J. OPERATOR THEORY
19(1988), 355–364

AGREEMENT OF WEAK TOPOLOGIES ON CONVEX SETS

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1. INTRODUCTION

Let S be a weak* (= ultraweakly) closed subspace of the algebra $L(H)$ of Hilbert space operators. In answer to a question of D. Larson [4], B. Chevreau, J. Esterle [2] and P. Dixon [3] showed that if the relative weak* and weak operator topologies agree on S, then S is in fact weakly closed. Looking at the proof, several immediate generalizations come to mind: $L(H)$ can be replaced by any Banach space L with a separable predual T, and the weak operator topology can be replaced by topologies induced by many subsets F of T.

In this paper, we consider a further generalization of this result by allowing S to be an arbitrary convex subset of L. The obstacle to carrying over the original proof to this setting is that the classical description, in terms of linear functionals, of when two relative topologies agree on a linear subspace of L is no longer available for arbitrary convex sets. Theorem 3.4 of this paper develops such a characterization. Once this is done, it is easy to conclude (Corollary 3.7) that if the relative weak* and F-topologies agree on a weak* closed convex subset S of L, then S is F-closed.

In this paper the term *weak topology* will always denote the *F-topology* rather than the topology induced on L by the dual of L.

The structure of the remainder of this paper is as follows. Section 2 is devoted to preliminaries needed for the main theorem and corollaries. In Section 3 we present the main results along with the further corollary that if the relative weak* and weak topologies have a common base at a single point of a convex subset S of L, then the two relative topologies agree on all of S. The first four examples in Section 4 show that the theorems and corollary are sharp; this is followed by further consequences of the main results.

Larson's original question was motivated by recent developments in invariant subspace theory. For more details, consult [1], [4], [6] and the references listed there.

This paper is based on the author's dissertation [6]. The author would like to thank Professor E. Azoff for his advice and encouragement.

2. PRELIMINARIES

If (X, τ) is a topological space and S is a subset of X, then the relative topology on S is the topology, $\{A \cap S \mid A \in \tau\}$, induced by the members of τ on S.

It is clear that if the relative weak* and weak topologies agree on a subset S of L, then the two relative topologies agree on every subset of S. Moreover, the two relative topologies agree on every translate of S.

For the proof of Theorem 3.4 some tools and terminology are needed which we state here without proof. For more details and proofs see [1].

DEFINITION 2.1. A topological space (X, τ) is said to be *Polish* if it is separable and τ is induced by a complete metric. A subset of a topological vector space is called *analytic* if it is a continuous image of a Polish space.

The analytic sets have the following properties:

(1) Continuous images of analytic sets are analytic.

(2) All Borel sets are analytic.

(3) The Cartesian product of countably many analytic sets is analytic.

(4) The collection of analytic subsets of a space X is closed under countable unions and countable intersections.

Note that if X is a separable Banach space, then the linear span of any countable collection of closed subsets of X is analytic.

THEOREM 2.2. (B. J. Pettis [5]). *Let X be a Banach space. Suppose $S \subset X$ is analytic and is of second category. Then $S - S$ is a neighborhood of the origin in X.*

The definitions and results of this paper are valid regardless of the field of scalars, $\mathbf{R}$ or $\mathbf{C}$, for the Banach spaces involved. Throughout the paper, T will be a separable Banach space, L its dual, and F will denote a dense linear manifold in T which is an analytic set. We write $\langle x, t\rangle$ for $t(x)$ when $x \in L$ and $t \in T$. The motivating example discussed in the Introduction is when $L = L(H)$, $T = T(H)$, the space of trace class operators and $F = F(H)$, the set of all finite rank operators. The examples in the sequel will be in the simpler setting $L = \ell_\infty(\mathbf{R})$, $T = \ell_1(\mathbf{R})$, and $F = \{x \in \ell_1 \mid x \text{ has at most finitely many non-zero entries}\}$.

3. MAIN RESULTS

DEFINITION 3.1. Let S be a convex subset of L.

(1) S is said to be *confined* by F if for every $t \in T$ there exists a constant $K > 0$ and a finite subset $F_t \subset F$ such that

$$(*) \qquad |\langle x, t\rangle| \leqslant K \cdot \|t\| + \sum_{f \in F_t} |\langle x, f\rangle| \quad \text{for all } x \in S.$$

K and F_t may depend on t.

(2) S is said to be *uniformly confined* by F if (*) holds with K independent of t and F_t can be chosen such that $\sum_{f \in F_t} \|f\| \leqslant K\|t\|$.

Note that the weak* closure of a (uniformly) confined convex subset of L is also (uniformly) confined.

PROPOSITION 3.2. (1) *Every norm bounded subset of L is confined by F.*
(2) *Let S be a cone in L. If S is confined by F, then K in (*) can be taken to be equal to zero.*
(3) *Let S be linear manifold in L. Then S is confined by F if and only if for every $t \in T$ there exists $f \in T$ such that $\langle x, t\rangle = \langle x, f\rangle$ for all $x \in S$.*

Proof. (1) Every norm bounded subset of L is a subset of a ball in L, so take K to be the radius of the ball and $F_t = \emptyset$.
(2) Since S is a cone then $n \cdot x \in S$ for all $x \in S$ and all $n \in \mathrm{N}$. Hence $|\langle nx, t\rangle| \leqslant K + \sum_{f \in F_t} |\langle nx, f\rangle|$ for all $x \in S$ and all $n \in \mathrm{N}$. Divide both sides of the inequality by n, then let $n \to \infty$ to get the conclusion.
(3) ($\Leftarrow$) Clear.
($\Rightarrow$) If $t \equiv 0$, set $f \equiv 0$. Otherwise there exists a finite subset $F_t \equiv \{f_1, f_2, \ldots, f_n\} \subset F$ such that $|\langle x, t\rangle| \leqslant \sum_{i=1}^{n} |\langle x, f_i\rangle|$ for all $x \in S$. Therefore $\left[\bigcap_{i=1}^{n} \mathrm{Ker}(f_i)\right] \cap S \subset \mathrm{Ker}(t)$. Define $M \equiv \{(\langle x, f_1\rangle, \ldots, \langle x, f_n\rangle) \mid x \in S\}$; then M is a subspace of $\mathbf{C}_n$. Define the functional $\varphi : M \to \mathbf{C}$ by $(\langle x, f_1\rangle, \ldots, \langle x, f_n\rangle) \mapsto \langle x, t\rangle$; then φ is well defined. Hence there exist $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}_n$ such that $\langle x, t\rangle = \sum_{i=1}^{n} \lambda_i \langle x, f_i\rangle$ for all $x \in S$. Set $f \equiv \sum_{i=1}^{n} \lambda_i f_i$. ▨

LEMMA 3.3. *Suppose S is a convex subset of L which is uniformly confined by F and $x_0 \in L$. Then, given a weak* neighborhood U of x_0, there exists a weak neighborhood V of x_0 such that $V \cap S \subset U \cap S$.*

Proof. First we should point out that the collection, $\mathscr{B} \equiv \{P_t \mid t \in T\}$ where $P_t \equiv \{y \in L \mid |\langle y - x_0, t\rangle| < 1\}$ forms a subbasis for the weak* neighborhoods of x_0. Since some *finite* intersection of members of $\mathscr{B}$ is contained in U, it suffices (and simplifies the proof) to assume $U \in \mathscr{B}$.

We determine the constant $K > 0$ from the definition of uniformly confined; then we find $g \in F$ such that $\|t_0 - g\| \leqslant \dfrac{1}{4(1 + K)(1 + \|x_0\|)}$. By the triangle inequality,

$$|\langle x_0 - x, t_0\rangle| \leqslant |\langle x_0 - x, g\rangle| + |\langle x_0, t_0 - g\rangle| + |\langle x, t_0 - g\rangle| \quad \text{for all } x \in L.$$

Apply the definition of uniformly confined to $t_0 - g$ and apply the triangle inequality once more to get

$$|\langle x_0 - x, t_0\rangle| \leqslant |\langle x_0 - x, g\rangle| + \|x_0\|\cdot\|t_0 - g\| + K\cdot\|t_0 - g\| +$$

$$\sum_{f\in F_{t_0-g}} |\langle x_0 - x, f\rangle| + \sum_{f\in F_{t_0-g}} |\langle x_0, f\rangle| < \frac{3}{4} + |\langle x_0 - x, g\rangle| + \sum_{f\in F_{t_0-g}} |\langle x_0 - x, f\rangle|$$

for all $x \in S$.

Let $V \equiv \left\{x \in L \mid |\langle x_0 - x, g\rangle| + \sum_{f\in F_{t_0-g}} |\langle x_0 - x, f\rangle| < \frac{1}{4}\right\}$; then $V \cap S \subseteq$ $\subseteq U \cap S$. ▧

THEOREM 3.4. *Let S be a convex subset of L. Then the following are equivalent:*

(1) *The relative weak* and weak topologies agree on S.*

(2) *S is confined by F.*

(3) *S is uniformly confined by F.*

Before proving this theorem we require a lemma.

LEMMA 3.5. *The three statements in Theorem 3.4 are translation invariant.*

Proof. (1) Clear.

(2) Suppose S is confined by F. We show $S - x_0$ is confined for any fixed $x_0 \in L$. Let $0 \neq t \in T$ and find the corresponding K and F_t. Then

$$|\langle x - x_0, t\rangle| \leqslant |\langle x_0, t\rangle| + |\langle x, t\rangle| \leqslant$$

$$\leqslant \|x_0\|\cdot\|t\| + K\cdot\|t\| + \sum_{f\in F_t} |\langle x, f\rangle| \leqslant \qquad \text{for all } x \in S$$

$$\leqslant \|x_0\|\cdot\|t\| + K\cdot\|t\| + \sum_{f\in F_t} |\langle x_0, f\rangle| + \sum_{f\in F_t} |\langle x - x_0, f\rangle| \leqslant$$

$$\leqslant \|x_0\|\cdot\|t\| + K\cdot\|t\| + \Big(\sum_{f\in F_t} \|f\|\Big)\|x_0\| + \sum_{f\in F_t} |\langle x - x_0, f\rangle| \leqslant$$

$$\leqslant K_1\cdot\|t\| + \sum_{f\in F_t} |\langle x - x_0, f\rangle|$$

where

$$K_1 = \|x_0\| + K + \|x_0\|\frac{\sum_{f\in F_t}\|f\|}{\|t\|}\,.$$

(3) Repeat the argument of (2), noticing that $K_1 \leqslant \|x_0\| + K + K\|x_0\|$ is independent of t if K is. ▨

Proof of Theorem 3.4. By Lemma 3.5, without loss of generality, we may assume that S contains the origin.

(1) $\Rightarrow$ (2). Suppose (2) fails. Then there exists $t \in T$ such that for every $K>0$ and every finite subset $J \subset F$ there exists $x_J \in S$ such that $|\langle x_J, t\rangle| > K\cdot\|t\| + \sum |\langle x_J, f\rangle|$. Fix K such that $K\cdot\|t\| > 1$ and set $y_J = \dfrac{x_J}{|\langle x_J, t\rangle|}$. Then each $y_J \in S$ because S is convex and $|\langle y_J, t\rangle| = 1$ for all finite subsets $J \subset F$. Let $A \equiv \{y_J \mid y_J$ as defined above and J is a finite subset of $F\}$. Clearly 0 is not a weak* accumulation point of A. To see that 0 is a weak accumulation point of A, let $g_1, \ldots, g_n \in F$ and $\varepsilon > 0$ be given. Set $J = \{(1/\varepsilon)(g_1, \ldots, g_n)\}$; then $\sum_{i=1}^{n} |\langle y_J, g_i\rangle| < \varepsilon$, so every weak neighborhood of zero does intersect A.

(2) $\Rightarrow$ (3). Set $A = \{(t, f, \ldots, f) \in T^{(n+1)} \mid |\langle x, t\rangle| \leqslant n + \sum_{i=1}^{n} |\langle x, f_i\rangle|$ for all $x \in S$; $\|f_i\| \leqslant 1$, $i = 1, \ldots, n\}$. Each A_n is closed.

Let $B_n = A_n \cap (T \times F^{(n)})$. Then each B_n is analytic. Let E_n denote the range of the projection of B_n onto T. By hypothesis $\bigcup_{n=1}^{\infty} E_n = T$, and since projection is continuous, each E_n is analytic. By Baire category Theorem there exists $N \in \mathrm{N}$ such that E_N is of second category. By Pettis' theorem, $E_N - E_N$ is a neighborhood of the origin in T; therefore there exists a constant $c > 0$ such that the unit ball of T is contained in $c(E_N - E_N)$. Therefore, given $t \in T$ there exist t_1 and t_2 in E_N such that $|\langle x, t\rangle| = c\cdot\|t\|\cdot|\langle x, t_1 - t_2\rangle|$ for all $x \in S$. Let F_{t_1} and F_{t_2} be the finite sets corresponding t_1 and t_2 respectively, by definition of E_N. Then $\sum_{f \in F_{t_1}} \|f\| \leqslant N$ and $\sum_{f \in F_{t_2}} \|f\| \leqslant N$, and

$$|\langle x, t\rangle| \leqslant c\cdot\|t\| \, [2N + \sum_{f \in F_{t_1}} \|f\|\cdot|\langle x, f\rangle| + \sum_{f \in F_{t_2}} \|f\|\cdot|\langle x, t\rangle|]$$

for all $x \in S$. Take $F_t \equiv \{2c\|t\|f \mid f \in F_{t_1} \cup F_{t_2}\}$ and set $K = 2Nc$. Then $|\langle x, t\rangle| \leqslant K\cdot\|t\| + \sum_{f \in F_t} |\langle x, f\rangle|$ for all $x \in S$ and

$$\sum_{f \in F_t} \|f\| = c\cdot\|t\|(\sum_{f \in F_{t_1}} \|f\| + \sum_{f \in F_{t_2}} \|f\|) \leqslant 2Nc\cdot\|t\| = K\cdot\|t\|.$$

(3) $\Rightarrow$ (1). By Lemma 3.3, if $x_0 \in S$ and $U \subset L$ is a weak* neighborhood of x_0 then there exists a weak neighborhood $V \subset L$ such that $x_0 \in V \cap S \subset U \cap S$. ▨

The proof of Theorem 3.4 establishes the following remark:

REMARK 3.6. Given a convex subset S of L which is confined by F, there exists a fixed $n_0 \in \mathrm{N}$ such that the cardinality $|F_t| \leqslant n_0$ for all $t \in T$; n_0 may depend on S.

COROLLARY 3.7. *If S is a weak* closed convex subset of L and the relative weak* and weak topologies agree on it, then S is weakly closed.*

Proof. By Lemma 3.3, if x_0 is a weak accumulation point of S, then x_0 is a weak* accumulation point of S. Hence, if S is weak* closed then $x_0 \in S$, i.e., S is weakly closed. ▨

COROLLARY 3.8. *Let S be a convex subset of L. Suppose the relative weak* and weak topologies have a common base at a single point of S. Then the two relative topologies coincide on all of S. In particular the weak* and weak closures of S are the same.*

Proof. Let $x_0 \in S$ be the point at which the two relative topologies have the same base. Then $S - x_0$ is convex, contains the origin and has the property that every net in it which converges weakly to 0 must converge weak* to 0. By the proof of (1) $\Rightarrow$ (2) of Theorem 3.4, $S - x_0$ is confined by F. Thus S is confined by F and the proof of the first statement follows from Theorem 3.4. The second statement follows from Lemma 3.3. ▨

4. EXAMPLES AND FURTHER RESULTS

Examples 4.1—4.4 show that Theorem 3.4 and Corollary 3.7 are sharp.

EXAMPLE 4.1. This example shows the necessity of density of F in T. Set $S = F =$ a common one-dimensional subspace of $\mathbf{R}^2$, and $L = T = \mathbf{R}^2$. Then S is T-closed but not F-closed; in fact the F-closure of $S = L$.

The following example shows that the finite subset $F_t \subset F$ appearing in Definition 3.1 and consequently in Theorem 3.4, cannot always be reduced to a single element in F.

EXAMPLE 4.2. Let $S \equiv \{x = (x_1, x_2, \ldots) \in \ell^\infty \mid x_1 \leqslant x_n \leqslant x_2 \text{ for all } n\}$. S is a cone with vertex at origin. To see that S is confined by F, note that each $x \in S$ satisfies $\|x\| \leqslant |x_1| + |x_2|$. Let $t = (t_1, t_2, \ldots) \in \ell_1$. Define f and g in F such that $f \equiv (\|t\|, 0, \ldots)$ and $g \equiv (0, \|t\|, 0, \ldots)$, then

$$|\langle x, t\rangle| \leqslant \|t\| \cdot \|x\| \leqslant \|t\|(|x_1| + |x_2|) = \|t\| \cdot |x_1| + \|t\| \cdot |x_2| \leqslant$$

$$\leqslant |\langle x, f\rangle| + |\langle x, g\rangle| \quad \text{for all } x \in S.$$

Hence, S is confined by F.

Let $t_0 \equiv (t_1, t_2, \ldots)$, $t_n > 0$ for all n. We show no single $f \in F$ with t_0 makes the inequality in Definition 3.1 (consequently in Theorem 3.4), hold with $K = 0$. It suffices to show that for every $f \in F$ there exists an $x \in S$ such that $|\langle x, t_0\rangle| > |\langle x, f\rangle|$. Fix $f = (f_1, f_2, \ldots, f_{n_0}, 0, \ldots) \in F$. If $f_1 = f_2 = 0$ set $x \equiv (0,1,0,\ldots)$. Otherwise

Case 1. If $f_1 \geqslant 0$ and $f_2 \geqslant 0$, with one at least strictly positive, set $x \equiv (-f_2, f_1, 0, \ldots)$.

Case 2. If $f_1 < 0$ and $f_2 < 0$, set $x \equiv (f_2, -f_1, 0, \ldots)$.

Case 3. If f_1 and f_2 have different signs, define $x = (x_1, x_2, \ldots) \in S$ such that

$$x_n = \begin{cases} \|f^-\| & \text{if } f_n \geqslant 0 \\ \|f^+\| & \text{if } f_n < 0 \end{cases}$$

where

$$\|f^-\| = -\sum\{f_i \mid f_i < 0\}$$

and

$$\|f^+\| = \sum\{f_i \mid f_i > 0\}.$$

In all cases $\langle x, f\rangle = 0$ but $\langle x, t_0\rangle \neq 0$, i.e., $|\langle x, t_0\rangle| > |\langle x, f\rangle|$.

The following two examples show the necessity of the convexity hypothesis in Theorem 3.4 and Corollary 3.7.

EXAMPLE 4.3. Let $S \equiv \{n^2 e_n - e_1\}_{n=1}^{\infty} \subset \ell_\infty$. Then the relative weak* and weak topologies agree on S because every point in S is weakly isolated. We show that S is not confined by F. Let $t = (1, 2^{-3/2}, 3^{-3/2}, \ldots) \in \ell_1$. For any choice of $K > 0$ and finite subset $F_t \subset F$, choose n_0 to be sufficiently large such that, $\sqrt{n_0} - 1 > \max\{$(largest index of the nonzero entry for all $f \in F_t$), $K + \sum_{f \in F_t} |f_1|$ where f_1 is the first entry of $f \in F_t\}$. Then

$$|\langle n^2 e_n - e_1, t\rangle| > K + \sum_{f \in F_t} |\langle n^2 e_n - e_1, f\rangle| \quad \text{for all } n \geqslant n_0,$$

i.e., S is not confined by F.

EXAMPLE 4.4. Let S be as in Example 4.3. Then S is weak* closed because it has no weak* accumulation point and the relative weak* and weak topologies agree on it but it is not weakly closed because $-e_1$ is a weak limit point of S which does not belong to S.

No subspace of L with interior in L can be confined by F because L itself is the only subspace with interior in L. Also, Proposition 3.2 tells us that every norm bounded subset of L is confined by F. The following example shows the existence of unbounded convex subsets of L with interior in L and confined by F.

EXAMPLE 4.5. Let $S \equiv \{x = (x_1, x_2, \ldots) \in \ell_\infty \mid 0 \leqslant x_n \leqslant x_1, \text{ for all } n\}$. For any $t \in \ell_1$, let $F_t \equiv \{f\}$ where $f \equiv (\|t\|, 0, \ldots)$; then $|\langle x, t\rangle| \leqslant |\langle x, f\rangle|$ for all $x \in S$, so S is confined by F.

NOTE. The set $S \equiv \{x = (x_1, x_2, \ldots) \in \ell_\infty \mid 0 \leqslant x_n \text{ for all } n\}$ is not confined by F. To see that, note that $\{n^2 e_n\}_{n=1}^{\infty} \subseteq S$ converges weakly to 0, but not weak*—to see that, let $t = (1, 2^{-2}, 3^{-2}, \ldots)$.

The following example shows that the relative weak* and weak topologies may agree on a subset S of L without agreeing on its convex hull co(S).

EXAMPLE 4.6. Let $S \equiv \{n^2 e_n\}_{n=0}^{\infty} \cup \{-e_1\}$. The only nets in S which converges weakly to a point in S are those which are eventually constant, so the relative weak* and weak topologies agree on S. However $0 \in \text{co}(S)$ and we have just seen that $n^2 e_n \xrightarrow{\text{weakly}} 0$ but $n^2 e_n \overset{\text{weakly*}}{\not\to} 0$. Hence the two relative topologies do not agree on co(S).

We conclude this paper with some further results, the first of which is a generalization of Proposition 3.2(1).

PROPOSITION 4.7. *Suppose S is a subset of L such that every point of S has a norm bounded relative weak neighborhood. Then the relative weak* and weak topologies agree on S.*

Proof. Let $\{x_\alpha\}_{\alpha \in \Gamma} \subset S$ be a net such that $x_\alpha \xrightarrow{\text{weakly}} x_0$, $x_0 \in S$. Then there exists a norm bounded, say by M, relative weak neighborhood U of x_0 in S. Therefore there exists α_0 such that $\alpha \geqslant \alpha_0$ implies $x_\alpha \in U$. Now if $t \in T$ and $\varepsilon > 0$ are given, choose $g \in F$ such that $\|t - g\| < \varepsilon$. By the triangle inequality,

$$|\langle x_\alpha - x_0, t\rangle| \leqslant |\langle x_\alpha - x_0, g\rangle| + |\langle x_\alpha - x_0, t - g\rangle|.$$

Clearly, $|\langle x_\alpha - x_0, g\rangle| \to 0$ and for $\alpha > \alpha_0$, we have $|\langle x_\alpha - x_0, t - g\rangle| \leqslant 2M\cdot\varepsilon$ Since ε is arbitrary, $|\langle x_\alpha - x_0, t\rangle| \to 0$, i.e. $x_\alpha \xrightarrow{\text{weak*}} x_0$. ▨

The following proposition shows that the converse of Proposition 4.7 holds for convex subsets of ℓ_∞.

PROPOSITION 4.8. *Let S be a convex subset of ℓ_∞. If the relative weak* and weak topologies agree on S, then every point of S has a norm bounded relative weak neighborhood.*

Proof. Review the proof of Theorem 3.4, setting

$$E_n \equiv \{t \in \ell_1 \mid |\langle x, t\rangle| \leqslant n + n \sum_{i=1}^{n} |x_i| \text{ for some } n \in \mathrm{N} \text{ and all}$$

$$x = (x_1, x_2, \ldots, x_n, \ldots) \in S\}.$$

Then $\ell_1 = \bigcup_{n=1}^{\infty} E_n$, so by the Baire category theorem, there exists $N \in \mathrm{N}$ such that E_N is of second category; an application of Pettis' theorem shows that

$$|\langle x, t\rangle| \leqslant 2N \cdot \|t\| \left(1 + \sum_{i=1}^{N} |x_i|\right) \quad \text{for all } x \in S.$$

Thus the set $\{x \in S \mid |x_i| < 1, \ i = 1, \ldots, N\}$ is a norm bounded relative weak neighborhood of 0. By translation invariance the proposition is proved. ▨

The following proposition and example show that the analogue of Proposition 4.8 does not hold for convex subsets of $L(H)$.

PROPOSITION 4.9. *Suppose S is a weak* closed subspace of L such that the origin has a norm bounded relative weak* neighborhood. Then S is finite dimensional.*

Proof. Since the origin has a norm bounded relative weak* neighborhood in S, the closed unit ball of S is also a relative weak* neighborhood of the origin. But the closed unit ball in S is relatively weak* compact. Hence S is finite dimensional by Problem 9 on page 99 of [18]. ▨

EXAMPLE 4.10. The converse of Proposition 4.7 fails, even for operator algebras. Let H be an infinite dimensional Hilbert space and take $S \equiv (L(H))^{(\infty)}$, the infinite ampliation of $L(H)$. Then the relative weak* and weak topologies agree on S [1], but by Proposition 4.9, the origin has no norm bounded relative weak neighborhood.

PROPOSITION 4.11. *Let M and K be linear manifolds in L such that $M \subset K$ and M is confined by F. If* $\dim(K/M) < \infty$, *then K is confined by F.*

Proof. Without loss of generality we may assume $\dim(K/M) = 1$. Choose $x_0 \in K \setminus M$. Given $t \in T$, apply Proposition 3.2(3) to find $f \in F$ such that $t - f \perp M$. Since M is confined by F, there exists $f_1 \in F$ such that $f_1(x_0) = 1$ and $f_1 \perp M$. So there exists $\lambda \in \mathbf{C}$ such that $(t - f)(x_0) = \lambda = \lambda f_1(x_0)$. Then $(t - f - \lambda f_1)(x_0) = 0$ and $(t - f - \lambda f_1) | \hat{M} \equiv 0$. Therefore $(t - f - \lambda f_1) \perp K$, i.e., $t | \hat{K} \equiv (f - \lambda f_1) | \hat{K}$ Hence K is confined by F. ▨

PROPOSITION 4.12. *Suppose S is a subspace of L and $t \in T$ is given. Then S is confined by F if and only if the set $A \equiv \{x \in S \mid \mathrm{Re}\langle x, t\rangle \geqslant \alpha\}$ is confined by F.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Without loss of generality we may assume $\alpha = 0$.

Let $A_1 \equiv \{x \in S \mid \mathrm{Re}\langle x, t\rangle \leqslant 0\}$. Then $-A = A_1$ and A is confined by F if and only if A_1 is. Since A is confined by F and $|\langle -x, t\rangle| = |\langle x, t\rangle|$ for all $x \in S$, then S is confined by F. ▨

DEFINITION 4.13. A subset S of a linear space X is said to be *x_0-star*, $x_0 \in S$, if the line segment

$$\{\lambda x_0 + (1-\lambda)x \mid 0 \leqslant \lambda \leqslant 1\} \subset S$$

whenever $x \in S$.

REMARK 4.14. If S is a subset of L which is x_0-star for some $x_0 \in S$, then all results in this paper which hold for general convex sets still hold for S.

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Received June 30, 1987; revised October 14, 1987.