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THE K-THEORY OF TOEPLITZ EXTENSIONS

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In this note we study the Toeplitz extension along the orbits of a minimal flow on a torus, $\mathbf{T}^n$. There has been considerable recent work in this area, [2], [4], [5], [6], [7], [9]. We will completely determine the K-theory exact sequence for this extension as well as the K-groups of each of the algebras involved. The method is based on the relation between the Toeplitz extension and a "smoothed" version of it. This has been used in [6], [7] and in some unpublished work of Arveson. For the smoothed Toeplitz extension the exact sequence is analyzed by using the Thom isomorphism theorem of Connes, [3], [8]. It is noted that from this perspective the Pimsner-Voiculescu Toeplitz extension (or rather the version studied by Rieffel in [10]) is dual in the sense of KK-theory to the smoothed Toeplitz extension. The authors want to thank Jingbo Xia for many illuminating conversations, and Ron Douglas for many insights. We would like to take note of the fact that Jingbo Xia has also obtained a proof of Proposition 3.2 which will appear in [11].

1. TOEPLITZ EXTENSIONS

Let $\mathbf{R}$ act on the n-torus, $\mathbf{T}^n = \mathbf{R}^n/2\pi Z^n$, by $\beta_t(\theta_1, \ldots, \theta_n) = (\theta_1 + t\alpha_1, \ldots \ldots, \theta_n + t\alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are linearly independent over the rationals. The resulting flow is uniquely ergodic. The numbers $\alpha_1, \ldots, \alpha_n$ generate a dense subgroup $\Gamma \subseteq \mathbf{R}$ of rank n. Conversely, given a dense subgroup $\Gamma \subseteq \mathbf{R}$ of rank n, one obtains a flow on $\mathbf{T}^n$ precisely as above by taking Pontryagin duals, $\mathbf{R} \cong \hat{\mathbf{R}} \subseteq \subseteq \hat{\Gamma} \cong \mathbf{T}^n$ and letting $\mathbf{R}$ act by translation on its cosets. We construct an algebra of Toeplitz operators along the flow as a subalgebra of $\mathscr{L}(L^2(\mathbf{T}^n))$. The construction could also have been done on $L^2(\mathbf{R})$, but in the present context the algebras would be isomorphic.

Three C^*-algebra extensions will be constructed. One will be the genuine Toeplitz extension, which is our principal interest. The other two are a smooth invariant of the Toeplitz extension, and an extension containing both of these which we construct first.

Let $\delta(f)(x) = \lim\limits_{t\to 0}\dfrac{\beta_t(f)(x) - f(x)}{t}$ be the generator of the action. It is given by the unbounded skew-adjoint operator on $L^2(\mathbf{T}^n)$, $iD = \sum \alpha_j \dfrac{\partial}{\partial\theta_j}\cdot$ Let $P \in \mathscr{L}(L^2(\mathbf{T}^n))$ denote the spectral projection for D corresponding to the interval $[0,\infty)$. We define $\hat{\mathscr{T}}$ to be the C^*-algebra generated by P and $C(\mathbf{T}^n)$, where $C(\mathbf{T}^n)$ acts as multiplication operators on $L^2(\mathbf{T}^n)$. Let $\hat{\mathscr{C}}$ denote the commutator ideal of $\hat{\mathscr{T}}$.

PROPOSITION 1.1. *The quotient algebra $\hat{\mathscr{T}}/\hat{\mathscr{C}}$ is isomorphic to $C(\mathbf{T}^n_+) \oplus C(\mathbf{T}^n_-)$, where $T^n_\pm$ denote two copies of $\mathbf{T}^n$. Thus, there is a short exact sequence of C^*-algebras*

$$0 \to \hat{\mathscr{C}} \to \hat{\mathscr{T}} \xrightarrow{\hat{\sigma}} C(\mathbf{T}^n_+) \oplus C(\mathbf{T}^n_-) \to 0,$$

where the map $\hat{\sigma}$ satisfies $\hat{\sigma}(P\varphi P) = \varphi$ on the first summand and $\hat{\sigma}((I-P)\psi(I-P)) = \psi$ on the second.

Proof. Define $\tau\colon C(\mathbf{T}^n_+)\oplus C(\mathbf{T}^n_-) \to \hat{\mathscr{T}}/\hat{\mathscr{C}}$ by $\tau(\varphi,\psi) = P\varphi P + (I-P)\psi(I-P) + \hat{\mathscr{C}}$. Then τ is a surjective $*$-homomorphism. Since $C(\mathbf{T}^n_+) \oplus C(\mathbf{T}^n_-) \cong C(\mathbf{T}^n_+ \cup T^n_-)$ we see that the kernel of τ consists of all functions vanishing on some closed set $A \subseteq \mathbf{T}^n_+ \cup \mathbf{T}^n_-$. Moreover, since the kernel is invariant under the action of $\mathbf{R}$ on $\mathbf{T}^n_+ \cup \mathbf{T}^n_-$ induced by the original action on each of $\mathbf{T}^n_\pm$, A is an invariant set. By the minimality of the action, $A \cap \mathbf{T}^n_\pm$ is either $\mathbf{T}^n_\pm$ or is empty. Since $\tau(1,1) = I + \mathscr{C} \neq 0$ we see that $A \cap \mathbf{T}^n_\pm = \mathbf{T}^n_\pm$ and, hence, τ is injective. Setting $\hat{\sigma} = \tau^{-1}$ completes the construction. ▨

Next we construct the Toeplitz extension. Let $\mathscr{T}$ be the C^*-subalgebra of $\hat{\mathscr{T}}$ generated by $\{T_\varphi = P\varphi P : \varphi \in C(\mathbf{T}^n)\}$, and let $\mathscr{C}$ be the commutator ideal of $\mathscr{T}$. Then $\mathscr{T}$ is the algebra of Toeplitz operators. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C} & \longrightarrow & \mathscr{T} & \longrightarrow & C(\mathbf{T}^n_+) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle \bar{j}} & & \\
0 & \longrightarrow & \hat{\mathscr{C}} & \longrightarrow & \hat{\mathscr{T}} & \xrightarrow{\hat{\sigma}} & C(\mathbf{T}^n_+)\oplus C(\mathbf{T}^n_-) & \longrightarrow & 0
\end{array}$$

where the vertical maps are induced by inclusions. Note that $\bar{j}$ is the identity on $C(\mathbf{T}^n_+)$.

The algebras $\hat{\mathscr{C}}$ and $\mathscr{C}$ are related to the crossed-product $C(\mathbf{T}^n) \rtimes \mathbf{R}$, as represented on $L^2(\mathbf{T}^n)$. We must make this connection precise, (cf. [6]). First, there is a representation of $C(\mathbf{T}^n) \rtimes \mathbf{R}$ on $L^2(\mathbf{T}^n)$ coming from the covariant representation of $C(\mathbf{T}^n)$ by multiplication operators and $\mathbf{R}$ as translation operators. Let Γ denote the subgroup of $\mathbf{R}$ corresponding to the action. The Fourier transform provides an isomorphism $C_0(\hat{\mathbf{R}}) \times\!\!\mid \Gamma \to C(\mathbf{T}^n) \times\!\!\mid \mathbf{R}$. The corresponding maps of $C_0(\mathbf{R})$ and Γ into $\mathscr{L}(L^2(\mathbf{T}^n))$ are given as follows. Let $\alpha_1, \ldots, \alpha_n$ be a basis for Γ and let $r \in \Gamma$ be given by $r = \sum m_i \alpha_i$. Then define $f_r \in C(\mathbf{T}^n)$ by $f_r(z_1, \ldots, z_n) = z_1^{m_1} \ldots z_n^{m_n}$. The functions f_r are a complete orthonormal set of eigenfunctions for the flow. The required maps are given by sending $r \in \Gamma$ to the multiplication operator M_{f_r}, and $g \in C_0(\hat{\mathbf{R}})$ to $g(D)$. Let $\mathscr{B}_0$ denote the C^*-subalgebra of $C(\mathbf{R})$ generated by characteristic functions $\chi_{[r,s)}$, where $r, s \in \Gamma$, and $\mathscr{B}_0^+$ the analogous algebra where $r, s \geqslant 0$. Note that $C_0(\mathbf{R}) \subseteq \mathscr{B}_0$. The next result is simply a translation from [6] to the present context.

Proposition 1.2. *The following hold.*

i) *$\hat{\mathscr{C}}$ is the C^*-algebra generated by finite sums $\sum \varphi_r(D) M_{f_r} \psi_r(D)$, $\varphi_r, \psi_r \in \mathscr{B}_0$.*

ii) *$\mathscr{C}$ is the C^*-algebra generated by finite sums $\sum \varphi_r(D) M_{f_r} \psi_r(D)$, $\varphi_r, \psi_r \in \mathscr{B}_0^+$.*

iii) *$C(\mathbf{T}^n) \times\!\!\mid \mathbf{R}$ is the C^*-algebra generated by finite sums*

$$\sum \varphi_r(D) M_{f_r} \psi_r(D), \quad \varphi_r, \psi_r \in C_0(\mathbf{R}).$$

Proof. It follows from [9] that the algebra $\hat{\mathscr{C}}$ is the crossed-product $\mathscr{B}_0 \times\!\!\mid \Gamma$. Since $M_{f_r} \chi_{[s,t)} = \chi_{[s-r,t-r)} M_{f_r}$ the generators described in (i) suffice. Statement (ii) is a direct computation. Finally, we note that $C_0(\mathbf{R}) \subseteq \mathscr{B}_0$ and the representation described earlier of $C_0(\mathbf{R}) \times\!\!\mid \Gamma$ is generated as in (iii). ▨

Finally, we construct the smoothed Toeplitz extension. Let h be a smooth function on $\mathbf{R}$ satisfying

$$h(t) = \begin{cases} 0 & t \leqslant 0 \\ 1 & t \geqslant \varepsilon \end{cases}, \quad h'(t) \geqslant 0, \quad |h(t)| \leqslant 1 \quad \text{for all } t. \tag{1.3}$$

Using the functional calculus on $L^2(\mathbf{T}^n)$ we set $P_\varepsilon = h(D)$. This operator can be viewed as a smooth approximation to P.

Proposition 1.4. *Let $P = \chi_{[0,\infty)}(D)$ and $P_\varepsilon = h(D)$. Then one has*

i) $P - P_\varepsilon \in \hat{\mathscr{C}}$,

ii) $P_\varepsilon^2 - P_\varepsilon \in C(\mathbf{T}^n) \times\!\!\mid \mathbf{R}$.

Proof. Let Γ be the subgroup of $\mathbf{R}$ generated by $\alpha_1, \ldots \alpha_n$. We will show that $\chi_{[0,r)}(D) \in \hat{\mathscr{C}}$ for $r \in \Gamma$, where $\chi_{[0,r)}$ is the characteristic function of the interval $[0, r)$. Approximating $\chi_{[0,\infty)} - h$ by such characteristic functions will then complete the proof of (i).

Recall that each element $r \in \Gamma$ is an eigenvalue for D, for, if $r = \sum m_i \alpha_i$, then $f_r(z_1, \ldots, z_n) = z_1^{m_1} \ldots z_n^{m_n}$ satisfies $Df_r = rf_r$. Moreover, the f_r form an orthonormal basis for $L^2(\mathbf{T}^n)$. We claim that $[(f_r P)^*, (f_r P)] = \chi_{[0,r)}(D)$, if $r > 0$. For this note that the commutator is equal to $P - f_r P \bar{f}_r$, and one checks directly that

$$(P - f_r P \bar{f}_r) f_s = \begin{cases} f_s & \text{if } 0 \leqslant s < r \\ 0 & \text{otherwise.} \end{cases}$$

Since this is the same as $\chi_{[0,r)}(D) f_s$, (i) follows.

For (ii), one observes that $h^2 - h \in C_0(\mathbf{R})$ and applies Proposition 1.2 (iii). ▨

We now define the smoothed Toeplitz extension. Let $\mathcal{T}_\varepsilon$ be the C^*-algebra generated by $\{P_\varepsilon \varphi P_\varepsilon : \varphi \in C(\mathbf{T}^n)\}$ and $C(\mathbf{T}^n) \times \mathbf{R}$.

PROPOSITION 1.5. *There is a short exact sequence*

$$0 \to C(\mathbf{T}^n) \rtimes \mathbf{R} \to \mathcal{T}_\varepsilon \xrightarrow{\sigma_\varepsilon} C(\mathbf{T}^n) \to 0$$

which fits into the commutative diagram

(1.6)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(\mathbf{T}^n) \rtimes \mathbf{R} & \longrightarrow & \mathcal{T}_\varepsilon & \xrightarrow{\sigma_\varepsilon} & C(\mathbf{T}^n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle \bar{j}'} & & \\
0 & \longrightarrow & \hat{\mathcal{C}} & \longrightarrow & \hat{\mathcal{T}} & \xrightarrow{\hat{\sigma}} & C(\mathbf{T}^n_+) \oplus C(\mathbf{T}^n) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \bar{j}} & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{T} & \xrightarrow{\sigma} & C(\mathbf{T}^n) & \longrightarrow & 0
\end{array}$$

where $\bar{j}'$ and $\bar{j}$ are the identity maps onto $C(\mathbf{T}^n)$.

Proof. The proof of exactness of the sequence on the top row is very similar to that of Proposition 1.1. Let $\mathcal{C}_\varepsilon$ denote the commutator ideal of $\mathcal{T}_\varepsilon$. Define $\tau : C(\mathbf{T}^n) \to \mathcal{T}_\varepsilon / \mathcal{C}_\varepsilon$ by $\tau(f) = P_\varepsilon f P_\varepsilon + \mathcal{C}_\varepsilon$. Since $C(\mathbf{T}^n) \times \mathbf{R}$ is simple one has that $C(\mathbf{T}^n) \times \mathbf{R} \subseteq \mathcal{C}_\varepsilon$. Using Proposition 1.4 it follows that τ is a surjective $*$-homomorphism. The kernel of τ is an invariant ideal, and since the action is minimal, it is zero. Finally, direct computation shows that $[T^\varepsilon_{f_r}, T^\varepsilon_{f_s}] \in C(\mathbf{T}^n) \rtimes \mathbf{R}$, where $T^\varepsilon_g = P_\varepsilon g P_\varepsilon$. Since these elements generate $\mathcal{C}_\varepsilon$, the result follows.

The vertical maps are inclusions so it is only necessary to note that $\mathcal{T}_\varepsilon \subseteq \hat{\mathcal{T}}$ since $P - P_\varepsilon \in \mathcal{C}$. ▧

2. THOM CLASSSES AND SMOOTHED TOEPLITZ EXTENSIONS

If $(A, \mathbf{R}, \alpha)$ is a C^*-dynamical system then A. Connes has shown that there is a Thom isomorphism, $\Phi\colon \mathrm{K}_i(A) \to \mathrm{K}_{i+1}(A \rtimes \mathbf{R})$, which depends functorially on the action, [3]. This has been reformulated in terms of KK-theory by Fack and Skandalis, [8], [1]. In their version, an isomorphism $\Phi^i_{(A,\alpha)}\colon \mathrm{KK}^i(B, A) \to \mathrm{KK}^{i+1}(B, A \rtimes \mathbf{R})$, for any B, is obtained by finding an element $U \in \mathrm{KK}^1(A, A \rtimes \mathbf{R})$ so that $\Phi^i_{(A,\alpha)}(x) = x \otimes_A U$. Of course, given $\Phi^i_{(A,\alpha)}$ one may take $U = \Phi^i_{(A,\alpha)}(1)$, $1 \in \mathrm{KK}(A, A)$. We will refer to any such U as a Thom class. There are a set of axioms which characterize a Thom class uniquely. Fack and Skandalis give an explicit way of associating to $(A, \alpha, \mathbf{R})$ a Thom class U. We will have need for this, so we review their construction (cf. [1]).

Given $(A, \alpha, \mathbf{R})$, let $\varphi\colon A \to \mathcal{M}(A \rtimes \mathbf{R})$ and $\mu\colon \mathcal{M}(C^*(\mathbf{R})) \to \mathcal{M}(A \rtimes \mathbf{R})$ be the standard maps, where $\mathcal{M}(B)$ denotes the multiplier algebra of B. The real-valued smooth function $f(t) = 2h(t) - 1$, where $h(t)$ is the function defined in (1.3) satisfies $f'(t) \geqslant 0$, $f(t) = \begin{cases} 1 & t \geqslant \varepsilon \\ -1 & t \leqslant -\varepsilon \end{cases}$, and $f \in C^{\mathrm{b}}(\mathbf{R}) \cong \mathcal{M}(C^*(\mathbf{R}))$. The triple ([illegible] $\mu(f)$) satisfies that $\mu(f)^2 - I \in A \rtimes \mathbf{R}$ and $[\varphi(a), \mu(f)] \in A \rtimes \mathbf{R}$ for all $a \in A$. One may construct from this data an element of $\mathrm{KK}(A, A \rtimes \mathbf{R} \otimes C_1) \cong \mathrm{KK}^1(A, A \rtimes \mathbf{R})$ which we will denote U. Indeed, the triple $\left((C(\mathbf{T}^n) \rtimes \mathbf{R}) \hat{\otimes} C_1, \varphi \hat{\otimes} I, \mathrm{i}\begin{bmatrix} 0 & \mu(f) \\ -\mu(f) & 0 \end{bmatrix}\right)$ satisfies the requirements to define the element, U, of $\mathrm{KK}(A, A \rtimes \mathbf{R} \otimes C_1)$ whose class is independent of the choice of f as long as it satisfies the above requirements for some $\varepsilon > 0$. Then the map $\Phi_{A,\alpha}(x) = x \otimes_A U$ satisfies the axioms for a Thom isomorphism given in [8]. Hence, $\Phi_{(A,\alpha)}$ is an isomorphism.

Let $\mathbf{R}$ act on $C(\mathbf{T}^n)$ via the dense embedding of the rank n subgroup $\Gamma \subseteq \mathbf{R}$.

PROPOSITION 2.1. *Under the standard isomorphism of* $\mathrm{KK}^1(C(\mathbf{T}^n), C(\mathbf{T}^n) \rtimes \mathbf{R})$ *with* $\mathrm{Ext}(C(\mathbf{T}^n), C(\mathbf{T}^n) \rtimes \mathbf{R})$ *the element U corresponds to the extension* $\mathcal{T}_\varepsilon$.

Proof. Let $(C(\mathbf{T}^n) \rtimes \mathbf{R}, \varphi, \mu(f))$ be a triple as above, and $\left((C(\mathbf{T}^n) \rtimes \mathbf{R}) \hat{\otimes} C_1, \varphi \hat{\otimes} I, \mathrm{i}\begin{bmatrix} 0 & \mu(f) \\ -\mu(f) & 0 \end{bmatrix}\right)$ the associated element of $\mathrm{KK}^1(C(\mathbf{T}^n), C(\mathbf{T}^n) \rtimes \mathbf{R})$. The corresponding extension in $\mathrm{Ext}(C(\mathbf{T}^n), C(\mathbf{T}^n) \rtimes \mathbf{R})$ is defined by the map $\tau\colon C(\mathbf{T}^n) \to \mathcal{M}(C(\mathbf{T}^n) \rtimes \mathbf{R})/C(\mathbf{T}^n) \rtimes \mathbf{R}$ where $\tau(g) = \pi(P\varphi(g)P)$, $P = \dfrac{\mu(f) + I}{2}$ and $\pi\colon \mathcal{M}(C(\mathbf{T}^n) \rtimes \mathbf{R}) \to \mathcal{M}(C(\mathbf{T}^n) \rtimes \mathbf{R})/C(\mathbf{T}^n) \rtimes \mathbf{R}$ is the projection. We need to check that $P_\varepsilon = h(D)$ is equal to P. This follows by noting that the composition

$$C^b(\mathbf{R}) \to \mathcal{M}(C(\mathbf{T}^n) \rtimes \mathbf{R}) \to \mathcal{L}(L^2(\mathbf{T}^n))$$

is given by $g \mapsto g(D)$. From this it is easy to check that τ is the same map which defines the extension $\mathcal{T}_\varepsilon$. ▨

The conclusion we have reached is that $[\mathcal{T}_\varepsilon] \in \mathrm{KK}^1(C(\mathbf{T}^n), C(\mathbf{T}^n) \rtimes \mathbf{R})$ is a Thom class and, hence, $\otimes_{C(\mathbf{T}^n)}[\mathcal{T}_\varepsilon]\colon \mathrm{K}_*(C(\mathbf{T}^n)) \to \mathrm{K}_{*+1}(C(\mathbf{T}^n) \rtimes \mathbf{R})$ is an isomorphism.

We may now consider the six term K-theory sequence for the extension $\mathcal{T}_\varepsilon$.

$$\begin{array}{ccccc} \mathrm{K}_0(C(\mathbf{T}^n) \rtimes \mathbf{R}) & \longrightarrow & \mathrm{K}_0(\mathcal{T}_\varepsilon) & \longrightarrow & \mathrm{K}_0(C(\mathbf{T}^n)) \\ \uparrow \partial_1^\varepsilon & & & & \downarrow \partial_0^\varepsilon \\ \mathrm{K}_1(C(\mathbf{T}^n)) & \longleftarrow & \mathrm{K}_1(\mathcal{T}_\varepsilon) & \longleftarrow & \mathrm{K}_1(C(\mathbf{T}^n) \rtimes \mathbf{R}). \end{array}$$

Since the boundary maps are given by $\partial_*^\varepsilon(x) = x \otimes_{C(\mathbf{T}^n)} U$, [1], they are isomorphisms and we obtain

PROPOSITION 2.3. $\mathrm{K}_*(\mathcal{T}_\varepsilon) = 0$.

3. TOEPLITZ EXTENSIONS ON $\mathbf{T}^n$

We are now able to analyze the K-theory sequence for the Toeplitz extensions for flows on $\mathbf{T}^n$ associated to dense subgroups of rank n, $\Gamma \subseteq \mathbf{R}$. (This extension is referred to as $\mathcal{T}_\Gamma$ in [9] and [6].) We will need the following basic result from [9].

PROPOSITION 3.1. (R. Ji and J. Xia). *The inclusions* $\iota\colon C(\mathbf{T}^n) \rtimes \mathbf{R} \to \hat{\mathcal{C}}$ *induces a surjection* $\iota_*\colon \mathrm{K}_*(C(\mathbf{T}^n) \rtimes \mathbf{R}) \to \mathrm{K}_*(\hat{\mathcal{C}})$. *It is an isomorphism on* K_0 *and a surjection on* K_1 *with kernel* $\mathbf{Z}[e]$, *where* $[e]$ *is the image of* $[1] \in \mathrm{K}_0(C(\mathbf{T}^n))$ *under* ∂_0^ε.

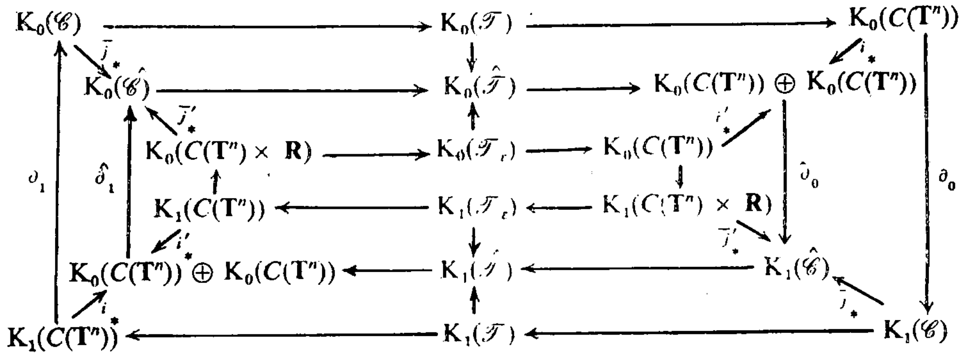
We can now state our main result.

PROPOSITION 3.2. *Let* $(C(\mathbf{T}^n), \alpha, \mathbf{R})$ *denote a dynamical system obtained from a dense subgroup* $\Gamma \subseteq \mathbf{R}$. *Then the* K-*theory exact sequence of the associated Toeplitz extension,*

$$\begin{array}{ccccc} \mathrm{K}_0(\mathcal{C}) & \longrightarrow & \mathrm{K}_0(\mathcal{T}) & \longrightarrow & \mathrm{K}_0(C(\mathbf{T}^n)) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ \mathrm{K}_1(C(\mathbf{T}^n)) & \longleftarrow & \mathrm{K}_1(\mathcal{T}) & \longleftarrow & \mathrm{K}_1(\mathcal{C}) \end{array}$$

satisfies that the boundary map ∂_1 *is an isomorphism and the boundary map* ∂_0 *is surjective with kernel* $\mathbf{Z}[1]$, *where* $[1] \in K_0(C(\mathbf{T}^n))$ *is the class of the unit in* $C(\mathbf{T}^n)$. *Moreover,* $K_0(\mathscr{T}) \cong \mathbf{Z}$ *and* $K_1(\mathscr{T}) = 0$.

Proof. We obtain three six-term sequences from diagram (1.5).



First note that $\bar{j}_*$ and $\bar{j}'_*$ are injections onto the first summand so that $(\bar{j}'_*)^{-1}\bar{j}_* = 1$, Thus, one has $\partial_1 = i_*^{-1}\hat{\partial}_1\bar{j}_* = i_*^{-1}i'_*\partial_1^\varepsilon$, hence ∂_1 is an isomorphism. Similarly $\partial_0 = i_*^{-1}i'_*\partial_0^\varepsilon$ is onto, with kernel $(\partial_0^\varepsilon)^{-1}(\ker i'_*) \cong \mathbf{Z}[1]$. From this it follows from the exactness of the outer ring that $K_0(\mathscr{T}) \cong \ker \partial_0 \cong \mathbf{Z}$ and $K_1(\mathscr{T}) = 0$. ▨

4. REMARKS

1) We have formulated our results for flows on a finite dimensional torus that was induced by a dense *finite rank* subgroup $\Gamma \subseteq \mathbf{R}$. However, all the results still hold even if the subgroup Γ does not have finite rank by taking direct limits. We did this because in the finite rank case the situation is more analogous to a general theory of Toeplitz extensions on foliated manifolds.

2) In [10], Rieffel has considered an extension associated to $(A, \alpha, \mathbf{R})$ which he calls the Wiener-Hopf extension:

$$0 \to A \otimes \mathscr{K} \to \mathscr{T}_{\mathbf{R}} \to A \rtimes \mathbf{R} \to 0. \tag{4.1}$$

It is the analog of the Pimsner-Voiculescu extension for a dynamical system $(B, \mathbf{Z}, \beta)$. This relationship is best understood by considering the algebra $C_0(\mathbf{Z} \cup \{+\infty\})$ with action given by $\gamma(n) = n - 1$, $\gamma(\infty) = \infty$. $C_0(\mathbf{Z})$ is an ideal which is preserved by γ. Then the exact sequence

$$0 \to C(\mathbf{Z}) \otimes B \rtimes_{\gamma\otimes\beta} \mathbf{Z} \to C_0(\mathbf{Z} \cup \{+\infty\}) \otimes B \rtimes_{\gamma\otimes\beta} \mathbf{Z} \to \text{Quotient} \to 0$$

can be identified with the Pimsner-Voiculescu extension

$$0 \to B \otimes \mathscr{K} \to \mathscr{T}_{\mathrm{PV}} \to B \rtimes_\beta \mathbf{Z} \to 0$$

(cf. [1], 10.2.1). If one now starts with an $\mathbf{R}$ action $(A, \alpha, \mathbf{R})$ and does the analogous construction then one obtains (4.1).

One can check that associating $[\mathscr{T}_{\mathbf{R}}] \in \mathrm{KK}^1(A \times \mathbf{R}, A)$ to $(A, \alpha, \mathbf{R})$ enables one to define a homomorphism $\Psi^i_{(A,\alpha)}: \mathrm{K}_i(A \times \mathbf{R}) \to \mathrm{K}_{i+1}(A)$ by $\Psi^i_{(A,\alpha)}(x) = x \otimes_{A \times \mathbf{R}} [\mathscr{T}_{\mathbf{R}}]$ which satisfies the axioms in [8]. Specializing to the case of a minimal $\mathbf{R}$ action on $\mathbf{T}^n$ we obtain, from the uniqueness of the Thom isomorphism, that

$$[\mathscr{T}_\varepsilon] \otimes_{C(\mathbf{T}^n) \times \mathbf{R}} [\mathscr{T}_{\mathbf{R}}] = 1_{C(\mathbf{T}^n)} \quad \text{and} \quad [\mathscr{T}_{\mathbf{R}}] \otimes_{C(\mathbf{T}^n)} [\mathscr{T}_\varepsilon] = 1_{C(\mathbf{T}^n) \times \mathbf{R}} .$$

This can be interpreted as saying that the Wiener-Hopf extension of Rieffel is dual, in the sense of KK-theory, to the somoothed Toeplitz extension along the orbits of the flow.

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