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ORDER PRESERVING OPERATOR INEQUALITIES

EDWARD BACH and TAKAYUKI FURUTA

1. INTRODUCTION

In this paper, operators mean bounded linear operators on a Hilbert space H. An operator A is said to be positive if $(Ax, x) \geqslant 0$ for every vector x in H. It is well known that $A \geqslant B \geqslant 0$ does not always ensure $A^2 \geqslant B^2$. Let J be a subset of the real axis and $f(x)$ a real-valued function defined on J. What functions preserve the ordering of positive operators? In other words, what must f satisfy so that

$$(*) \qquad A \geqslant B \geqslant 0 \quad \text{implies} \quad f(A) \geqslant f(B)?$$

A function f is said to be a *monotone operator function* if f satisfies the property stated above $(*)$ when J contains the spectra of A and B. This problem was first studied by K. Löwner [10] who has given a complete description of monotone operator functions in the case of a finite dimensional space H.

Recently, Furuta [8] has obtained the following two order preserving inequalities.

THEOREM A. [8]. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$*

$$\text{(i)} \qquad (B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$$

and

$$\text{(ii)} \qquad A^{(p+2r)/q} \geqslant (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geqslant 0$, $q \geqslant 1$ and $(1 + 2r)q \geqslant p + 2r$.

In this paper, we shall attempt to extend Theorem A.

2. STATEMENT OF RESULTS

THEOREM 1. *If* $A \geqslant B \geqslant 0$, *then for each* $a, b, c \geqslant 0$ *and each* $r \geqslant 0$

(i) $$\{B^r(aA^\alpha + bB^\beta + c)^p B^r\}^{1/q} \geqslant \{B^r(aB^\alpha + bB^\beta + c)^p B^r\}^{1/q}$$

(ii) $$\{B^r(aA^\alpha + bA^\beta + c)^p B^r\}^{1/q} \geqslant \{B^r(aB^\alpha + bB^\beta + c)^p B^r\}^{1/q}$$

(iii) $$\{A^r(aA^\alpha + bA^\beta + c)^p A^r\}^{1/q} \geqslant \{A^r(aA^\alpha + bB^\beta + c)^p A^r\}^{1/q}$$

and

(iv) $$\{A^r(aA^\alpha + bA^\beta + c)^p A^r\}^{1/q} \geqslant \{A^r(aB^\alpha + bB^\beta + c)^p A^r\}^{1/q}$$

hold for each p, q *and each* α, β *such that* $p \geqslant 0$, $q \geqslant 1$, $\alpha, \beta \in [0, 1]$ *and* $2rq \geqslant p + 2r \geqslant p + \mathrm{Max}\{p\alpha, p\beta\}$.

COROLLARY 1. *If* $A \geqslant B \geqslant 0$, *then for each* $a, b, c \geqslant 0$ *and each* $r \geqslant 0$

(i) $$\{B^r(aA^\alpha + bB^\beta + c)^p B^r\}^{1/2} \geqslant \{B^r(aB^\alpha + bB^\beta + c)^p B^r\}^{1/2}$$

(ii) $$\{B^r(aA^\alpha + bA^\beta + c)^p B^r\})^{12} \geqslant \{B^r(aB^\alpha + bB^\beta + c)^p B^r\}^{1/2}$$

(iii) $$\{A^r(aA^\alpha + bA^\beta + c)^p A^r\}^{1/2} \geqslant \{A^r(aA^\alpha + bB^\beta + c)^p A^r\}^{1/2}$$

and

(iv) $$\{A^r(aA^\alpha + bA^\beta + c)^p A^r\}^{1/2} \geqslant \{A^r(aB^\alpha + bB^\beta + c)^p A^r\}^{1/2}$$

hold for each p *such that* $2r \geqslant p \geqslant 0$ *and for each* α, β *such that* $\alpha, \beta \in [0, 1]$.

COROLLARY 2. *If* $A \geqslant B \geqslant 0$, *then for each* $a, b, c \geqslant 0$ *and each* $r \geqslant 0$

(i) $$\{B^r(aA + bB + c)^p B^r\}^{1/q} \geqslant \{B^r(aB + bB + c)^p B^r\}^{1/q}$$

and

(ii) $$\{A^r(aA + bA + c)^p A^r\}^{1/q} \geqslant \{A^r(aA + bB + c)^p A^r\}^{1/q}$$

hold for each p *and* q *such that* $p \geqslant 0$, $q \geqslant 1$ *and* $2rq \geqslant p + 2r \geqslant 2p$.

COROLLARY 3. *If* $A \geqslant B \geqslant 0$, *then* $(BA^2B)^{1/2} \geqslant B^2$ *and* $A^2 \geqslant (AB^2A)^{1/2}$.

Corollary 3 is just an affirmative answer to the conjecture posed in Chan and Kwong [3].

THEOREM 2. *If* $A \geqslant B \geqslant 0$, *then for each* $a_k \geqslant 0$ $(k = 1, 2, \ldots, n)$ *and each* $r \geqslant 0$

(i) $$\{B^r h(A, B)^p B^r\}^{1/q} \geqslant \{B^r h(B, B)^p B^r\}^{1/q}$$

(ii) $$\{B^r h(A, A)^p B^r\}^{1/q} \geqslant \{B^r h(B, B)^p B^r\}^{1/q}$$

(iii) $$\{A^r h(A, A)^p A^r\}^{1/q} \geqslant \{A^r h(A, B)^p A^r\}^{1/q}$$

and

(iv) $$\{A^r h(A, A)^p A^r\}^{1/q} \geqslant \{A^r h(B, B)^p A^r\}^{1/q}$$

hold for each p, q *and each* α_k *such that* $p \geqslant 0$, $q \geqslant 1$, $\alpha_k \in [0, 1]$ $(k = 1, 2, \ldots, n)$ *and* $2rq \geqslant p + 2r \geqslant p + \operatorname{Max}_k\{p\alpha_k\}$, *where* $h(A, B)$ *is defined by*

$$h(A, B) = \sum_{i=1}^{k} a_i A^{\alpha_i} + \sum_{i=k+1}^{n-1} a_i B^{\alpha_i} + a_n.$$

3. PROOFS OF THE RESULTS

We need the following lemmas, Theorem B and Theorem C in order to prove Theorem 1.

LEMMA 1. *Let* $p \geqslant 1$, $\alpha, \beta \in [0, 1]$ *and also let* r *be a positive number such that* $2r \geqslant \operatorname{Max}\{p\alpha, p\beta\}$. *Let* $f(z)$ *and* $g(z)$ *be defined for all complex* z *as follows:*

$$f(z) = (az^{(1/p - \alpha/2r)} + bz^{(1/p - \beta/2r)} + cz^{1/p})^p$$

$$g(z) = (az^{(1/p + \alpha/2r)} + bz^{(1/p + \beta/2r)} + cz^{1/p})^{2pr/(p+2r)}$$

where a, b *and* c *are all nonnegative numbers. Then, if* z *lies in the upper half plane, then both* $f(z)$ *and* $g(z)$ *take values in the same half plane, that is,* $f(z)$ *and* $g(z)$ *are Pick functions such that*

$$g(z) = (zf(z^{-1})z)^{2r/(p+2r)}.$$

LEMMA 2. *Let* H *and* K *be invertible positive operators. Then the following statements are equivalent.*

(1) $H \geqslant (H^{1/2} K H^{1/2})^{1/2}$.

(2) *There exists a positive invertible operator* $V \geqslant 1$ *such that*

$$VKV = H.$$

THEOREM B. [10], [12]. *If* S *and* T *are bounded positive linear operators such that* $S \geqslant T \geqslant 0$, *then* $S^\alpha \geqslant T^\alpha$ *for each* $\alpha \in [0, 1]$.

THEOREM C. [1], [2], [5], [9], [10]. *Let f be a real valued continuous function on a finite or infinite interval (α, β). In order that f is a monotone operator function it is necessary and sufficient that it admits an analytic continuation $\hat{f}$ to the upper and lower half planes such that* $\operatorname{Im}\hat{f}(z) > 0$ *for* $\operatorname{Im}(z) > 0$.

Further extensions of Theorem C are obtained in [4] and [6].

Proof of Lemma 1. First of all, $f(z)$ and $g(z)$ are both real valued continuous functions whenever z varies on the real line. The hypothesis easily implies $\operatorname{Max}(1/p + \alpha/2r, 1/p + \beta/2r) \leqslant 1/p + 1/2r$, then $g(z)$ turns out to be a Pick function, that is, $g(z)$ is analytic and if z lies in the upper half plane, then $g(z)$ takes values in the same half plane. Also $f(z)$ turns out to be a Pick function by the hypothesis and the rest of the lemma is an immediate consequence of easy calculation.

Proof of Lemma 2. The proof is a slight modification of [11] and [13].

(1) $\Rightarrow$ (2). Using Douglas' result [7], condition (1) ensures the existence of an invertible operator S such that

$$(H^{1/2}KH^{1/2})^{1/4} = H^{1/2}S = S^{*}H^{1/2}.$$

Put $T = SS^*$. Obviously $\|T\| \leqslant 1$. Then

$$H^{1/2}KH^{1/2} = (H^{1/2}SS^*H^{1/2})(H^{1/2}SS^*H^{1/2})$$

therefore $K = THT$ because H is invertible, that is, $H = VKV$ since $V = T^{-1} \geqslant 1$.

(2) $\Rightarrow$ (1). Condition (2) yields

$$(H^{1/2}KH^{1/2})^{1/2} = H^{1/2}V^{-1}H^{1/2} \leqslant H.$$

Proof of Theorem 1. In case $0 \leqslant p \leqslant 1$, the result is obvious, so we have only to consider $p > 1$. We may assume $r > 0$ because the result is easy in case $r = 0$. Also we may assume that A and B are both invertible because $A + \varepsilon$ and $B + \varepsilon$ are both invertible for any $\varepsilon > 0$. If $a = b = c = 0$, the result is trivial, so we may assume that $aA^\alpha + bB^\beta + c$ and $aB^\alpha + bB^\beta + c$ are both invertible such that

$$aA^\alpha + bB^\beta + c \geqslant aB^\alpha + bB^\beta + c \geqslant 0$$

by the hypothesis $A \geqslant B \geqslant 0$ and Theorem B. Putting $q = 2$ and $r = p/2$ in (ii) of Theorem A, there exists $V \geqslant I$ such that

$$(aA^\alpha + bB^\beta + c)^p = V(aB^\alpha + bB^\beta + c)^p V$$

by Lemma 2. Then $X \equiv B^rVB^r \geqslant B^{2r}$ and $X^{-1} \leqslant B^{-2r}$. We have only to consider the case $q = (p + 2r)/2r$ in Theorem 1 since (i) of Theorem 1 for larger values q

than $(p + 2r)/2r$ follows by Theorem B. Recall that $f(Y)$ and $g(Y)$ in Lemma 1 are both monotone operator functions whenever $Y \geqslant 0$ by Lemma 1 and Theorem C. Then we have

$$\{B^r(aA^\alpha + bB^\beta + c)^p B^r\}^{1/q} = \{XB^{-r}(aB^\alpha + bB^\beta + c)^p B^{-r}X\}^{1/q} =$$

$$= [X\{B^{-2r/p}(aB^\alpha + bB^\beta + c)\}^p X]^{1/q} = \{Xf(B^{-2r})X\}^{1/q} \geqslant$$

$$\geqslant \{Xf(X^{-1})X\}^{1/q} = g(X) \geqslant g(B^{2r}) =$$

$$= \{B^r(aB^\alpha + bB^\beta + c)^p B^r\}^{1/q}.$$

Whence the proof of (i) in Theorem 1 is complete and (ii), (iii) and (iv) in Theorem 1 are also obtained by the same method.

Proof of Corollary 1. Put $q = 2$ in Theorem 1.

Proof of Corollary 2. Obvious by Theorem 1.

Proof of Corollary 3. Obvious by Corollary 1.

Proof of Theorem 2. Using the same methods as in Theorem 1 we obtain the desired results, so we have omitted the proofs.

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EDWARD BACH
School of Mathematics,
University of Dublin,
39 Trinity College,
Dublin 2,
Ireland.

TAKAYUKI FURUTA
Department of Mathematics,
Faculty of Science,
Hirosaki University,
Bunkyo-cho 3,
Hirosaki 036 Aomori,
Japan.

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