on functions $\{F = F_\varsigma\}$ and assume that top order perturbation coefficients $\{b_\alpha\}_{|\alpha| = m}$ belong to $L_0^\infty(1/\rho)$. Both operators are considered on the minimal domain $C_0^\infty(\mathbf{R}^n)$.

THEOREM 1. (i) *Operators A_0 and A are closeable in all L^p-spaces:* $1 < p < \infty$ *for A_0 and* $1 < p \leqslant \min\{p_\alpha\}$ *for A (the minimum taken over L^p-classes of coefficients b_α of B).*

(ii) *The L^p-domains of operators A_0, A, B satisfy*

$$\operatorname{Dom}_p(B) \supseteq \operatorname{Dom}_p(A_0) = \operatorname{Dom}_p(A).$$

(iii) *A formally symmetric operator A, with* $\min_\alpha\{p_\alpha\} \geqslant 2$ *is essentially selfadjoint on L^2*, i.e. $\operatorname{Dom} A^* = \operatorname{Dom} A =$ *closure of the minimal operator.*

(iv) *Series* (2.2)—(2.3) *converge absolutely in the region Ω given by* (2.16) *or* (2.17) *and define resolvents* $R_\varsigma^0 = (\varsigma - A_0)^{-1}$ *and* $R_\varsigma = (\varsigma - A)^{-1}$.

The main ingredients of the proof are already contained in Lemma 1 and subsequent discussions, in particular, convergence of series (2.2) — (2.3).

The rest will follow from two resolvent identities

$$\begin{aligned} &(\varsigma - A_0) R_\varsigma^0 f = f; \quad (\varsigma - A) R_\varsigma f = f, \quad \text{all } f \in L^p \\ (2.18) \qquad & \\ &R_\varsigma^0 (\varsigma - A_0) f = f; \quad R_\varsigma(\varsigma - A) f = f, \quad \text{all } f \text{ in } \operatorname{Dom} A_0 \text{ or } A. \end{aligned}$$

For the operator A both identities are verified straightforward by definition of series (2.3). For A_0 the 1-st identity (2.18) also follows directly from (2.2).

The second is less obvious because of the apparent asymmetry of (2.2). To prove it we construct the left resolvent of A_0, treating the latter as an operator of left type

$$A_0[u] = \sum D^\alpha[\rho a_\alpha u] + B[u]$$

and transforming all expansions and estimates of the lemma from the "right-type" to "left-type" ΨDO's.

Once both (right and left) resolvents of A_0 are shown to exist they must be equal.

The relation between domains, part (ii), follows from the a priori estimate of the lemma,

$$\|Bf\|_p \leqslant \varepsilon\|A_0 f\|_p + C\|f\|_p \quad \text{all } f \in \operatorname{Dom} A_0,$$

or any $\varepsilon > 0$ and $C = C(\varepsilon)$.

This estimate could also be used to establish essential selfadjointness of A (cf. [20], [21]) provided the formally symmetrized operator A_0 is shown to be selfadjoint. In fact, selfadjointness of both (symmetrized A_0 and formally symmetric A) follows directly from the existence of resolvents in $\mathbf{C}\setminus\Omega$.

Finally one can show (cf. [15]) that the domain of A_0 is the same as of the "model operator": $\rho(-\Delta)^{m/2} + V$, i.e.

$$\operatorname*{Dom}_p(A_0) = \{u : \| V(1 - \Delta_\varphi)^{m/2} u\|_{L^p} < \infty\}.$$

The latter space is a modification of the L^p-weighted Sobolev space $L^p_m(w)$ of [15], (see also [29], Chapter 6) with the standard Laplacian $-\Delta$ being replaced by the new "conformally stretched" Laplacian $\Delta_\varphi = \nabla\cdot\varphi^2\nabla$ with $\varphi = \sqrt[m]{\rho/V}$.

An obvious consequence of Theorem 1 are bounds on L^p-spectra of operators A: spec $A \subset \mathbf{C}\setminus\Omega$. These will be discussed in more detail later on (Section 3).

Another application is to so-called "resolvent summability".

THEOREM 2. *The resolvent* $R_\varsigma = (\varsigma - A)^{-1}$ *of an operator* A *of Theorem* 1 *has*

(i) *the "maximal rate" of decay in any nonzero direction:* $\|R_\varsigma\| \leqslant \dfrac{C}{|\varsigma|}$ *as* $\varsigma = re^{i\theta} \to \infty$, $\theta \neq 0$,

(ii) *it satisfies*

$$\varsigma R_\varsigma[u] \to u \quad \textit{in } L^p\textit{-norm} \quad \textit{as } \varsigma \to \infty$$

uniformly in any sector $\{\varsigma = re^{i\theta} : |\theta| \geqslant \theta_0 > 0\}$.

Proof. (i) It suffices to show that

$$\|\varsigma K_\varsigma\| \leqslant \text{Const} \tag{2.19}$$

for the operator K_ς.

Indeed, by (2.2) – (2.3)

$$R_\varsigma = R_\varsigma^0(I - BR_\varsigma^0)^{-1} \quad \text{and} \quad R_\varsigma^0 = K_\varsigma(I - L_\varsigma)^{-1}$$

and by Lemma 1 both operators $(I - BR_\varsigma^0)^{-1}$ and $(I - L_\varsigma)^{-1}$ remain bounded as $\varsigma \to \infty$. In fact both L_ς and BR_ς^0 go to 0 as $\varsigma \to \infty$.

The kernel of the operator ςK_ς is of the form

$$\frac{\varsigma}{|\varsigma - V|} M_\delta(x; z), \quad \text{where the symbol } \sigma_M = \frac{e^{i\theta}}{e^{i\theta} - a}$$

in notations of Lemma 1. Now (2.19) easily follows as in the lemma.

(ii) We first show L^p-convergence

$$\varsigma K_\varsigma u \to u \quad \text{as } \varsigma = re^{i\theta} \to \infty \tag{2.20}$$

and then estimate differences $\varsigma(K_\varsigma - R_\varsigma^0)u$ and $\varsigma(R_\varsigma^0 - R_\varsigma)u$.

The estimates of differences follow directly from series expansion (2.2)—(2.3) and Lemma 1 as in [14], [15]. The proof of convergence (2.20) is however, different from the "uniformly elliptic case" [14], because of the lack of global L^1-radial bound of A.

Writing ςK_ς as $\frac{\varsigma}{|\varsigma - V|} M_\delta(x; z)$ we observe the following properties of a ΨDO kernel $M(x; z)$, whose symbol $\varphi = \frac{e^{i\theta}}{e^{i\theta} - a}$,

(1) $\int M(x; x - y)\mathrm{d}y = 1$,

(2) $M(x;z)$ admits an L^1-radial convolution bound $H(|z|)$ (2.10). Now we estimate the difference

$$|\varsigma K_\varsigma u(x) - u(x)| \leqslant \left| \frac{\varsigma}{\varsigma - V(x)} - 1 \right| |u(x)| + \int H_\delta(|x - y|)|u(x) - u(y)| \, \mathrm{d}y. \tag{2.21}$$

Since the family of L^p-operators $\{\varsigma K_\varsigma\}_\varsigma$ was shown to be bounded in part (i) it suffices to prove convergence $\varsigma K_\varsigma u \to u$ on a dense subset of compactly supported $u \in C_0(\mathbf{R}^n)$. Assume $\operatorname{supp} u \subseteq \{|x| \leqslant r_0;\ \operatorname{dist}(x; \Sigma) > \varepsilon\}$.

Notice that on any compact set $Q \subset \mathbf{R}^n \setminus \Sigma$, the dilating factor $\delta(x) = \sqrt[m]{\rho(x)/|\varsigma - V(x)|} \to 0$, as $\varsigma \to \infty$ uniformly in $x \in Q$. So $(M_\delta u - u) \to 0$ in L^∞-norm and consequently any L^p-norm on Q.

It remains to show the convergence $M_\delta u(x) \to u(x)$ in L^p-norm outside any ball $\{|x| \geqslant r_1\}$. Taking $r_1 = 2r_0$ (twice the radius of the support of u), we observe that $u(x)$ in the RHS of (2.21) becomes 0. The resulting integral $\int H_\delta(|x-y|)|u(y)|\,dy$, after the change of variable $y \to y' = \dfrac{x-y}{\delta(\alpha)}$, can be estimated by

$$\|u\|_\infty \int_B H(y')\,dy', \tag{2.22}$$

integration over the ball $B = \{|y' - x'| < r_0/\delta\}$ centered at $x' = x/\delta(x)$.

Next we observe that the centers $x' = x/\delta(x)$ remain bounded away from 0, as x varies over the region $\{|x| > 2r_0\}$, by the "finite propagation speed" condition (H7) of Section 1, i.e. $\delta(x) = O(|x|)$. At the same time the radii of balls B are less than $|x'|/2$, so the region swept by the whole family $\{B\}$ is bounded away from 0.

Remembering now explicit radial bound of $H(z) \leqslant C|z|^{-(n+\varepsilon)}$ for large z we estimate

$$\int_B H\,dy' < \text{Const}\; H(x_0)\, r_0^n/\delta^n = \frac{\text{Const}\, r \quad \delta^\varepsilon}{|x|^{n+\varepsilon}} \lessapprox \frac{\text{Const}}{|\varsigma - V(x)|^\varepsilon |x|^n} = g_\varsigma(x).$$

The function $g_\varsigma(x)$ obviously belongs to all L^p-spaces $(1 < p < \infty)$ in the region $\{|x| \geqslant 2r_0\}$ and its L^p-norm goes to 0 as $\varsigma \to \infty$.

Thus we have shown that for any compactly supported u the L^p-norms of the "local" and "global" parts of $(\varsigma K_\varsigma u - u)$ converge to 0, as $\varsigma \to \infty$, which proves (2.20) and Theorem 2 altogether.

After Theorems 1 and 2 we can get a variety of other "analytic multipliers" $f(A)$, and "summation families" $\{f_\varepsilon(A)\}_\varepsilon$, by Cauchy integration of the resolvent,

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\varsigma)\,(\varsigma - A)^{-1}\,d\varsigma. \tag{2.23}$$

According to Theorem 1, the contour Γ can be chosen to consists of two rays $\{re^{\pm i\theta_0} : r > r_0\}$ with an arbitrary small opening $\theta_0 > 0$ and the arc $\{r_0 e^{i\theta} : |\theta| \geqslant \theta_0\}$ of sufficiently large radius $r_0 = r(\theta_0)$.

One such example is a family of multipliers $\{\varphi_t(\varsigma) = e^{-t\varsigma}\}$ that gives the semigroup kernel.

THEOREM 3. *An operator A of Theorem 1 generates a holomorphic semigroup $\{e^{-tA}\}$ in the half plane* $\operatorname{Re} t > 0$, *which is strongly continuous at $t = 0$, $e^{-tA}u \to u$ (as $t \to 0$) in all admissible L^p-spaces.*

The proof is fairly standard and straightforward after Theorems 1 and 2 (cf. [20], [14], [15].)

Various special cases of Theorems 1—3 relevant to L^p-theory were proved in our earlier work on uniformly elliptic operators [14], and strictly elliptic operators subject to "finite propagation speed" condition [15]. Other related results can be found in [25], [31], [3], [29].

The third statement of Theorem 1 (essential selfadjointness) has an extensive literature ([23], [26], [7], [6], [11], [20], [21] [9], [29]) both for 2-nd order (Schrödinger) and higher order elliptic operators. Our result extends all those in various directions, though it may not cover them completely because of the number of incompatible conditions existing in the literature. As a special case it includes a wide range of Schrödinger operators (Examples 1 — 3) as well as their powers.

An interesting problem related to resolvent and semigroup kernels is their "smoothing properties" in $\{L^p\}$ or other natural scales, so called "hypercontractivity" or "supercontractivity" (cf. [26], [8]). We give a sufficient condition for supercontractivity in our context.

THEOREM 4. *If the weight function* $w = \rho^{-s/m} V^{s/m-1}$ *is bounded for some* $0 < s \leq m$, *then the resolvent* $R_\varsigma = (\varsigma - A)^{-1}$ *is* $\varepsilon = \frac{s}{m}$*-smoothing in the* L^p*-scale*: $\frac{1}{p} \to \frac{1}{p} - \varepsilon$. *Consequently, the semigroup* e^{-tA} *is* ∞*-smoothing*, i.e. e^{-tA} *maps* L^p *into* $\bigcap_{1<q\leq\infty} L^q$.

Semigroup smoothing follows immediately from resolvent smoothing, by the semigroup property: $\exp(-tA) = \left[\exp\left(-\frac{t}{n}A\right)\right]^n$. Indeed each $\exp\left(-\frac{t}{n}A\right)$ is ε-smoothing and n can be taken arbitrary large.

The resolvent smoothing reduces, as above, to smoothing by the operator K_ς, since $R_\varsigma = K_\varsigma \times$ "invertible operator."

A ΨDO K_ς has a negative order symbol $\frac{1}{\varsigma - \rho a - V}$. So multiplying and dividing it by a suitable fractional Laplacian $\Lambda^s = (-\Delta)^{s/2}$ (as in Lemma 1) we get, after pulling out $|\varsigma - V|$,

$$\varphi = \text{"symbol } K_\varsigma \Lambda^s\text{"} = \rho^{-s/m} |\varsigma - V|^{s/m-1} \sigma(x; \delta\xi), \quad \text{where } \sigma = \frac{|\xi|^s}{e^{i\vartheta} - a}, \ \delta = \sqrt[m]{\rho/|\varsigma - V|}.$$

So $K_\varsigma \Lambda^s = w\, M_\delta$ (in notations of Lemma 1) factors into the product of a bounded function w and L^p-bounded operator M_δ, $1 < p < \infty$. Therefore $K_\varsigma \Lambda^s$ is L^p-bounded. The remaining negative Laplacian Λ^{-s} on the right side of the product $(K\Lambda^s)\Lambda^{-1}$ implements the required L^p-smoothing.

We would like to comment on the main condition of Theorem 3

$$w = \rho^{-s/m} V^{s/m-1} \in L^{\infty}. \tag{2.24}$$

Case $s = m$ corresponds to ρ bounded from below, $\rho(x) \geqslant \rho_0 > 0$, i.e. strictly elliptic operator A.

The combined form (2.24) is more flexible as it allows some degree of degeneracy of ρ, that must be "cancelled out" by a suitable singularity of V. One example of this type was already discussed in Section 1. Namely, $\rho(x) = \text{dist}(x; \Sigma)^{\alpha}$, $V(x) = \text{dist}(x; \Sigma)^{-\beta}$, with $\alpha + \beta \geqslant m$, where Σ is a surface (closed submanifold) in $\mathbf{R}^n$.

Condition (2.24) then becomes

$$\frac{s}{m}\cdot\alpha - \left(1 - \frac{s}{m}\right)\beta \leqslant 0 \quad \text{or} \quad 0 < s \leqslant \frac{\beta m}{\alpha + \beta}. \tag{2.25}$$

An immediate application of Theorem 4 and (2.25) is

COROLLARY 2. (i) *The resolvent* R_ς *of the Laguerre operator:* $L = -\partial x \partial + \frac{x^2 + \alpha^2}{4x}$ *with sufficiently large* α *and its multivariable version (Example 3) is smoothing of degree* $= 1/2$, *in the* L^p*-scale*, i.e. $\frac{1}{p} \to \frac{1}{p} - 1/2$.

(ii) *(Supercontractivity): The semigroup* e^{-tL} *is infinitely smoothing,*

$$e^{-tL}(L^p) \subseteq \bigcap_{1<q\leqslant\infty} L^q.$$

3. SPECTRAL THEORY

In this section we shall discuss the structure of L^p-spectra of operators A The main emphasis will be on discrete spectra, as very little is known in genera about continuous spectra, unless A is "sufficiently close" to a constant coefficient operator.

We shall give conditions for compactness of R_ς or discreteness of spec A, establish L^p-stability of discrete spectra and derive asymptotic distribution of large eigenvalues.

The following theorem plays the central role here.

THEOREM 5. *Let* $A = A_0 + B$ *be an operator of Section* 1, *with parameters* ρ, V *satisfying*

$$V^{-1+n/m}\rho^{-n/m} \in L^1_{\text{loc}}. \tag{3.1}$$

Then

(i) *A sufficient condition for compactness of $R_\varsigma - R_\varsigma^0$ in all admissible L^p-spaces is the L_0^∞-hypothesis (*i.e. *vanishing at ∞ and Σ) on leading perturbation coefficients: $b_\alpha/\rho \in L_0^\infty$ for $|\alpha| = m$.*

(ii) *A sufficient condition for compactness of R_ς, resp. R_ς^0, K_ς, is*

$$\text{Vol}\{x : V(x) \leqslant \lambda\} < \infty \quad \textit{for all } \lambda > 0. \tag{3.2}$$

Results of the type of Theorem 5 are well known for perturbations of constant coefficient or uniformly elliptic operators (see [25], [31]). Here the only relevant condition is (3.2), "large potential V".

In [16] we extended these results to strictly elliptic operators, $\rho(x) \geqslant \rho_0 > 0$, with possibly "large" ρ, subject to "finite propagation speed" condition, $\rho(x) = O(|x|^m)$ (see also [29], Chapters 6, 7).

The new feature of Theorem 5 is condition (3.1), which allows certain degeneracy of ρ and singularity of V.

The proof of Theorem 5 involves some standard steps (cf. [25], [16], [29]), which reduce the problem to the proof of compactness of an operator

$$T = f(x)H_\delta(|z|), \quad \text{on } L^p(D)$$

in a bounded region $D \subset \mathbf{R}^n$, where $f = \dfrac{1}{V(x)}$, the function $H(z)$ is compactly supported and bounded, and $\delta(x) = \sqrt[m]{\rho/V}$ is the dilating factor.

We use the standard L^p-equicontinuity test for compactness of the integral kernel $T(x, y)$. Let B denote a sufficiently small ball in D and let functions $u \in L^p(D)$, $v \in L^{p'}(B)$ $\left(p' = \dfrac{p}{p-1}\right)$.

Then we estimate

$$|\langle Tu; v\rangle u| \leqslant \sup_{y \in D} \left(\int |T(x, y)| \, dx\right) \|u\|_p \|v\|_{p'}.$$

In the "uniformly elliptic case" (nondegenerate ρ) and for B outside of the degeneracy set $\Sigma = \{x : \delta(x) = 0\}$ the dilating factor $\delta(x) \geqslant C > 0$ and L^p-equicontinuity follows immediatly, as the RHS of (3.3) is bounded by

$$\sup_y \int_B |T(x, y)| dx \leqslant \left\| \int_B H(y - x) f(x) dx \right\|_{L^\infty}.$$

So it becomes small as $\operatorname{vol} B \to 0$, for any pair $H \in L^q$, $f \in L^{q'}$.

In our cases $T(x, y) = V^{-1}(x)\delta^{-n}(x)\, H\left(\frac{|x-y|}{\delta(x)}\right)$ and the estimate of the RHS of (3.3) takes the form

$$(3.4) \qquad \|H\|_\infty \sup_{y\in D} \int_B \frac{dx}{V\delta^n} = \text{Const} \int_B \frac{dx}{V^{1-n/m}\rho^{n/m}}.$$

Due to hypothesis (3.1) the integral in (3.4) becomes small as $\operatorname{vol} B \to 0$, which proves L^p-equicontinuity of the family $\{T[u] : \|u\|_p \leqslant 1\}$ everywhere in D and thus completes the proof of Theorem 5.

REMARK 3. To illustrate condition (3.1) of Theorem 5 we assume that the degeneracy set $\Sigma = \{x : \rho(x) = 0\}$ is a smooth submanifold and functions $\rho(x)$, $V(x)$ are of the type discussed in Section 1, i.e. $\rho(x) \approx \operatorname{dist}(x; \Sigma)^\alpha$, $V(x) \approx \approx \operatorname{dist}(x; \Sigma)^{-\beta}$ near Σ, where $\alpha + \beta \geqslant m$. Then the integrability condition (3.1) on the function

$$V^{n/m-1}\rho^{-n/m} \approx \operatorname{dist}^{\beta - n/m(\alpha+\beta)}$$

becomes

$$(3.5) \qquad \frac{n}{m}(\alpha+\beta) - \beta < n - \dim \Sigma.$$

In the critical case $\alpha + \beta = m$ (Laguerre-type operators) it reduces to

$$\dim \Sigma < \beta.$$

The latter condition obviously holds in dimension 1. So we immediately derive L^p-compactness of R_ς for a large class of Schrödinger operators $-\partial\rho\partial + V$ on $\mathbf{R}$ with parameters

$$\rho \approx |x - x_0|^\alpha, \quad V \approx |x - x_0|^{-\beta} + \text{“large regular } V_0\text{”}$$

so that $\alpha + \beta \geqslant 2$, $\beta > 0$.

In the multidimensional case of Example 3 (Section 1) we have $m = 2, n = 2$, $\dim \Sigma = 1$, so (3.5) reduces to $\alpha < 1$. Theorem 5 applies to all such Laguerre-type operators. However the critical case $\alpha = 1$ remains open.

As a corollary of Theorem 5 one can show that operators of the Laguerre-type (Examples 2—3) have purely discrete spectra in all L^p-spaces, $1 < p < \infty$, whence L^p-stability of spectra follows (Corollary 1, Section 1).

After Theorem 5 we can also study L^p-spectra of operators A in general.

The first statement of Theorem 5 implies by the standard "Weyl" lemma that the difference of two spectra $\sigma(A)\setminus\sigma(A_0)$ is a discrete set of eigenvalues $\{\lambda_j\}_1^\infty$ of A.

The spectrum of A_0 may be fairly complicated and very little is known in general. As for eigenvalues we can prove the following "L^p-stability" result.

COROLLARY 3. *Discrete spectra* $\sigma_d(A) = \{\lambda_j\}_1^\infty$ *are identical in all admissible* L^p*-spaces. Moreover, all* L^p*-eigenfunctions* $(A - \lambda_j)u = 0$ *and root functions* $(A - \lambda_j)^k u = 0$, *belong to* $\bigcap_{1<q\leqslant\infty} L^q$.

This follows immediately from Theorem 4 (L^p-smoothing of semigroup e^{-tA}).

Theorem 5 and Corollary 3 extend some of earlier known results ([25], [31] [16]) to a wider class of differential operators.

Our final result gives an asymptotic distribution of large eigenvalues for self-adjoint operators $A = A_0 + B$ with purely discrete spectrum.

Let $N(\lambda)$ = "number of eigenvalues $\lambda_j \leqslant \lambda$" denote a counting function of A. The celebrated "Weyl" (or "volume counting") principle states that

$$(3.6)\qquad N(\lambda) \sim \mathrm{Vol}(\lambda) = \mathrm{Vol}\{(x, \xi) : \text{principal symbol} \leqslant \lambda\} \quad \text{as } \lambda \to \infty.$$

In our case (3.6) takes the form

$$(3.7)\qquad N(\lambda) \sim \mathrm{Vol}(\lambda) = \iint_{\rho a + V \leqslant \lambda} d\xi\, dx = \int \left(\frac{\lambda - V}{\rho}\right)_+^{n/m} \omega(x) dx,$$

where $\omega(x) = \mathrm{Vol}\{\xi : a(x;\xi) \leqslant 1\}$.

Formula (3.7) will be established under three additional hypotheses:

(A) volume function $f(\lambda) = \mathrm{Vol}(\lambda)$ satisfies

$$\alpha f(\lambda) < \lambda f'(\lambda) < \beta f(\lambda) \quad \text{for some } \alpha, \beta > 0.$$

(B) There exists an integer $l > \frac{n}{m}$ so that the following integral converges

$$\int \frac{dx}{(\varsigma + V)^{l - n/m} \rho^{n/m}} < \infty.$$

(C) Let $H(|z|)$ denote the radial function $e^{-c|z|}$ for a sufficiently large $c > 0$, and let $H_t = t^{-n}H(z/t)$ be its L^1-dilation. Then we require that the functions

$$F(x) = w(x) \sup_{t>0} \{H_t * \rho^{(\alpha)}\} \in L_0^\infty, \quad \text{for all } |\alpha| \leqslant m$$

where $w = \rho^{-1+|\alpha|/m} V^{-|\alpha|/m}$.

Conditions (A) and (B) are essential in the Tauberian Theorem of M.V. Keldysh, which is used in the proof in order to translate large ς-asymptotics of R^l_ς (Cauchy-Stieltjes transform of $N(\lambda)$) into λ-asymptotics of $N(\lambda)$. Let us observe that (B) is essentially a "global integrability" condition at $\{\infty\}$ on the function $(\varsigma + V)^{-l+n/m}\rho^{-n/m}$, since its "local integrability" is already implied by the "compactness hypothesis" (3.1) of Theorem 5. Condition (A) allows function $\mathrm{Vol}(\lambda)$ to grow at most polynomially as $\lambda \to \infty$, which in turn implies a polynomial lower bound of $V(x)/\rho(x)$ for large x. So it covers all Schrödinger operators $-\Delta + V(x)$ with $V(x) \geqslant |x|^\varepsilon$ ($\varepsilon > 0$), but $V(x) \sim \log|x|$ is not allowed.

Condition (B) implies that a power $l > n/m$ of the resolvent $(\varsigma + A)^{-l}$, equivalently a ΨDO $K^{(l)}$ with symbol $(\varsigma + \rho a + V)^{-l}$, belongs to the trace class. Indeed,

$$\operatorname{tr} K^{(l)} = \iint \frac{\mathrm{d}\xi\,\mathrm{d}x}{(\varsigma + \rho a + V)^l} = \int (\varsigma + V)^{-l+n/m}\rho^{-n/m}\omega_l\,\mathrm{d}x \tag{3.8}$$

with

$$\omega_l(x) = \int \frac{\mathrm{d}\xi}{[1 + a(x,\xi)]^l} \in L^\infty.$$

Finally, condition (C) represents a strengthening of the basic hypothesis (H1), "L_0^∞-condition" on $w\rho^{(\alpha)}$ with $w = \rho^{-1+|\alpha|/m}V^{-|\alpha|/m}$. Indeed, $w\rho^{(\alpha)}$ is formally equal to $\lim wH_t * \rho^{(\alpha)}$, as $t \to 0$.

Now we can state the result.

THEOREM 6. *A selfadjoint operator $A = A_0 + B$ of Section 1 satisfying hypotheses* (A — C) *has a purely discrete spectrum $\{\lambda_k\}_1^\infty$, whose asymptotics obeys the classical "Weyl principle"*,

$$N(\lambda) \sim \int \left(\frac{\lambda - V}{\rho}\right)_+^{n/m} \omega(x)\mathrm{d}x, \quad \textit{as } \lambda \to \infty.$$

Asymptotics of the spectral function $N(\lambda)$ can be transformed into asymptotics of other related kernels, for instance, the semigroup e^{-tA}. Thus we get

COROLLARY 4. *The semigroup e^{-tA} is of trace class for all $t > 0$ and*

$$\operatorname{tr}(\mathrm{e}^{-tA}) \sim \Gamma\left(\frac{n}{m} + 1\right) t^{-n/m} \int \mathrm{e}^{-tV}\rho^{-n/m}\omega\,\mathrm{d}x, \quad \textit{as } t \to 0.$$

Proof of Theorem 6 is based on the Cauchy-Stieltjes transform of the counting function,

$$f(\varsigma) = \int \frac{1}{(\varsigma + \lambda)^l}\,\mathrm{d}N(\lambda), \tag{3.9}$$

with an integer $l > n/m$ of hypothesis (B). Notice that (3.9) represents the trace of the l-th power of the resolvent: $f(\varsigma) = \mathrm{tr}(R_\varsigma^l)$, the exponent l was chosen high enough to make R_ς^l of the trace class. The latter will be shown to admit an approximation by the trace of K_ς^l or equivalently a ΨDO $K_\varsigma^{(l)}$, whose symbol $= \dfrac{1}{(\varsigma + \rho a + V)^l}$ and whose trace is computed explicitly:

$$f(\varsigma) \sim \mathrm{tr}\, K_\varsigma^{(l)} = \iint \frac{d\lambda\, dx}{(\varsigma + \rho a + V)^l}. \tag{3.10}$$

Hypothesis (B) guarantees the convergence of the integral.

Once the asymptotic relation (3.10) is establishd for large $\varsigma \to +\infty$ we can go back to asymptotics of $N(\lambda)$ via the Tauberian Theorem of M. P. Keldysh [22]. Hypothesis (A) provides a sufficient condition for validity of the Tauberian Theorem.

To establish (3.10) we introduce intermediary operators K_ς^l, R_ς^{0l}, R_ς^l (the l-th powers of K, R_ς^0 and R) and prove that traces of all four: $K_\varsigma^{(l)}$, K_ς^l, R_ς^{0l} and R_ς are asymptotically equal as $\varsigma \to \infty$, or phrased in a different way

$$\frac{\mathrm{tr}(R_\varsigma^l - R_\varsigma^{0l})}{\mathrm{tr}\, R_\varsigma^{0l}} \sim \frac{\mathrm{tr}(R^{0l} - K_\varsigma^l)}{\mathrm{tr}\, K_\varsigma^l} \sim \frac{\mathrm{tr}(K_\varsigma^l - K_\varsigma^{(l)})}{\mathrm{tr}\, K_\varsigma^l} \to 0 \quad \text{as } \varsigma \to \infty. \tag{3.11}$$

The rest of the proof is to establish (3.11).

We write $K_\varsigma^l = K_\varsigma^{(l)}(I + \tilde{M}_\varsigma)$; $R_\varsigma^0 = K_\varsigma(I + \tilde{L}_\varsigma)$ and $R_\varsigma = R_\varsigma^0(I + \tilde{B}_\varsigma)$, where the remainders $\tilde{M}_\varsigma, \tilde{L}_\varsigma, \tilde{B}_\varsigma$ are constructed and estimated as in Lemma 1. In particular, all three can be shown to go to zero as $\varsigma \to \infty$ in L^2 (L^p)-operator norm.

The differences of the l-th powers is represented by

$$R_\varsigma^{0l} - K_\varsigma^l = \sum_{\substack{l+\dots+l_k = l \\ 1 \leqslant k \leqslant l}} K^{l_1}\tilde{L} \dots K^{l_k}\tilde{L}, \tag{3.12}$$

similarly for $R^l - R^{0l}$ and $K^l - K^{(l)}$, from which we derive the following estimates

$$\begin{aligned} &\mathrm{tr}(R_\varsigma^{0l} - K_\varsigma^l) \leqslant \|K_\varsigma\|_l^l \sum_{k=1}^{l} \|\tilde{L}_\varsigma\|^k \\ &\mathrm{tr}(R_\varsigma^l - R_\varsigma^{0l}) \leqslant \|R_l\|_l^l \sum_{1}^{l} \|\tilde{B}_\varsigma\|^k \\ &\mathrm{tr}(K_\varsigma^l - K_\varsigma^{(l)}) \leqslant \|K^{(l)}\|_{\mathrm{tr}} \|\tilde{M}_\varsigma\|. \end{aligned} \tag{3.13}$$

Here $\|K\|_l$ denotes the Shatten-von Neumann l-norm of K, and $\|K^{(l)}\|_{tr}$ its trace class norm.

In order to deduce (3.11) from (3.13) it suffices to show that the ratios

(3.14) $$\frac{\|K_\varsigma\|_l^l}{\operatorname{tr}(K_\varsigma^l)}; \quad \frac{\|K_\varsigma^0\|_l^l}{\operatorname{tr}(R_\varsigma^0)^l}; \quad \frac{\|K^{(l)}\|_{tr}}{\operatorname{tr} K^{(l)}} \quad \text{are bounded as } \varsigma \to +\infty.$$

Then the remaining factors (norms of "twiddled" operators) will provide the required convergence to 0 in (3.11).

In fact we shall prove more: all three ratios (3.14) approach 1 as $\varsigma \to \infty$.

If operators K_ς, R_ς^0, $K_\varsigma^{(l)}$ were selfadjoint positive, the ratios would be equal to 1 and the result would hold trivially. Though neither one is actually selfadjoint we shall show that all three become very close to positive selfadjoint operators as $\varsigma \to \infty$.

Precisly, let us write $K = P + iQ$ as the sum of the real and imaginary parts: $P = (1/2)(K + K^*)$ and $Q = (1/2i)(K - K^*)$, and let P_+, P_- denote positive and negative part of a selfadjoint operator $P = P_+ - P_-$. The main step in the proof of (3.14) is

LEMMA 2. *Operators P_- and Q are "small" relative to P (or P_+), i.e.*

(3.15) $\|P_-\| + \|Qu\| \leqslant \varepsilon \|P_+u\|$, $(u \in L^2)$ *with* $\varepsilon = \varepsilon(\varsigma) \to 0$ *as* $\varsigma \to \infty$.

In other words the operator $(Q^*Q)^{1/2} + (P_-^*P)^{1/2} \leqslant \varepsilon P_+$ in the sense of comparison of positive operators.

Similar estimates hold for positive real parts of operators R_ς^0 and $K^{(l)}$.

From Lemma 2 relations (3.14) follow immediately. Indeed, the operator inequality $(Q^*Q)^{1/2} \leqslant \varepsilon P$ implies a factorization formula

$$K = P + Q = P(I + W), \quad \text{with } \|W\| < \varepsilon.$$

Hence

$$\|K\|_l^l = \operatorname{tr}[(I + W^*)P^2(I + W)]^{l/2} \leqslant (1 + \varepsilon) \operatorname{tr} P^l \leqslant (1 + \varepsilon') \operatorname{tr} K^l$$

with $\varepsilon' = \varepsilon'(\varsigma) \to 0$ as $\varsigma \to \infty$. So $\|K_\varsigma\|_l^l \sim \operatorname{tr}(K_\varsigma^l)$ as $\varsigma \to \infty$, Q.E.D.

Similar argument applies to R_ς^0 and $K^{(l)}$.

It remains to prove Lemma 2. We shall restate it separately for P_- and Q. Notice that inequalities

(3.16) $$\|Q\| \leqslant \varepsilon \|P_+\| \leqslant \varepsilon' \|P\|$$

(3.17) $$\|P_-\| \leqslant \varepsilon \|P_+\| \leqslant \varepsilon' \|P\|$$

for positive real and imaginary parts of K_ς are equivalent to those of R^0. Indeed, $K_\varsigma = R_\varsigma^0(I + L_\varsigma)$ with $\|L_\varsigma\| \to 0$. Inequality (3.17) for R_ς^0 holds trivially, since its negative real part is 0 for sufficiently large ς as a consequence of semiboundedness of A_0 from below (Theorem 1).

Relation (3.16) in its turn is equivalent to norm estimates

$$\|Qu\| \leqslant \varepsilon''\|Ku\| \leqslant \varepsilon''\|R^0u\|$$

and the latter can be written as

$$(3.18) \qquad \|Q(\varsigma + A_0)\| \to 0 \quad \text{or} \quad \|(\varsigma + A_0^*)Q\| \to 0, \quad \text{as } \varsigma \to \infty.$$

In this form it becomes very close to estimates of Lemma 1. Precisely, $A_0^* = A_0 +$ "small perturbation $Q = (1/2i)(K - K^*)$", and the product $(\varsigma + A_0)K_\varsigma = I - L_\varsigma$, where $\|L_\varsigma\| \to 0$ by Lemma 1.

We need to show that the same holds for the product of $(\varsigma + A_0)K_\varsigma^*$. Namely,

$$(3.19) \qquad (\varsigma + A_0)K_\varsigma^* = I - L'_\varsigma \quad \text{and} \quad \|L'_\varsigma\| \to 0 \quad \text{as } \varsigma \to \infty.$$

The difference between (3.19) and Lemma 1 is in the type of two ΨDO's: A_0 of the standard (left) type and K^* of the right type, which means its symbol $\sigma_K^* = \sigma(y, \xi)$. The corresponding product formula is

$$(3.20) \qquad \text{symb}_{(\varsigma+A_0)K^*} = 1 + \sum_{1 \leqslant |\alpha| \leqslant m} \frac{1}{\alpha!}\left[\frac{\sigma_{(\alpha)}(y;\, \xi)}{\sigma(y;\, \xi)}\right]^{(\alpha)} + t_m,$$

where $\sigma = \rho a + V + \varsigma$ and the remainder has the form,

$$(3.21) \qquad t_m(x, y, \xi) = \sum_{|\alpha| = m} \frac{1}{\alpha!}\int_0^1\left[\frac{\sigma_{(\alpha)}(y + \tau z;\, \xi)}{\sigma(y;\, \xi)}\right]^{(\alpha)}(1 - \tau)^{m-1}\, d\tau.$$

All regular terms of (3.20) represent adjoint ΨDO's to operators of Lemma 1. Hence they all are L^p-bounded $(1 < p < \infty)$ and become small as $\varsigma \to \infty$.

The remainder is treated similar to Lemma 1. The basic steps include "iterated chain rule", dilating symbols by $\delta(y) = \sqrt[m]{\rho/V + \varsigma}$ exploiting homogeneity in ξ etc. However, the resulting integrals kernels are somewhat different. Namely,

$$(3.22) \quad T_m = T(x,y) = w_\varsigma(y)\rho^{(\alpha)}(y + \tau z)H_{\delta(y)}(z), \text{ with } w_\varsigma = \rho^{-1+|\alpha|/m}(\varsigma + V)^{-|\alpha|/m}.$$

Here $H(z) = e^{-c|z|}$ is the radial bound (2.10) of a uniformly elliptic symbol $\psi = = \frac{\xi^{\nu}}{[C + a(x, \xi)]^k}$ of order $-m$.

We estimate the operator (3.22) in L^p-spaces as in the proof of Theorem 5:

$$\|T\| \leqslant \sup_y \int |T(x, y)| \, dx = \sup_y \{ w_\varsigma(y) \int_0^1 (\rho^{(\alpha)} * H_{\tau\delta(y)})(y) \tau^{m-1} \, d\tau) \}.$$

Now hypothesis (C) applies to show that the RHS of (3.22) goes to 0 as $\varsigma \to \infty$.

This proves convergence of the remainder $\|T_m(\varsigma)\| \to 0$ as $\varsigma \to \infty$. Combining the latter with estimates of regular terms of $(\varsigma + A_0)K_\varsigma^*$ (3.20) we establish (3.18) and complete the proof of Lemma 2 and Theorem 6 altogether.

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Added in the proof. After the paper was submitted I learned about the recent work of M.M.H. Park "On spectral properties of singular second order elliptic operators" (Preprint, 1987), which studies in detail the effect of "singular-degenerate" leading coefficients of Laplace-Beltrami operators A and their L^p-spectral properties.