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ON THE HOMOTOPY GROUPS OF THE AUTOMORPHISM GROUP OF AF-C^*-ALGEBRAS

VICTOR NISTOR

INTRODUCTION

In this paper we study the homotopy groups of the automorphism group of an AF-C^*-algebra. Results on this line were previously obtained by J. Dixmier and A. Douady [8] and K. Thomsen [22]. Our results concerning the computation of homotopy groups contain as special cases the above mentioned results.

Our method of computation reduces completely the computation of the groups $\pi_k(\mathrm{Aut}(A))$, $k > 0$, to the computation of the homotopy groups of unitaries (A is an AF-C^*-algebra, $\mathrm{Aut}(A)$ is the group of $*$-automorphisms of A endowed with the point norm topology). Using standard results concerning $\pi_k(U(n))$ we succeded to make a complete computation for $\pi_k(\mathrm{Aut}(A))$ for a large class of AF-C^*-algebras A. If A is simple, $A \neq K$ (the algebra of compact operators on a separable Hilbert space) the results are as follows: $\pi_{2k}(\mathrm{Aut}(A)) \simeq \simeq \mathrm{Hom}(\mathrm{K}_0(A)/\mathbf{Z}[1], \mathrm{K}_0(A))$, $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}(\mathrm{K}_0(A)/\mathbf{Z}[1], \mathrm{K}_0(A))$ for A unital ($k \geqslant 1$) and $\pi_{2k}(\mathrm{Aut}(A)) \simeq \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(A))$, $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_0(A))$ if A is not unital ($k \geqslant 1$). Note the similarity with results obtained by J. Cuntz in [6]; also there exist a few points of resemblence in the techniques used there by J. Cuntz and by us. If A is not simple the results are more complicated depending in a nontrivial way on the ideal structure of A. In order to handle these situations we were led to introduce the groups Hom_c and Ext_c which take into account the restrictions introduced by the ideals of $\mathrm{K}_0(A)$.

The method of proof is the following. First we study $\pi_k(\mathrm{End}(A))$, the homotopy groups of the semigroup of all $*$-homomorphisms $A \to A$ endowed with the pointwise convergence. It turns out then that the natural embedding $\mathrm{Aut}(A) \to \to \mathrm{End}(A)$ induces an isomorphism $\pi_k(\mathrm{Aut}(A)) \to \pi_k(\mathrm{End}(A))$ for any $k \geqslant 1$, and this is the crux point of the proof. The computation of $\pi_k(\mathrm{End}(A))$ requires the knowledge of $\pi_k(U(A_n'))$ (A_n' is the commutant of the finite dimensional C^*-algebra A_n

in A). This type of questions enter in what is called "nonstable K-theory" (see [17], [18]); in the same order of ideas we prove that certain C^*-algebras obtained from locally trivial fields of AF-C^*-algebras on spheres satisfy the cancellation property for finitely generated projective modules, and also we classify the positive cone of K_0 of these C^*-algebras.

The first section contains general results: the isomorphism $\pi_k(\mathrm{Aut}(A)) \to \pi_k(\mathrm{End}(A))$ for $k \geqslant 1$ and the reduction of the computation of $\pi_k(\mathrm{End}(A))$ to $\pi_k(U(n))$. In the second section we introduce the class of ordered groups with large denominators and show that $\pi_k(U(A_n')) \simeq K_0(A_n')$ if $K_0(A)$ has large denominators. Also we introduce Hom_c and Ext_c and develop briefly their properties, showing that $K_0(A_n') \simeq \mathrm{Hom}_c(K_0(A_n), K_0(A))$. Next to a k-loop f in $\mathrm{Aut}(A)$ we associate as usual a locally trivial field of AF-C^*-algebras on S^{k+1} and show that for k odd this defines an element in $\mathrm{Ext}_c(K_0(A), K_0(A))$ which is trivial if and only if f is inner. The final result is Theorem 2.12.

1.

In this section we shall prove same general results about the homotopy groups of the group of automorphisms of an AF-C^*-algebra.

For the basic results concerning AF-algebras and for the definitions not explained, such as ordered group, ideal of an ordered group, the interested reader may consult [3] or [9].

1.1. Let us introduce first some notations and fixe some conventions to be used from now on.

a) $K_i(A)$, $i = 0, 1$ will denote the K-theory groups of a C^*-algebra A ([3], [21]). If A is an AF-C^*-algebra, $\geqslant$ will denote the order on $K_0(A)$, $K_0(A)_+$ will denote the positive cone of $K_0(A)$ and $\Sigma(A)$ will denote the scale of $K_0(A)$ ([3], [9]). If $f: A \to B$ is a $*$-morphism of C^*-algebras $K_i(f): K_i(A) \to K_i(B)$ denotes the natural group morphism.

b) If A is a C^*-algebra $M(A)$ is the multiplier C^*-algebra of A ([15]).

c) Let us fix a base point $p_0 \in S^k$ for $k \geqslant 1$. If (X, x) is a pointed topological space a k-loop in X is a continuous base point preserving function $f: (S^k, p_0) \to (X, x)$. The class of this function in $\pi_k(X)$ will be denoted by $[f]$.

d) Let A be a C^*-algebra. A^+ denotes the algebra A with adjoined unit, $\chi : A^+ \to \mathbf{C}$ is the quotient map. $\tilde{A}$ denotes A if A has unit and A^+ otherwise. $U(A)$ is the set of those unitaries $u \in A^+$ such that $\chi(u) = 1$. $\mathrm{ad}_u(x) = uxu^*$ is the inner automorphism of A induced by $u \in U(A)$.

e) If (X, x) is a pointed topological space X^0 denotes the path component of the base point.

f) If A and B are C^*-algebras $\mathrm{Hom}(A, B)$ will denote the set of all $*$-morphisms $f : A \to B$. We shall topologise this set with the topology of norm-pointwise convergence. If $i : A \to B$, $\mathrm{Hom}(A, B, i)$ is the pointed topological space $(\mathrm{Hom}(A, B), i)$. $\mathrm{End}^0(A)$ denotes $(\mathrm{Hom}^0(A, A), \mathrm{id})$. id denotes various identity morphisms.

g) If $B \subset A$ are two C^*-algebras B' denotes the relative commutant of B in A.

h) Let G_n, $n \in \mathbf{N}$ be abelian groups, $\varphi_{nm}: G_m \to G_n$, $m > n$ an inverse system of homomorphisms. Let $\delta : \prod_{n \in \mathrm{N}} G_n \to \prod_{n \in \mathrm{N}} G_n$ given by $\delta((X_n)_{n \in \mathrm{N}}) = (X_n - \varphi_{n,n+1}(X_{n+1}))_{n \in \mathrm{N}}$. Te shall denote by $\underleftarrow{\lim}^1(G_n, \varphi_{nm})$ the cokernel of this morphism. Of course $\underleftarrow{\lim}(G_n, \varphi_{nm}) = \ker \delta$. If $\psi_{nm}: X_m \to X_n$, $m \geqslant n$ is an inverse system of topological spaces $\underleftarrow{\lim}(X_n, \psi_{nm})$ is the subspace $\{(x_n)_{n \in \mathrm{N}}, x_n = \psi_{n,n+1}(x_{n+1})\}$ of $\prod_{n \in \mathrm{N}} X_n$. It has the induced product topology.

i) From now on A will always denote an AF-C^*-algebra, $A = \overline{\bigcup A_n}$ and $A_n = A_n^{(1)} \oplus \ldots \oplus A_n^{(k_n)}$, $A_n^{(j)}$ being factors of type $\mathrm{I}_{p_{nj}}$. Also we shall denote by $\alpha_{mn}: \mathrm{K}_0(A_n) \to \mathrm{K}_0(A_m)$ the natural morphism induced by the inclusion $i_{mn}: A_n \to A_m$, for $m \geqslant n$. The inclusion $A_n \to A$ will be denoted by i_n and $\mathrm{Hom}^0(A_n, A, i_n)$ by $\mathrm{Hom}^0(A_n, A)$.

j) Other notations: $I = [0, 1]$, $IL = [0, 1] \times L$, $SA = C_0(\mathbf{R}, A)$. $\mathbf{B}_n$ is the standard n cell, $\mathrm{S}^{n-1} = \partial \mathbf{B}_n$.

k) By "ideal" we shall mean "closed two-sided ideal".

1.2. Let us denote by ψ_n and ψ_{nm} the mappings $\psi_n : U(A) \to \mathrm{Hom}^0(A_n, A)$, $\psi_n(u) = \mathrm{ad}_u | A_n$, $\psi_{nm}: \mathrm{Hom}^0(A_m, A) \to \mathrm{Hom}^0(A_n, A)$, $\psi_{nm}(f) = f | A_n$.

LEMMA. $(U(A), \varphi_n, \mathrm{Hom}^0(A_n, A))$ *is a locally trivial principal* $U(A_n')$-*bundle and* $(\mathrm{Hom}^0(A_m, A), \psi_{nm}, \mathrm{Hom}^0(A_n, A))$ *is a fibration.*

Proof. The second assertion follows from the first.

Let us prove now the first part. ψ_n is obviously surjective and the function $U(A)*U(A) \ni (u, v) \to \tau(u, v) = u^{-1}v \in U(A_n')$ is continuous (we have denoted, as usual, by $U(A)*U(A)$ the set of those pairs $(u, v) \in U(A) \times U(A)$ such that $\psi_n(u) = \psi_n(v)$). Also $\tau(u, \cdot)$ and $\tau(\cdot, v)$ are onto for any fixed u and v. This shows that $(U(A), \psi_n, \mathrm{Hom}^0(A_n, A))$ is a principal $U(A_n')$-bundle.

Let us show that there exists a cross section for ψ_n defined in a neighbourhood of i_n. Let V be the set of those $\varphi \in \mathrm{Hom}^0(A_n, A)$ such that $\|\varphi - i_n\| < 1$. If e and f are two selfadjoint projections such that $\|e - f\| < 1$ then fe has a polar decomposition $(efe \geqslant (1 - \|e - f\|)e)$. Denote by $\theta(f, e)$ the partial isometry arising in this polar decomposition, thus $fe = \theta(f, e)(efe)^{1/2}$. It follows that

$\theta(f, e)\theta(e, f) = f$ and $\theta(f, e) = \theta(e, f)^*$. Let (e^k_{ij}) be a matrix unit for A_n. The required cross section is defined as follows: $u(\varphi) = \sum_{k=1}^{k_n} \sum_{i=1}^{p_{nk}} \varphi(e^k_{i1})\theta(\varphi(e^k_{11}, e^k_{11})e^k_{1i}$, (see [4]). Since $U(A)$ acts transitively on $\mathrm{Hom}^0(A_n, A)$ a local cross section exists in the neighbourhood of each point.

1.3. LEMMA. $\mathrm{End}^0(A)$ *is homeomorphic to the inverse limit* $\varprojlim(\mathrm{Hom}^0(A_n, A), \psi_{mn})$.

Proof. Denote by $\varphi_n(f) = f | A_n$, $\varphi_n\colon \mathrm{End}^0(A) \to \mathrm{Hom}^0(A_n, A)$; then $\varphi_n = \psi_{nm} \circ \varphi_m$ for any $n \leqslant m$. Since each of φ_n is continuous they define a continuous function $\varphi = \varprojlim \varphi_n\colon \mathrm{End}^0(A) \to \varprojlim(\mathrm{Hom}^0(A_n, A), \psi_{mn})$. φ is obviously one-to-one and onto. φ is a homeomorphism from the very definition of the topology on $\mathrm{End}^0(A)$.

1.4. LEMMA. *Let L be a finite cell complex,* $f : L \to \mathrm{End}^0(A)$. *Then there exists a continuous function* $g : IL \to \mathrm{End}^0(A)$ *such that* $g | \{1\} \times L = f$ *and* $g(t, x) \in \mathrm{Aut}(A)$ *for any* $0 \leqslant t < 1$ *and* $x \in L$.

Proof. Denote by $\mathbf{B}_n$ the standard n-cell, $\mathbf{S}^{n-1} = \mathbf{B}_n$. By induction on the number of cells we reduce the problem to the following: given $f : \{1\} \times \mathbf{B}_n \cup I\partial\mathbf{B}_n \to \mathrm{End}^0(A)$ a continuous function, extend this function to a continuous function g on $I\mathbf{B}_n$ such that $g(x) \in \mathrm{Aut}^0(A)$ for any $x \in I\mathbf{B}_n \setminus (\{1\} \times \mathbf{B}_n \cup I\partial\mathbf{B}_n)$. But since the pair $(I\mathbf{B}_n, \{1\} \times \mathbf{B}_n \cup I\partial\mathbf{B}_n)$ is homeomorphic to the pair $(I\mathbf{B}_n, \{1\} \times \mathbf{B}_n)$ it follows that we may suppose that L itself is a cell, $L = \mathbf{B}_n$.

Since $\mathbf{B}_n$ is contractible and $(U(A), \psi_n, \mathrm{Hom}^0(A_n, A))$ is a fibration there exists $\theta_m\colon \mathbf{B}_n \to U(A)$ such that $f(x) | A_m = \mathrm{ad}_{\theta_m(x)} | A_m$. Let $\theta_0(x) = 1$. Using that $U(A'_m)$ is connected and $\mathbf{B}_n$ is contractible we may choose a continuous function $\theta_m\colon I\mathbf{B}_n \to U(A'_m)$ such that $\theta_m(t, x) = 1$ for $x \in \mathbf{B}_n$, $t \in [0, 1 - 1/m]$ and $\theta_m(t, x) = \theta^*_{m-1}(x)\theta_m(x)$ for $t \in [1 - 1/(m + 1), 1]$, $x \in \mathbf{B}_n$, $m \geqslant 1$. Set as in [1] $g(t, x) = \mathrm{ad}_{\theta_1(t, x)\theta_2(t, x)\ldots\theta_n(t, x)}$ for $t \leqslant 1 - 1/(n + 1)$ and $g(1, x) = f(x)$.

We have to prove the continuity of $g(t, x)$. It is enough to show that $(t, x) \to g(t, x)|A_m$ is continuous. But $g(t, x)|A_m = \mathrm{ad}_{\theta_1(t, x)\ldots\theta_m(t, x)}|A_m$ which is obviously continuous since θ_j are continuous.

1.5. THEOREM. a) *The natural inclusion* $j : \mathrm{Aut}^0(A) \to \mathrm{End}^0(A)$ *induces isomorphisms* $\pi_k(j) : \pi_k(\mathrm{Aut}^0(A)) \to \pi_k(\mathrm{End}^0(A))$, $k \geqslant 1$.

b) *There exists a short exact sequence of groups:*

$$0 \to \varprojlim{}^1(\pi_{k+1}(\mathrm{Hom}^0(A_n, A)), \pi_{k+1}(\psi_{nm})) \to \pi_k(\mathrm{Aut}^0(A)) \to$$

$$\to \varprojlim(\pi_k(\mathrm{Hom}^0(A_n, A)), \pi_k(\psi_{nm})) \to 0 \quad (k \geqslant 1).$$

c) $\pi_0(\mathrm{Aut}(A))$ *is isomorphic to the group of the automorphisms of the scaled ordered group* $(\mathrm{K}_0(A), \Sigma(A))$.

Proof. a) follows from Lemma 1.4.

b) follows from a), Lemmata 1.2 and 1.3 and [24], Theorem 4.8, p. 433.

There is an obvious morphism $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{K}_0(A), \Sigma(A))$. The kernel of this morphism is $\mathrm{Aut}^0(A)$ ([2], Theorem 3.1). This morphism is surjective by a theorem of Elliott ([10]). This proves c).

1.6. REMARK. Let us note that a nontrivial element of $\varprojlim^1(\pi_2(\mathrm{Hom}^0(A_n, A)), \pi_2(\psi_{nm})) \subset \pi_1(\mathrm{Aut}(A))$, for a certain AF-$C^*$-algebra A is implicitly contained in the construction of Proposition 5.1 of [7].

1.7. REMARK. Using the exact sequence of a fibration we obtain if $A = K$ (the algebra of compact operators on a separable Hilbert space) $\pi_k(\mathrm{Hom}^0(A_n, A)) \simeq \simeq \{0\}$ for $k \neq 2$ and $\pi_2(\mathrm{Hom}^0(A_n, A)) = \pi_2(\mathrm{Hom}^0(A_{n+1}, A)) \simeq \mathbf{Z}$, the isomorphism being induced by $\pi_2(\psi_{n,n+1})$.

It follows from Theorem 1.5 b) that $\pi_2(\mathrm{Aut}(K)) \simeq \mathbf{Z}$ and $\pi_k(\mathrm{Aut}(K)) \simeq \{0\}$ for $k \neq 2$. This also follows from results in [8].

2.

In this section we shall go further into the structure of the homotopy groups of a certain class of AF-C^*-algebras, a class which contains, for example, all simple, non type I AF-C^*-algebras.

2.1. We shall need the following results concerning the homotopy groups of the unitary group $U(n) = U(M_n(\mathbf{C}))$.

Denote by i and j the following functions $i, j : U(n) \to U(m)$ $i(u) = u \oplus \oplus I_{m-n}$, $j(u) = u \oplus \ldots \oplus u \oplus I_p$ (i is defined for $m \geqslant n$, j is defined for $p = m - - nl \geqslant 0$, u occurs l-times).

PROPOSITION ([13]). $\pi_k(j) = l\pi_k(i)$ *and* $\pi_k(j)$ *is an isomorphism for* $k/2 < n$. *Also*

$$\pi_k(U(n)) = \begin{cases} \mathbf{Z} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \qquad k/2 < n.$$

2.2. DEFINITION. Let (G, G_+) be an ordered group. We shall say that G has *large denominators* if for any $a \geqslant 0$ and $n \in N$ there exists $b \in G$ and $m \in \mathbf{N}$ such that $nb \leqslant a \leqslant mb$.

2.3. PROPOSITION. *Suppose that* A *is simple, infinite dimensional,* $A \neq K$*; then* $K_0(A)$ *has large denominators.*

Proof. Let $e \neq 0$ be a projection, $a = [e]$; replacing A by $eM_n(A)e$ for some large n we may suppose that $a = [1]$. Let $k \in N$. Denote by $J_n = \bigoplus_{p_{nj} \geq k} A_n^{(j)} \subset A_n$. Then $i_{mn}(J_n) \subset J_m$ and hence $J = \overline{\bigcup_n J_n}$ is an ideal of A. Since A is simple it follows that $J = A$ or $J = \{0\}$. But A/J has only finite dimensional irreducible representations; this shows that $J \neq \{0\}$ is possible only if $A = K$. It follows from the above discussion that $1 \in J = A$. Choose n such that $1 \in J_n$. Let (e_{ij}^k) be a matrix unit for $J_n = A_n = \bigoplus_{j=1}^{k_n} A_n^{(j)}$ with $A_n^{(j)}$ finite dimensional factors. $b = \sum_{j=1}^{k_n} [e_{11}^{(j)}]$ will satisfy the requirements of Definition 2.2.

2.4. PROPOSITION. *Suppose* $K_0(A)$ *has large denominators. Then:*

a) $K_0(A'_m)$ *has large denominators,* $m \geq 1$.

b) *The natural morphisms* $\pi_k(U(A)) \to K_1(S^kA)$ *are isomorphisms.*

c) *The isomorphisms of* b) *give a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 \to & \pi_{2k}(\mathrm{Hom}^0(A_{n+1}, A)) & \to & K_0(A'_{n+1}) & \to & K_0(A) & \to & \pi_{2k-1}(\mathrm{Hom}^0(A_{n+1}, A)) & \to 0 \\
& \downarrow & & \downarrow & & \| & & \downarrow & \\
0 \to & \pi_{2k}(\mathrm{Hom}^0(A_n, A)) & \to & K_0(A'_n) & \to & K_0(A) & \to & \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) & \to 0.
\end{array}$$

d) $\varprojlim^1(\pi_{2k-1}(\mathrm{Hom}^0(A_n, A)), \pi_{2k-1}(\psi_{n,n+1})) = 0$.

Proof. a) Suppose that A is not unital, $1 \in M(A) \setminus A$.

Let (e_{ij}^k) be a matrix unit for A_n. Denote by e_n the unit of A_n. An easy computation (see [1]) shows that A'_n is isomorphic to $(1 - e_n)A(1 - e_n) + \bigoplus_{k=1}^{k_m} e_{11}^k A e_{11}^k$, the isomorphism being $\varphi(a \oplus \bigoplus_{k=1}^{k_m} a_k) = a + \sum_{k=1}^{k_m} \sum_{i=1}^{p_{nk}} e_{i1}^k a_k e_{1i}^k$. Let J_k be the ideal generated in A by $(1 - e_n)$ for $k = 0$ and by e_{11}^k for $k > 0$. It follows that $K_0(A'_n) \simeq$ $\simeq K_0(J_0) \oplus \bigoplus_{k=1}^{k_m} K_0(J_k)$ since $e_{11}^k A e_{11}^k$ $((1 - e_n)A(1 - e_n)$ is a full corner in J_k (J_0); to prove this, use [5]. Since $K_0(J)$ has large denominators for any ideal J of A it follows that $K_0(A'_n)$ has large denominators.

For A unital the proof is similar.

b) We shall repeatedly use Proposition 2.1. There exists a commutative diagram of isomorphisms

$$\begin{tikzcd} \varinjlim \pi_k(U(n)) \arrow[r] \arrow[dr] & \mathrm{K}_1(S^k\mathbf{C}) \arrow[d] \\ & \tilde{\mathrm{K}}^0(\mathrm{S}^{k+1}) \end{tikzcd}$$

(see [13]).

For each A_n denote by e_n its unit. Let $l_0 > k/2$ and f_n such that $l_0[f_n] \leqslant$ $\leqslant [e_n] \leqslant m[f_n]$ for some $m \in \mathbf{N}$. Replacing $(A_n)_{n\in\mathbf{N}}$ by a subsequence and the f_n's, by some equivalent projections we may suppose that $f_n \in A_{n+1}$. Replace again A_n by $e_nA_{n+1}e_n$. It follows that $A_n \simeq M_{r_1} \oplus \ldots \oplus M_{r_j}$ and $r_1, \ldots, r_j \geqslant l_0 > k/2$. Then $\pi_k(U(A_n)) \to \mathrm{K}_1(S^kA_n)$ is an isomorphism and we have isomorphisms $\pi_k(U(A)) \simeq$ $\simeq \varinjlim \pi_k(U(A_n)) \simeq \varinjlim \mathrm{K}_1(S^kA_n) \simeq \mathrm{K}_1(S^kA)$ (recall the convention made for $U(A)$ in 1.1 d)).

c) This follows from the exact sequences of the fibration $U(A_n') \to U(A) \to \mathrm{Hom}^0(A_n, A)$ and from the commutativity of the diagram

$$\begin{tikzcd} U(A_{n+1}') \arrow[r] \arrow[d] & U(A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}^0(A_{n+1}, A) \arrow[d] \\ U(A_n') \arrow[r] & U(A) \arrow[r] & \mathrm{Hom}^0(A_n, A). \end{tikzcd}$$

(Note that $\pi_{2k+1}(U(A)) \simeq \mathrm{K}_0(A)$ and $\pi_{2k}(U(A)) \simeq \{0\}$ by b).)

d) follows from the surjectivity of $\pi_{2k-1}(\psi_{n,n+1})$ as is apparent from c).

The previous lemma shows that it is important to know $\mathrm{K}_0(A_n')$ and, in view of Theorem 1.5, to compute also the morphisms $\mathrm{K}_0(A_{n+1}') \to \mathrm{K}_0(A_n')$. The following definition and Definition 2.9 are an attempt to give a satisfactory framework for our computations.

2.5. Definition. Let H_1, H_2 be ordered groups, $i : H_1 \to H_2$ a positive morphism, $\varphi : H_1 \to H_2$ a group morphism. We shall say that φ *is compatible with* i if for every $x \in H_1$, $x \geqslant 0$ there exists $m \in \mathbf{N}$ such that $-mi(x) \leqslant \varphi(x) \leqslant mi(x)$.

We shall denote by $\mathrm{Hom}_c(H_1, H_2, i)$ the set of morphisms $\varphi: H_1 \to H_2$ compatible with i. In the same spirit as before $\mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A), \mathrm{K}_0(i_n))$ will be denoted by $\mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$ and $\mathrm{Hom}_c(G, G, \mathrm{id})$ by $\mathrm{End}_c(G)$.

This definition is suggested by the computation of $\mathrm{K}_0(A_n')$ in the proof of Proposition 2.4 a).

2.6. The following proposition gives the basic proprieties of Hom_c needed in the computation of $\pi_k(\mathrm{Aut}(A))$.

Proposition. a) *Let H_1, H_2, and H_3 be ordered groups, $l_1: H_1 \to H_2$, $l_2: H_2 \to H_3$ be positive morphisms. Then there exist natural morphisms* $l_1^* : \mathrm{Hom}_c(H_2, H_3, l_2) \to$

$\to \mathrm{Hom}_c(H_1, H_3, l_2 \circ l_1)$ *and* $i_{2*}\colon \mathrm{Hom}_c(H_1, H_2, l_1) \to \mathrm{Hom}_c(H_1, H_3, l_2 \circ l_1)$ *given by* $l_1^*(\varphi) = \varphi \circ l_1$ *and* $l_{2*}(\varphi) = l_2 \circ \varphi$.

b) *If* H_1, H_2, *and* l_1 *are as before,* H_2 *is a simple ordered group, and* $l_1(x) \neq 0$ *for* $x \geqslant 0$, $x \neq 0$, *then* $\mathrm{Hom}_c(H_1, H_2, l_1) = \mathrm{Hom}(H_1, H_2)$.

c) *Suppose* H_n, $n \in \mathbf{N}$ *and* H' *are ordered groups,* $j_{mn} : H_n \to H_m$ *are positive morphisms for* $n \leqslant m$ *and* $H = \varinjlim(H_n, j_{mn})$. *Also let* $i : H \to H'$ *be an order morphism. Denote by* l_n *the composition* $H_n \to H \xrightarrow{i} H'$; *then*

$$\mathrm{Hom}_c(H, H', i) \simeq \varprojlim(\mathrm{Hom}_c(H_n, H', l_n), j^*_{mn}).$$

Proof. a) Let $\varphi_2 \in \mathrm{Hom}_c(H_2, H_3, l_2)$, $\varphi_1 \in \mathrm{Hom}_c(H_1, H_2, l_1)$. We have to prove that $\varphi_2 \circ l_1, l_2 \circ \varphi_1 \in \mathrm{Hom}_c(H_1, H_3, l_2 \circ l_1)$. Let $x \in H_1$, $x \geqslant 0$ then $l_1(x) \geqslant 0$. Choose m such that $-ml_2(l_1(x)) \leqslant \varphi_2(l_1(x)) \leqslant ml_2(l_1(x))$. This proves the first part. Choose m such that $-ml_1(x) \leqslant \varphi_1(x) \leqslant ml_1(x)$. Since l_2 preserves the inequalities we obtain the desired conclusion.

b) Since H_2 is simple and $l_1(x) \geqslant 0$, $l_1(x) \neq 0$ for $x \neq 0$, $x \geqslant 0$ it follows that $l_1(x)$ is on order unit for H_2, namely for any $y \in H_2$ there exists an $m \in \mathbf{N}$ such that $-ml_1(x) \leqslant y \leqslant ml_1(x)$ (see [9]). This concludes the proof.

c) Denote by j_n the positive morphism $H_n \to H$; j_n defines a morphism j_n^*: $:\mathrm{Hom}_c(H, H', i) \to \mathrm{Hom}_c(H_n, H', l_n)$. Since $j_m \circ j_{mn} = j_n$ it follows that $j_n^* = j^*_{mn} \circ j_m$ and hence j_n^* collect to define a morphism $f\colon \mathrm{Hom}_c(H, H', i) \to \varprojlim(\mathrm{Hom}_c(H_n, H', l_n), j^*_{mn})$. Let $\varphi \in \mathrm{Hom}_c(H, H', i)$. If $f(\varphi) = 0$ then $\varphi \circ j_n = 0$ for any n and hence $\varphi = 0$. Let $\varphi_n \in \mathrm{Hom}_c(H_n, H', l_n)$ such that $j^*_{mn}(\varphi_m) = \varphi_n$. This means that $\varphi_m \circ j_{mn} = \varphi_n$. Define $\varphi\colon \varinjlim H_n \to H'$ using the universal property of the inductive (direct) limit: $\varphi \in \mathrm{Hom}(H, H')$. We need to check that φ is actually in $\mathrm{Hom}_c(H, H', i)$. Let $x \in H$, $x \geqslant 0$. Then there exist n and $x_n \in H_n$, $x_n \geqslant 0$ such that $j_n(x_n) = x$. By the assumption that $\varphi_n \in \mathrm{Hom}_c(H_n, H', i \circ l_n)$ it follows that there exists $m \in \mathbf{N}$ such that $-mi(l_n(x_n)) \leqslant \varphi_n(x_n) \leqslant mi(l_n(x_n))$ and hence $-mi(x) \leqslant \varphi(x) \leqslant mi(x)$.

2.7. LEMMA. *Let* A *be a unital* AF-C^*-*algebra such that* $\mathrm{K}_0(A)$ *has large denominators. Suppose also that* $A_0 = \mathbf{C}1$.

a) *There exist isomorphisms* $\beta_n\colon \mathrm{K}_0(A'_n) \to \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$ *and a commutative diagram*

$$\begin{tikzcd}
\mathrm{K}_0(A'_m) \arrow[rr] \arrow[dd, "\mathrm{K}_0(j)"'] \arrow[dr] & & \mathrm{Hom}_c(\mathrm{K}_0(A_m), \mathrm{K}_0(A)) \arrow[d, "\alpha^*_{0m}"] \arrow[dd, bend left, "\alpha^*_{mn}"] \\
& \mathrm{K}_0(A) \arrow[r] & \mathrm{Hom}_c(\mathbf{Z}, \mathrm{K}_0(A)) \\
\mathrm{K}_0(A'_n) \arrow[rr] \arrow[ur] & & \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \arrow[u, "\alpha^*_{0n}"']
\end{tikzcd} \qquad m > n$$

($j\colon A'_m \to A'_n$ *is the natural inclusion and* α_{pq} *is as in* 1.1 i)).

b) *There exist morphisms* $\mu\colon \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to \mathrm{Hom}(\mathrm{K}_0(A_n)/\mathbf{Z}, \mathrm{K}_0(A))$ *and* $\varepsilon\colon \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to \mathrm{Ext}(\mathrm{K}_0(A_n)/\mathbf{Z}, \mathrm{K}_0(A))$ *and a commutative diagram with exact rows:*

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \arrow[r] \arrow[d, "\mu"] & \pi_{2k-1}(U(A_n')) \arrow[r] \arrow[d, "\delta"] & \pi_{2k-1}(U(A)) \arrow[r] \arrow[d, "\approx"] & \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathrm{K}_0(A_n)/\mathbf{Z},\mathrm{K}_0(A)) \arrow[r] & \mathrm{Hom}(\mathrm{K}_0(A_n),\mathrm{K}_0(A)) \arrow[r] & \mathrm{Hom}(\mathbf{Z},\mathrm{K}_0(A)) \arrow[r, "-1"] & \mathrm{Ext}(\mathrm{K}_0(A_n)/\mathbf{Z},\mathrm{K}_0(A)) \arrow[r] & 0
\end{tikzcd}$$

($\delta\colon \pi_{2k-1}(U(A_n')) \to \mathrm{Hom}(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$ *is the composition*

$$\pi_{2k-1}(U(A_n')) \to \mathrm{K}_0(A_n') \to \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \to \mathrm{Hom}(\mathrm{K}_0(A_n), \mathrm{K}_0(A)),$$

$\mathbf{Z}$ *is embedded as* $n \to n[1]$*).*

Proof. It follows from the proof of Proposition 2.4 a) that $\mathrm{K}_0(A_n')$ is a subgroup of $\mathrm{K}_0(A)^{k_n} \simeq \mathrm{Hom}(\mathbf{Z}^{k_n}, \mathrm{K}_0(A)) \simeq \mathrm{Hom}(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$. The previous isomorphism maps $\mathrm{K}_0(A_n')$ onto the set of those morphisms $\varphi\colon \mathrm{K}_0(A_n) \to \mathrm{K}_0(A)$ such that $\varphi([e_{11}^k])$ belongs to the ideal generated in $\mathrm{K}_0(A)$ by $[e_{11}^k]$, namely the set of those $a \in \mathrm{K}_0(A)$ such that there exists $m \in \mathbf{N}$ such that $-m[e_{11}^k] \leqslant a \leqslant m[e_{11}^k]$. This shows that $\mathrm{K}_0(A_n')$ is isomorphic to $\mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$.

We shall prove that $\alpha_{mn}^*\beta_m = \beta_n\mathrm{K}_0(j)$, the other relations being similar. Let $\alpha_{mn} = (a_{pq})$ the matrix representation of the morphism $\alpha_{mn}\colon \mathrm{K}_0(A_n) \to \mathrm{K}_0(A_m)$ $(1 \leqslant p \leqslant k_m,\ 1 \leqslant q \leqslant k_n)$. Let $([e], 0, \ldots, 0) \in \mathrm{K}_0(A)^{k_m} \cap \mathrm{Hom}_c(\mathrm{K}_0(A_m), \mathrm{K}_0(A))$, i.e. $[e] \in \mathrm{K}_0(J_1)$ (we use the notations introduced in the proof of 2.4 a)). Suppose $([e], 0, \ldots, 0)$ is represented in A_m' by $f_1 = \sum_{i=1}^{p_{m1}} e_{i1}^1 f e_{1i}^1$ for a projection f equivalent to $[e]$ (e_{ij}^k is a matrix unit of A_m). We want to find the class of this projection in $\mathrm{K}_0(A_n')$. Let (e_{st}^r) be a matrix unit of A_n. We may suppose that the matrix units e_{st}^r and (e_{ij}^k) are compatible in the sense that each of e_{st}^r is a sum of some of e_{ij}^k. To be more precise in such a sum for e_{11}^r appear a_{kr} projections from $e_{11}^k, e_{22}^k, \ldots, e_{p_{mk}p_{mk}}^k$. Let $[g]$ be the r^{th} component in $\mathrm{K}_0(A_n')$ of f_1, this is the r^{th} component of $\alpha_{mn}^*([e], 0, \ldots, 0)$ in $\mathrm{Hom}_c(\mathrm{K}_0(A_n'), \mathrm{K}_0(A))$. $[g]$ is represented by $\sum_{s=1}^{p_{m1}} e_{s1}^r h e_{1s}^r$ and h is a projection equivalent to g. It is clear now that h is of the form $h = e_{11}^r f_1 e_{11}^r = \sum_{e_{ii}^1 \leqslant e_{11}^r} e_{i1}^1 f e_{1i}^1$, a_{1r} terms occuring in the sum and hence $[g] = [h] = a_{1r}[f] = a_{1r}[e]$. It follows that there exists a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(A_m') \arrow[r] \arrow[d, "\mathrm{K}_0(j)"] \arrow[rrr, bend left, "\simeq"] & \mathrm{K}_0(A_m)^* \otimes \mathrm{K}_0(A) \arrow[r, "\simeq"] \arrow[d, "\alpha_{mn}^{\mathrm{T}} \otimes 1"] & \mathrm{Hom}(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) & \mathrm{Hom}_c(\mathrm{K}_0(A_m), \mathrm{K}_0(A)) \arrow[l] \arrow[d, "\alpha_{mn}^*"'] \\
\mathrm{K}_0(A_n') \arrow[r] \arrow[rrr, bend right, "\simeq"'] & \mathrm{K}_0(A_n)^* \otimes \mathrm{K}_0(A) \arrow[r, "\simeq"] & \mathrm{Hom}(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) & \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)). \arrow[l]
\end{tikzcd}$$

This diagram gives the desired conclusion.

b) Let $f\colon \mathbf{S}^{2k-1}, p_0 \to U(A_n'), 1$ be a $2k-1$ loop. We identify $\mathbf{S}^{2k-1}$ with $\partial \mathbf{B}_{2k}$. Choose $g\colon \mathbf{B}_{2k} \to U(M_2(A))$ an extension of $f \oplus f^*$ to $\mathbf{B}_{2k}$. ad_g defines a morphism $A_n \to C(\mathbf{B}_{2k}, M_2(A)) \to C(\mathbf{B}_{2k}, K \otimes A)$. Since f takes values in $U(A_n')$, the range of the previous morphisms is actually in $C(\mathbf{B}_{2k}/\partial\mathbf{B}_{2k}, K \otimes A) \simeq C(\mathbf{S}^{2k}, K \otimes A)$. Denote by $\varphi_f\colon A_n \to C(\mathbf{S}^{2k}, K \otimes A)$ the previous defined morphism and by $\psi\colon A_n \to$ $\to C(\mathbf{S}^{2k}, K \otimes A)$ the morphism $\psi(a)(x) = a \oplus 0$ (the upper left corner embedding by constant functions). Using a Künneth theorem ([3], [19]) or by direct computation $\mathrm{K}_0(C(\mathbf{S}^{2k}, A)) \simeq \mathrm{K}_0(A) \otimes \mathrm{K}_0(A)$, the first summand being $\mathrm{K}_0(\psi)(\mathrm{K}_0(A))$ and the second being the kernel of the morphism $\mathrm{K}_0(e)\colon \mathrm{K}_0(C(\mathbf{S}^{2k}, A)) \to \mathrm{K}_0(A)$ induced by the evaluation at $\mathbf{S}^{2k-1}/\mathbf{S}^{2k-1}$ (the point obtained by collapsing $\mathbf{S}^{2k-1} = \partial \mathbf{B}_{2k}$ to a point).

It follows that $\mathrm{K}_0(\varphi_f) - \mathrm{K}_0(\psi)$ defines a morphism $\delta([f]) = \mathrm{K}_0(\varphi_f) - \mathrm{K}_0(\psi)$: $:\mathrm{K}_0(A) \to \ker(\mathrm{K}_0(e)) \simeq \mathrm{K}_0(A)$. This morphism depends only on the class of f in $\pi_{2k-1}(U(A_n'))$. Ths shows that δ is a well defined function.

Let us show that δ is actually a morphism.

Denote by $f * g$ the operation of concatenation of loops and by $\sigma\colon \mathbf{S}^{2k} \to$ $\to \mathbf{S}^{2k} \vee \mathbf{S}^{2k} \simeq \mathbf{S}^{2k}/\text{equator}$ the obvious morphism. Note that there exists a homotopy commutative diagram:

$$
\begin{tikzcd}
 & C(\mathbf{S}^{2k}, A \otimes K) \arrow[dr] & & \\
A_n \arrow[ur, "\varphi_f"] \arrow[dr, "\varphi_g"'] \arrow[dd, equal] & & C(\mathbf{S}^{2k} \vee \mathbf{S}^{2k}, A \otimes K) \arrow[r, "\sigma^*"] & C(\mathbf{S}^{2k}, A \otimes K) \arrow[dd, equal] \\
 & C(\mathbf{S}^{2k}, A \otimes K) \arrow[ur] & & \\
A_n \arrow[rrr, "\varphi_{f*g}"] & & & C(\mathbf{S}^{2k}, A \otimes K).
\end{tikzcd}
$$

The corresponding commutative diagram of K_0-groups looks as follows:

$$
\begin{tikzcd}
 & \mathrm{K}_0(A) \oplus \mathrm{K}_0(A) \arrow[dr, "i_1"] & & \\
\mathrm{K}_0(A) \arrow[ur, "{(\mathrm{id}, \delta([f]))}"] \arrow[dr, "{(id, \delta([g]))}"'] \arrow[dd, equal] & & \mathrm{K}_0(A) \oplus \mathrm{K}_0(A) \oplus \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\sigma^*)"] & \mathrm{K}_0(A) \oplus \mathrm{K}_0(A) \arrow[dd] \\
 & \mathrm{K}_0(A) \oplus \mathrm{K}_0(A) \arrow[ur, "i_2"'] & & \\
\mathrm{K}_0(A) \arrow[rrr, "{(\mathrm{id}, \delta([f * g]))}"] & & & \mathrm{K}_0(A) \oplus \mathrm{K}_0(A),
\end{tikzcd}
$$

where $i_1(x, y) = (x, y, 0)$, $i_2(x, z) = (x, 0, z)$, $\mathrm{K}_0(\sigma^*)(x, y, z) = (x, y + z)$.

This gives the desired conclusion.

Note that $\delta([f])([1]) = \delta_0([f])$ = the index of the loop f regarded as an element of $K_1(S^{2k-1}A)$. Hence, if f is homotopic to the constant loop p_0 in $U(A)$ then $\delta([f])$ factors to give a well defined morphism $K_0(A_n)/\mathbf{Z} \to K_0(A)$. This is $\mu([f])$ under the identification $\pi_{2k}(\mathrm{Hom}^0(A_n, A)) = \ker(\pi_{2k-1}(U(A_n')) \to \pi_{2k-1}(U(A)))$.

Let us define now ε. Let f: $(S^{2k-1}, p_0) \to (U(A), 1)$ be a $2k-1$ loop. f defines a unital morphism $A_n \to C(S^{2k-1}, A)$. This morphism is the Busby invariant of a unital extension

$$0 \to S^{2k}A \to E \to A_n \to 0. \tag{1}$$

Denote by 1 the units of E and A_n as well. This gives an extension of groups

$$0 \to K_0(A) \to K_0(E)/\mathbf{Z} \to K_0(A_n)/\mathbf{Z} \to 0$$

($\mathbf{Z}$ is embedded as $n \to n\,[1]$).

The class of this extension in $\mathrm{Ext}(K_0(A_n)/\mathbf{Z}, K_0(A))$ will be denoted by $\varepsilon([f])$. We may show that ε is a group morphism as we did for δ or as we shall do for E in Theorem 2.11. However we shall confine ourselves to note that this will follow if we shall show that the diagram

$$\begin{array}{ccc} \pi_{2k-1}(U(A)) & \to & \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to 0 \\ \downarrow \approx & & \downarrow \varepsilon \\ \mathrm{Hom}(\mathbf{Z}, K_0(A)) & \xrightarrow{-1} & \mathrm{Ext}(K_0(A_n)/\mathbf{Z}, K_0(A)) \to 0 \end{array}$$

is commutative. To show this observe that (1) becomes a trivial extension after tensoring $C_0(\mathbf{B}_{2k}, A)$ by K. A lifting for $A_n \to C(S^{2k-1}, A) \to C(S^{2k-1}, M_2(A))$ to $C(\mathbf{B}_{2k}, M_2(A))$ is given by a lifting of $f \oplus f^*$. This shows that our extension of groups is isomorphic to

$$0 \to K_0(S^{2k}A) \to K_0(S^{2k}A) \oplus K_0(A_n)/\mathbf{Z}(\delta_0([f]), [1]) \to K_0(A_n)/\mathbf{Z}[1] \to 0$$

and hence it is the image of the morphism $\mathbf{Z} \to K_0(A)$ which sends 1 to $-\delta_0([f])$ in $\mathrm{Ext}(K_0(A_n)/\mathbf{Z}, K_0(A))$ (this also justifies the appearence of the sign -1).

The first row is exact since it is a segment of the exact sequence of homotopy groups of a fibration ([24]). The second row is a segment of the Ext-exact sequence of homological algebra ([14]).

The next lemma is the nonunital case of 2.7.

2.8. LEMMA. *Let A be a nonunital* AF-C^*-*algebra. There exists an exact sequence*

$$0 \to \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to \mathrm{Hom}_c(K_0(A_n^+), K_0(A_n^+)) \to \mathrm{Hom}_c(\mathbf{Z}, K_0(A^+)) \to$$

$$\to \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to 0.$$

Proof. There exist morphisms

$$\varphi_1\colon \mathrm{Hom}_c(\mathrm{K}_0(A_n^+), \mathrm{K}_0(A^+)) \to \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \simeq \mathbf{Z}$$

and

$$\varphi_2\colon \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A^+)) \to \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \simeq \mathbf{Z}$$

given by $\mathbf{Z} \rightrightarrows \mathbf{Z}[1] \subset \mathrm{K}_0(A_n^+)$ and $\mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbf{C}) \simeq \mathbf{Z}$. It follows that $\mathrm{K}_0(A_n') \simeq \simeq \mathrm{Ker}\,\varphi_1$ and $\mathrm{K}_0(A) \simeq \ker \varphi_2$ as easily seen from the proof of Proposition 2.4 a) (we have an argument similar to that given in the first paragraph of the proof of Lemma 2.7). The rest is an application of Proposition 2.4 c).

2.9. Definition. Let H_1, H_2 be ordered groups, $i : H_1 \to H_2$ an order morphism. An exact sequence of groups

$$[E] \qquad 0 \to H_2 \to E \xrightarrow{q} H_1 \to 0$$

in which E is an ordered group, q is a positive morphism, will be called *compatible* with i if

(a) $q(E_+) = H_{1+}$;

(b) if $x, y \geqslant 0$, $x \in H_1$, $y, z \in E$ are such that $x - q(y) = q(z) \geqslant 0$ then $z \geqslant 0$ if any only if there exists $m \geqslant 0$ such that $-mi(x) \leqslant z - y \leqslant mi(x)$ in H_2.

As in the usual case, two compatible extensions E_1 and E_2 will be called *isomorphic* if there exists a positive group morphism $\varphi\colon E_1 \to E_2$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H_2 & \to & E_1 & \to & H_1 & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \to & H_2 & \to & E_2 & \to & H_1 & \to & 0. \end{array} \tag{2.9.1}$$

It follows that φ is necessarily an isomorphism of ordered groups.

The set of isomorphism classes of extensions of H_1 by H_2, compatible with i, will be denoted by $\mathrm{Ext}_c(H_1, H_2, i)$.

An extension E compatible with i is called *trivial* if there exists a positive lifting for q. Note that in view of our definition two liftings differ by an element of $\mathrm{Hom}_c(H_1, H_2, i)$.

It is not true that a trivial extension in $\mathrm{Ext}_c(H_1, H_2, i)$ is isomorphic to $H_1 \oplus \oplus H_2$ as ordered groups.

We will need only the following results about this Ext_c.

2.10. Proposition. a) *Let H_1, H_2, H_3, H_4 be ordered groups, $l_j\colon H_j \to H_{j+1}$, $j \in \{1, 2, 3\}$ be order morphisms. Suppose that the ideal generated in H_{j+1} by $l_j(H_j)$ is H_{j+1} ($j \in \{1, 2, 3\}$).*

Then there exist functions

$$l_1^* : \mathrm{Ext_c}(H_2, H_3, l_2) \to \mathrm{Ext_c}(H_1, H_3, l_2 \circ l_1)$$

$$l_{3*} : \mathrm{Ext_c}(H_2, H_3, l_2) \to \mathrm{Ext_c}(H_2, H_4, l_3 \circ l_2)$$

with the property that $l_1^* \circ l_{3*} = l_{3*} \circ l_1^*$ *as functions from* $\mathrm{Ext_c}(H_2, H_3, l_2)$ *to* $\mathrm{Ext_c}(H_1, H_4, l_3 \circ l_2 \circ l_1)$.

b) *Let* $E_1, E_2 \in \mathrm{Ext_c}(H_1, H_2, i)$, $d: H_1 \to H_1 \oplus H_1$ $d(a) = a \oplus a$, $\sigma: H_2 \oplus \oplus H_2 \to H_2$, $\sigma(a, b) = a + b$ *then* $E_1 \oplus E_2 \in \mathrm{Ext_c}(H_1 \oplus H_2, H_2 \oplus H_2, i \oplus i)$ *and* $d^*(\sigma_*(E_1 \oplus E_2)) = \sigma_*(d^*(E_1 \oplus E_2)) \in \mathrm{Ext_c}(H_1, H_2, i)$ *defines a group structure on* $\mathrm{Ext_c}(H_1, H_2)$ *with the trivial extension as a neutral element.*

c) *Let* $H_n, n \in \mathbf{N}$, H', i, j_{mn}, l_n *and* H *be as in Proposition* 2.6e); *then there exists an exact sequence of groups*

$$0 \to \varprojlim{}^1(\mathrm{Hom_c}(H_n, H', l_n), j_{mn}^*) \to \mathrm{Ext_c}(H, H', i) \to \varprojlim(\mathrm{Ext_c}(H_n, H', l_n), j_{mn}^*) \to 0.$$

Proof. a) Let $[E] = 0 \to H_3 \xrightarrow{j} E \xrightarrow{q} H_2 \to 0$ be an element of $\mathrm{Ext_c}(H_2, H_3, l_2)$. Define $l_1^*([E])$ to be the class in $\mathrm{Ext_c}(H_1, H_3, l_2 \circ l_1)$ of the extension

$$0 \to H_3 \to E \amalg_{l_1} H_1 \to H_1 \to 0$$

Here $E \amalg_{l_1} H_1 = \{(x, h_1) \mid q(x) = l_1 h_1)\}$ and $(x, h_1) \geqslant 0$ if and only if $x \geqslant 0$ and $h_1 \geqslant 0$.

$l_{3*}([E])$ is the class in $\mathrm{Ext_c}(H_2, H_4, l_3 \circ l_2)$ of the extension

$$0 \to H_4 \to E \oplus H_4/(-j) \oplus l_3(H_3) \xrightarrow{q_1} H_2 \to 0.$$

The order on $E_1 = E \oplus H_4/(-j) \oplus l_3(H_3)$ has as positive cone the set P_1 o the classes of elements (x, h_4), $x \in E$, $x \geqslant 0$, $h_4 \in H_4$ such that there exists $m \geqslant 0$ for which $-ml_3 \circ l_2 \circ q(x) \leqslant h_4 \leqslant ml_3 \circ l_2 \circ q(x)$. Denote by $\widehat{(x, h_4)}$ the class of an element $(x, h_4) \in E \oplus H_4$ in E_1. We shall show that $[E_1] \in \mathrm{Ext_c}(H_2, H_4, l_3 \circ l_2)$. Let $\widehat{(x, h)} \in E_1$. There exist positive elements $x_1, x_2 \in E$ such that $x = x_2 - x_1$. Also, since the ideal generated by $l_3 \circ l_2(H_2)$ in H_4 is the whole of H_4, there exists

$x_3 \geqslant 0$ such that $-l_3 \circ l_2(x_3) \leqslant h \leqslant l_3 \circ l_2(x_3)$. It follows that $\widehat{(x, h)} = \widehat{(x_2 + x_3, h)} - \widehat{(x_1 + x_3, 0)}$ is the difference of two positive elements. If $(x, h) \in P_1 \cap (-P_1)$ then $q_1(\widehat{(x, h)}) = q(x) \in E_+ \cap (-E_+)$ (E_+ is the positive cone in E). This shows that $P_1 \cap (-P_1) = \{0\}$.

Let $\widehat{(x_1, h_1)}, \widehat{(x_2, h_2)} \in E_1$ be such that $\widehat{(x_1, h_1)}$ is positive and $h = q(x_1) = q(x_2)$. Suppose that $\widehat{(x_2, h_2)}$ is also positive. Then we may suppose that $x_1, x_2 \geqslant 0$, $-ml_2(h) \leqslant x_2 - x_1 \leqslant ml_2(h)$ and $-ml_3 \circ l_2(h) \leqslant h_j \leqslant ml_3 \circ l_2(h)$ for some $m \in \mathbf{N}$ and $j \in \{1, 2\}$. Then $\widehat{(x_1, h_1)} - \widehat{(x_2, h_2)} = \widehat{(0, h_1 - h_2 + l_3(x_1 - x_2))}$ satisfies $-3ml_3 \circ l_2(h) \leqslant h_1 - h_2 + l_3(x_1 - x_2) \leqslant 3ml_3 \circ l_2(h)$. Conversely, if $-ml_3 \circ l_2(h) \leqslant h' \leqslant ml_3 \circ l_2(h)$ then $\widehat{(x_1, h_1)} + \widehat{(0, h')}$ is positive from the definition. Since $q_1(\widehat{(x_1, h_1)}) = q(x_1)$, $q_1((x_2, h_2)) = q(x_2)$ it follows that (E_1, P_1) defines an element of $\mathrm{Ext}_c(H_2, H_4, l_3 \circ l_2)$.

Note that if in (2.9.1) we suppose only that φ is a group morphism then it follows that φ is actually a morphism of ordered groups. This shows that the natural function $\mathrm{Ext}_c(H_1, H_2, i) \to \mathrm{Ext}(H_1, H_2)$ is injective and hence that $\mathrm{Ext}_c(H_1, H_2, i)$ may be identified with a subset of $\mathrm{Ext}(H_1, H_2)$. This proves the rest of a) and b).

Let E_n and β_{mn} be such that the diagrams

$$\begin{array}{ccccccccc} 0 & \to & H' & \to & E_n & \to & H_n & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \beta_{mn}} & & \downarrow & & \\ 0 & \to & H' & \to & E_m & \to & H_m & \to & 0 \end{array}$$

are commutative, β_{mn} positive, then

$$0 \to H' \to \varinjlim (E_n, \beta_{mn}) \to \varinjlim (H_n, j_{mn}) \to 0$$

represents an element $[E] \in \mathrm{Ext}_c(H, H', i)$ such that its image in $\mathrm{Ext}_c(H_n, H', j_n)$ is $[E_n]$. This gives the surjectivity of $\mathrm{Ext}_c(H_n, H', j_n) \to \varprojlim (\mathrm{Ext}_c(H_n, H', j_n), j_{mn}^*)$.

Let $0 \to H' \to E \xrightarrow{q} H \to 0$ be an extension such that if $j_n \colon H_n \to H$ is the limit morphism then $j_n^*([E])$ is trivial. This means that there exist positive liftings $\tau_n : H_n \to E$ such that $q \circ \tau_n = j_n$. Let us observe that $\tau_{n+1} \circ j_{n+1,n} - \tau_n \in \mathrm{Hom}_c(H_n, H', j_n)$. If we choose other liftings τ'_n then $\tau'_{n+1} \circ j_{n+1,n} - \tau'_n = \tau_{n+1} \circ j_{n+1,n} - \tau_n + (\tau'_{n+1} - \tau_{n+1}) \circ j_{n+1,n} - (\tau'_n - \tau_n)$. It follows that $(\tau_{n+1} \circ j_{n+1,n} - \tau_n)_{n \in \mathbf{N}}$ and

$(\tau_{n+1} \circ j_{n+1,n} - \tau'_n)_{n\in\mathbf{N}}$ differ in $\prod_{n\in\mathbf{N}} \mathrm{Hom}_c(H_n, H', l_n)$ by an element in the range of δ (see 1.1h)). This gives the rest of the statement.

Let us observe that if H_1 and H_2 are unperforated then any ordered group representing an element in $\mathrm{Ext}_c(H_1, H_2, i)$ is unperforated.

We shall denote by $\mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A))$ the group $\mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A), \mathrm{id})$.

2.11. LEMMA. *Let $f: \mathbf{S}^{2k-1} \to \mathrm{Aut}(A)$ be a $2k-1$ loop and suppose that $\mathrm{K}_0(A)$ has large denominators. Let $E_f \subset C(\mathbf{B}_{2k}, A)$ be the C^*-algebra of those functions $\varphi: \mathbf{B}_{2k} \to A$ such that $\varphi(x) = f(x)(a)$ for some $a = \eta(\varphi) \in A$ and any $x \in \mathbf{S}^{2k-1}$. Then the semigroup $V(A)$ of projective finitely generated modules over E_f has cancellation. If $\mathrm{K}_0(E_f)_+$ is the positive cone of $\mathrm{K}_0(E_f)$ then $\mathrm{K}_0(\eta)(\mathrm{K}_0(E_f)_+) = \mathrm{K}_0(A)_+$ (η is the quotient map $E_f \to A$). Moreover, $\mathrm{K}_0(E_f)$ represents an element in $\mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A))$.*

Proof. We refer the reader to [16] for the notion of topological stable rank and for the theorems used in this proof.

There exists an exact sequence

$$0 \to S^{2k}A \to E_f \xrightarrow{\eta} A \to 0.$$

We denote as in [16] by $\mathrm{tsr}(B)$ the topological stable rank of a C^*-algebra B. It coincides with the Bass stable rank ([11]).

We know that $\mathrm{tsr}(S^{2k}A_n) \leqslant k+1$ and hence $\mathrm{tsr}(\varinjlim S^{2k}A_n) \leqslant k+1$. Also $\mathrm{tsr}(A) = 1$ and hence $\mathrm{tsr}(E_f) \leqslant k+1$. Analogously $\mathrm{tsr}(eE_fe) \leqslant k+1$ for any projection $e \in E_f$. Let e_1, e_2 be two projections in $K \otimes E_f$ such that $[e_1] = [e_2]$. Replacing A by some $M_n(A)$ we may suppose that $\eta(e_1), \eta(e_2) \in A$. Since close projections generate the same ideal and $\eta(e_1)$ and $\eta(e_2)$ are equivalent consider the ideal J generated by $\eta(e_1)$ and identify e_1 and e_2 with two functions φ_1 and φ_2 in $C(\mathbf{B}_{2k}, J) \cap E_f$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & S^{2k}J & \to & C(\mathbf{B}_{2k}, J) & \to & J & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^{2k}A & \to & E_f & \longrightarrow & A & \to & 0.
\end{array}$$

Since $\mathrm{K}_0(J) \to \mathrm{K}_0(A)$ and $\mathrm{K}_0(S^{2k}J) \to \mathrm{K}_0(S^{2k}A)$ are injective it follows that $\mathrm{K}_0(C(\mathbf{B}_{2k}, J) \cap E_f) \to \mathrm{K}_0(E_f)$ is injective and hence $[e_1]$ and $[e_2]$ represent the same class in $\mathrm{K}_0(C(\mathbf{B}_{2k}, J) \cap E_f)$. This shows that we may suppose that e_1 and e_2 are full projections in E_f.

Let $n = k + 1$. Choose a full projection $e \in A$ such that $n[e] \leqslant \mathrm{K}_0(\eta)([e_1])$. We may suppose that $e \in A_p$ for some large p. Also it follows from Proposition 2.4b) and c) and from Lemma 1.2 that there exists a loop of unitaries g such that $f|A_p = \mathrm{ad}_g|A_p$. Let h be a lifting of $g \oplus g^*$ to a unitary in $C(\mathbf{B}_{2k}, M_2(A))$. Then $\mathrm{ad}_h \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = e_0$ is a projection in $M_2(E_f)$ such that $\eta(e_0) = e$. Similarly we may find a projection $e' \in M_r(E_f)$, for some large r, such that $n[e] + \mathrm{K}_0(\eta)([e']) = \mathrm{K}_0(\eta)([e_1])$. We show now that we may replace $e_0 \oplus e'$ by some other projection e'' such that

$$(n - 1)[e_0] + [e''] = [e_1]. \tag{1}$$

Indeed let $x = [e_1] - n[e_0] - [e'] \in \mathrm{K}_0(S^{2k}A)$. Since e is full $p = e \oplus \eta(e')$ is also full and this means that the map $\pi_{2k-1}(U(pM_{r+1}(A)p)) \to \mathrm{K}_0(S^{2k}A)$ is surjective (use Proposition 2.4b)). Let g be a $(2k - 1)$ loop in $U(pM_{r+1}(A)p)$ representing x in $\mathrm{K}_0(S^{2k}A)$. Using g we may twist $e_0 \oplus e'$ such that the new projection e'' satisfies (1). Indeed, let h be a lifting of $g \oplus g^*$ to a unitary in $C(\mathbf{B}_{2k}, M_{2r}(A))$, $q = \mathrm{ad}_h \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $(n - 1)[e] + [q] = [e_1]$.

Since e is full in A, e_0 is also full in E_f and hence $(eE_f)^{n-1} \oplus qE_f \oplus (eE_f)^m$ and $(e_1E_f) \oplus (eE_f)^m$ are isomorphic as right E_f-modules for some $m \in \mathbf{N}$. We may use now the Warfield cancellation theorem ([16], [23]) to conclude that $(eE_f)^{n-1} \oplus qE_f$ and e_1E_f are actually isomorphic. The same argument shows that e_2E_f and $(eE)_f^{n-1} \oplus qE_f$ are isomorphic and hence we obtain that $[e_1]$ and $[e_2]$ are equivalent projections.

We have already proved that any projection in A has a lifting in E_f. This shows that $\mathrm{K}_0(\eta)(\mathrm{K}_0(E_f)_+) = \mathrm{K}_0(A)_+$. To prove that

$$0 \to \mathrm{K}_0(A) \to \mathrm{K}_0(E_f) \to \mathrm{K}_0(A) \to 0$$

is an element in $\mathrm{Ext_c}(\mathrm{K}_0(A), \mathrm{K}_0(A))$ we have to prove 2.9b).

Let $e_1, e_2 \in M_r(E_f)$ such that $\mathrm{K}_0(\eta)([e_1]) = \mathrm{K}_0(\eta)([e_2]) = [e]$. Then, as we did before, we note that e_1 and e_2 may be identified with functions $\varphi_1, \varphi_2 \colon \mathbf{B}_{2k} \to J$, J being the ideal generated in A by $\eta(e_1)$. Hence $[e_1]-[e_2]$ is an element of $\mathrm{K}_0(S^{2k}J)$. Conversely if $x \in \mathrm{K}_0(E_f)$ is such that $[e_1] - x \in \mathrm{K}_0(S^{2k}J) \subset \mathrm{K}_0(S^{2k}A)$ then we may find a $2k - 1$ loop g in $U(\eta(e_1)M_r(A)\eta(e_1))$ such that $\delta([g]) = x - [e_1]$. Using this g we may twist e_1 to obtain a new projection $e_2 \in M_1(E_f)$ such that $[e_2] - [e_1] = x - e_1$. This concludes the proof.

Note that E_f is a locally trivial field of AF-C^*-algebras.

2.12. THEOREM. *Let A be an* AF-C^**-algebra such that* $K_0(A)$ *has large denominators. Then there exists a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2k}(\mathrm{Aut}(A)) \arrow[r] \arrow[d, "M"] & \mathrm{End}_c(K_0(\tilde{A})) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbf{Z}, K_0(\tilde{A})) \arrow[r] \arrow[d, equal] & {} \\
0 \arrow[r] & \mathrm{Hom}(K_0(\tilde{A})/\mathbf{Z}, K_0(\tilde{A})) \arrow[r] & \mathrm{Hom}(K_0(\tilde{A}), K_0(\tilde{A})) \arrow[r] & \mathrm{Hom}(\mathbf{Z}, K_0(\tilde{A})) \arrow[r, "-1"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r, "E"] \arrow[d, "E_1"] & \mathrm{End}_c(K_0(\tilde{A}), K_0(\tilde{A})) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \mathrm{Ext}(K_0(\tilde{A})/\mathbf{Z}, K_0(\tilde{A})) \arrow[r] & \mathrm{Ext}(K_0(\tilde{A}), K_0(\tilde{A})) \arrow[r] & 0.
\end{tikzcd}$$

Proof. Let us suppose first that A is unital. $\mathrm{End}_c(K_0(\tilde{A})) \to \mathrm{Hom}(K_0(\tilde{A}), K_0(A))$ and $\mathrm{Ext}_c(K_0(\tilde{A}), K_0(\tilde{A})) \to \mathrm{Ext}_c(K_0(\tilde{A}), K_0(\tilde{A}))$ are the natural maps. M is defined analogously with μ of Lemma 2.7b). E associates to $x \in \pi_{2k-1}(\mathrm{Aut}(A))$, $x = [f]$, the class in $\mathrm{Ext}_c(K_0(A), K_0(A))$ of the extension

$$0 \to K_0(S^{2k}A) \to K_0(E_f) \to K_0(A) \to 0$$

constructed as in Lemma 2.11. (We identify using Bott periodicity $K_0(A)$ with $K_0(S^{2k}A)$.)

Let 1 denote the units of E_f and A as well. The extension of groups

$$0 \to K_0(S^{2k}A) \to K_0(E_f)/\mathbf{Z}[1] \to K_0(A)/\mathbf{Z}[1] \to 0$$

will be denoted by $E_1(x)$.

We prove now that E and E_1 are group morphisms.

Denote by $f * g$ the concatenation of loops. Also let $d\colon A \to A \oplus A$, $d(a) = a \oplus a$ and $\sigma\colon S^{2k}A \oplus S^{2k}A \to S^{2k}A$ the map induced by $C_0(R^{2k}) \oplus C_0(R^{2k}) \simeq C_0(R^{2k}_+) \oplus C_0(R^{2k}_-) \to C_0(R^{2k})$. There exists a commutative diagram of extensions:

$$\begin{tikzcd}
0 \arrow[r] & S^{2k}A \arrow[r] & E_{f*g} \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & S^{2k}A \oplus S^{2k}A \arrow[r] \arrow[u, "\sigma"] \arrow[d, equal] & E \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, "d"] & 0 \\
0 \arrow[r] & S^{2k}A \oplus S^{2k}A \arrow[r] & E_f \oplus E_g \arrow[r] & A \oplus A \arrow[r] & 0.
\end{tikzcd}$$

Since $K_0(\sigma)(a, b) = a + b$ and $K_0(d)(a) = a \oplus a$ we obtain that the extensions $E([f*g])$ and $E_1([f*g])$ are the Baer sums of the extensions $E([f])$ and $E([g])$ and, respectively, of the extensions $E_1([f])$ and $E_1([g])$.

The commutativity of the diagrams follows by the naturality of the definitions (compare with Lemmata 2.7 and 2.8).

We have to prove the exactness of the upper row. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{2k}(\mathrm{Aut}(A)) \arrow[d] & \varprojlim(\pi_{2k}(\mathrm{Hom}^0(A_n, A)), \pi_{2k}(\psi_{nm})) \arrow[l] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}_c(K_0(A)), K_0(A)) \arrow[r] \arrow[d] & \varprojlim(\mathrm{Hom}_c(K_0(A_n), K_0(A)), \alpha^*_{nm}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbb{Z}, K_0(A)) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbb{Z}, K_0(A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim{}^1(\pi_{2k}(\mathrm{Hom}^0(A_n, A)), \pi_{2k}(\psi_{nm})) \arrow[r] \arrow[d] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r] \arrow[d] & \varprojlim(\pi_{2k-1}(\mathrm{Hom}^0(A_n, A)), \pi_{2k-1}(\psi_{nm})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim{}^1(\mathrm{Hom}_c(K_0(A_n), K_0(A)), \alpha^*_{nm}) \arrow[r] \arrow[d] & \mathrm{Ext}_c(K_0(A), K_0(A)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In this diagram the first row is exact due to Theorem 1.5 b) and to Proposition 2.4 d). The second row is exact due to Proposition 2.6 c). It follows also from Theorem 1.5 b) that the fourth row is exact. The fifth row is exact due to Proposition 2.10 c). This shows that in the previous diagram all rows are exact. We want to show that the middle column is exact.

Let us first observe that the composition

$$\mathrm{Hom}(\mathbb{Z}, K_0(A)) \to \pi_{2k-1}(\mathrm{Aut}(A)) \to \mathrm{Ext}_c(K_0(A), K_0(A))$$

is zero. Indeed if f is a $2k-1$ loop in $\mathrm{Aut}(A)$ then there exists a $2k-1$ loop g in $U(A)$ such that $f = \mathrm{ad}_g$ for g a $2k-1$ loop in $U(A)$. Let e be a projection in $M_n(A)$. Choose a lifting h of $g \oplus g \oplus + \ldots \oplus g \oplus g^* \oplus \ldots \oplus g^*$ to a continuous function $h\colon \mathbb{B}_{2k} \to U(M_{2n}(A))$ (g appears n-times and g^* also n-times), then $\tau([e]) = \mathrm{ad}_h(e)$ defines a positive lifting for $K_0(\pi)\colon K_0(E_f) \to K_0(A)$.

Let us denote by $H_i^{(j)}$ the i-th cohomology group of the j-th column, $j \in \{1, 2, 3\}$, $i \in \{1, \ldots, 5\}$. We want to show that $H_i^{(2)} = 0$ for any $i \in \{1, \ldots, 5\}$. It is obvious that $H_1^{(1)} = H_2^{(1)} = H_3^{(1)} = \{0\}$. Also $H_1^{(3)} = H_2^{(3)} = \{0\}$ by direct computation using Lemma 2.7.

The computation of other cohomology groups requires the use of the $\lim^1$ exact sequence (see [20]). There exists an exact sequence (we omit writing the morphisms defining the inverse systems):

$$0 \to \varprojlim \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to \varprojlim \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \to$$

$$\to \varprojlim \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \xrightarrow{\delta} \varprojlim{}^1 \pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to$$

$$\to \varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \to \varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to 0.$$

We obtain that $H_4^{(1)} \simeq \mathrm{ran}(\delta)$ and that $H_5^{(1)} \simeq \varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/ /\pi_{2k}(\mathrm{Hom}^0(A_n, A))$.

There exists also a $\lim^1$ exact sequence obtained from the exact sequences

$$0 \to \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A)) \to$$

$$\to \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to 0,$$

$$0 \to \varprojlim \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A)) \to$$

$$\to \varprojlim \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to \varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \to 0.$$

This shows that $H_3^{(3)}$ is isomorphic to the cokernel of the map

$$\varprojlim \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \to \varprojlim \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)).$$

From the previous $\lim^1$ exact sequence this cokernel is isomorphic to $\mathrm{ran}(\delta)$ and hence $H^4_{(1)} \simeq H_3^{(3)}$. Similarly $H_4^{(3)} \simeq \varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A)) \simeq H_5^{(1)}$.

We have to show that the previous isomorphisms are induced by the connecting homomorphisms in the long exact sequence of cohomology groups:

$$(1) \qquad \ldots \to H_j^{(1)} \to H_j^{(2)} \to H_j^{(3)} \xrightarrow{\delta_j} H_{j+1}^{(1)} \to \ldots .$$

Let us prove first that $H_4^{(3)} \simeq H_5^{(1)}$ is the connecting morphism in (1). Let $x = (x_n)_{n \in \mathrm{N}} \in \varprojlim \pi_{2k-1}(\mathrm{Hom}^0(A_n, A))$. Each x_n is represented by an $y_n \in \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A))$, namely by a $(2k-1)$-loop of unitaries f_n in $U(A)$, such that $\mathrm{ad}_{f_{n+1}}|A_n = \mathrm{ad}_{f_n}|A_n$. It folows that $y_{n+1} - y_n$ comes from an element $z_n \in \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$. The class of $(z_1, z_2, \ldots, z_n, \ldots)$ in $\varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A))$ coincides with the image of x in $\varprojlim{}^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A))$

under both compositions $\varprojlim \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to H_4^{(3)} \simeq \varprojlim^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/$ $/\pi_{2k}(\mathrm{Hom}^0(A_n, A))$ and

$$\varprojlim \pi_{2k-1}(\mathrm{Hom}^0(A_n, A)) \to H_4^{(3)} \xrightarrow{\delta_4} H_5^{(1)} \simeq \varprojlim^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n, A).$$

This shows that the connecting map $\delta_4\colon H_4^{(3)} \to H_5^{(1)}$ is an isomorphism.

Let f be a $2k-1$ loop in $U(A)$ such that its class in $\varprojlim \pi_{2k-1}(\mathrm{Hom}^0(A_n, A))$ is 0. Then ad_f is in $\varprojlim^1 \pi_{2k}(\mathrm{Hom}^0(A_n, A))$ and is represented by the following element: for each n there exists a loop x_n in $U(A_n')$ such that $\mathrm{ad}_f|A_n$ and $\mathrm{ad}_{x_n}|A_n$ are homotopic. Also $\mathrm{ad}_f|A_n$ and $\mathrm{ad}_{x_{n+1}}|A_n$ are homotopic. The resulting homotopy from $\mathrm{ad}_{x_{n+1}}|A_n$ and $\mathrm{ad}_{x_n}|A_n$ defines a $2k$-loop in $\mathrm{Hom}^0(A_n, A)$. Denote the class of this loop by y_n. Then the class of ad_f in $\varprojlim^1 \pi_{2k}(\mathrm{Hom}^0(A_n, A))$ has as representative $([y_n])_{n\in\mathbf{N}}$ ([24]). x_n defines an element of $\pi_{2k-1}(U(A_n')) \simeq \mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$ such that x_n and x_{n+1} regarded as elements of $\mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$ have the property that $x_{n+1}([1]) = [f] = x_n([1])$ and hence $x_{n+1} - x_n$ is actually in the image of $\pi_{2k}(\mathrm{Hom}^0(A_n, A))$ in $\mathrm{Hom}_c(\mathrm{K}_0(A_n), \mathrm{K}_0(A))$. It follows from the definition that $x_{n+1} - x_n$ corresponds to y_n. It follows that we have a commutative diagram

$$\begin{tikzcd} H_3^{(3)} \arrow[rr, "\delta_3"] \arrow[dr] & & H_4^{(1)} \arrow[dl] \\ & \varprojlim^1 \pi_{2k}(\mathrm{Hom}^0(A_n, A)). & \end{tikzcd}$$

This shows that the connecting morphism δ_3 is an isomorphism. The nonunital case is similar requiring also the use of Lemma 2.8 and of the isomorphism $\varprojlim^1 \mathrm{Hom}_c((\mathrm{K}_0(A_n), \mathrm{K}_0(A)) \simeq \varprojlim^1 \mathrm{Hom}_c(\mathrm{K}_0(A_n^+), \mathrm{K}_0(A^+))$.

2.13. COROLLARY. *Let A be a simple AF-C^*-algebra, A infinite dimensional, $A \neq K$. Then, if A is unital*

$$\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}(\mathrm{K}_0(A)/\mathbf{Z}[1], \mathrm{K}_0(A))$$

$$\pi_{2k}(\mathrm{Aut}(A)) \simeq \mathrm{Hom}(\mathrm{K}_0(A)/\mathbf{Z}[1], \mathrm{K}_0(A))$$

and, if A is not unital

$$\pi_{2k-1}(\mathrm{Aut}(A)) = \mathrm{Ext}(\mathrm{K}_0(A), \mathrm{K}_0(A))$$

$$\pi_{2k}(\mathrm{Aut}(A)) \simeq \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(A)), \quad k \geqslant 1.$$

Proof. Use Theorem 2.12 and Proposition 2.3.

2.14. REMARK. a) Suppose that A is not unital. Let A be an AF-C^*-algebra with $\mathrm{K}_0(A)$ with large denominators. Let e_n denote the unit of A_n. Let us suppose also that $1-e_n$ is a full projection in A^+ then it is easily seen that $\pi_k(U(A_n')) \to \pi_k(U(A))$ is surjective for any $k \geqslant 1$ (see Lemma 2.7 a)). This shows that $\mathrm{End}_c(\mathrm{K}_0(A)) \to$ $\to \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A))$ is surjective and hence $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A))$.

b) If A is unital then $\pi_{2k-1}(\mathrm{Aut}(A))$ can be identified with isomorphism classes of compatible extensions with order unit

$$0 \to \mathrm{K}_0(A) \to (E, u) \xrightarrow{p} (\mathrm{K}_0(A), [1]) \to 0 \tag{1}$$

such that

$$0 \to \mathrm{K}_0(A) \to E \xrightarrow{p} \mathrm{K}_0(A) \to 0$$

is an exact sequence as in Definition 2.9 and u is a positive element in E such that $p(u) = [1]$. Two such extensions (E_1, u_1), (E_2, u_2) are isomorphic if there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}_0(A) \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\varphi"] & \mathrm{K}_0(A) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{K}_0(A) \arrow[r] & E_2 \arrow[r] & \mathrm{K}_0(A) \arrow[r] & 0 \end{tikzcd}$$

such that φ is a positive morphism (and hence necessarily an isomorphism of ordered groups) and $\varphi(u_1) = u_2$. The extension in (1) is trivial if there exists a positive lifting τ for π such that $\tau([1]) = u$.

We associate to a loop f representing $x \in \pi_{2k-1}(\mathrm{Aut}(A))$ the class of the extension $\mathrm{K}_0(E_f)$ with $[1] \in \mathrm{K}_0(E_f)$ as order unit. It turns out that there exists a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(A)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A)) \arrow[r] \arrow[d, equal] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r] \arrow[d] & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A)) \arrow[r] \arrow[d, equal] & 0 \\ \mathrm{Hom}_c(\mathrm{K}_0(A), \mathrm{K}_0(A)) \arrow[r] & \mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A)) \arrow[r] & \mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_0(A)) \arrow[r] & \mathrm{Ext}_c(\mathrm{K}_0(A), \mathrm{K}_0(A)) \arrow[r] & 0. \end{tikzcd}$$

($\mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_0(A))$ denotes the group of isomorphism classes of extensions as in (1), called *compatible unital extensions.*)

The morphism $\mathrm{Hom}(\mathbf{Z}, \mathrm{K}_0(A)) \to \mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_0(A))$ sends the morphism $n \to nu$ the class of the trivial extension

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}_0(A) \arrow[r] & E \arrow[r, shift left, "p"] & \mathrm{K}_0(A) \arrow[l, shift left, "\tau"] \arrow[r] & 0 \end{tikzcd}$$

with ordered unit $\tau([1]) + u$. It follows easily that the second row is exact and hence $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}_c^u(\mathrm{K}_0(A), \mathrm{K}_0(A))$ from the five lemma.

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VICTOR NISTOR
Department of Mathematics, INCREST,
Bdul Păcii 220, 79622 Bucharest,
Romania.

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