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PERTURBATION OF THE SPECTURM OF EVOLUTION OPERATORS

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If $T(r)$ is a family of bounded linear operators in a Banach space X, depending continuously in the uniform operator topology (that is, in the norm of the space $L(X)$) on a parameter r in a metric space (E, d) then the spectrum $\sigma(T(r))$ is an upper continuous function of r; this means that for any r_0 in E and any $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(r, r_0) < \delta$ then $\sigma(T(r))$ is contained in the ε-neighborhood of $\sigma(T(r_0))$. A consequence of this fact is that the asymptotic behavior of the linear discrete dynamical system $\{T^n(r),\ n = 0, 1, 2, \ldots\}$ is a continuous function of the parameter. However, if $T(r)$ is continuous in the strong operator topology (that is, $(r, u) \in E \times X \to T(r)u \in X$ is continuous) then $\sigma(T(r))$ may present an "explosion" as a function of r; to be more specific, if $\rho(T)$ denotes the spectral radius of T, we may have $\rho(T(r_0)) \leqslant 1/2$ and $\rho(T(r_n)) \geqslant 2$ for a sequence r_n converging to r_0. Of course this implies that the asymptotic behavior of the c orresponding discrete dynamical system is very sensitive to perturbations of the parameter. This phenomenon has been known in the context of difference equations ([2]) and it is called parametric unstability. Since mixed problems for hyperbolic systems in one space variable are closely related to difference equations ([3]) one expects that parametric unstability may occur also in that class of equations. Indeed, this is the case. In this paper we give conditions under which we are able to estimate the "explosion" of the spectrum of time periodic evolution operators continuously in the strong operator topology on a parameter r. The applicability of the method depends very heavily on the possibility of finding an evolution operator which is simpler than the given one but still related to it (in a sense which will be made precise). We start by stating the abstract result for the case of a linear discrete dynamical system.

THEOREM I. *Let $T, T_0 : (E, d) \to L(X)$ be operator valued functions defined in a metric space (E, d) with values in the set $L(X)$ of the bounded linear operators of a Banach space X and suppose the following conditions are satisfied:*

H_1) *$T_0(r)$ and $T_0^*(r)$ are strongly continuous in r;*

H_2) *for each* r, $T(r) - T_0(r)$ *is a compact operator;*

H_3) $T - T_0 : (E, d) \to L(X)$ *is continuous with respect to the norm of* $L(X)$*;*

H_4) *there are real constants* $\delta_0 > 0$, $M > 0$, $\tau > 0$ *and* α_0 *and a point* r_0 *in* E *such that* $|T_0^n(r)| \leqslant M e^{n\alpha_0\tau}$ *for any positive integer* n *and any* r *satisfying* $d(r, r_0) < \delta_0$*;*

H_5) *for some* $\alpha > \alpha_0$, $T(r_0)$ *has no spectrum on the circle* $|\lambda| = e^{\alpha\tau}$.

If $\lambda_1, \ldots, \lambda_m$ *are the eigenvalues of* $T(r_0)$ *outside the disk of radius* $e^{\alpha\tau}$ *and* $\varepsilon > 0$ *is given then there are* $\delta, 0 < \delta < \delta_0$ *and* $K(\varepsilon, M)$ *such that for* $d(r, r_0) < \delta$ *the following assertions hold:*

1) $T(r)$ *has no spectrum on the circle of radius* $e^{\alpha\tau}$ *and its spectrum outside the disk of radius* $e^{\alpha\tau}$ *consists of eigenvalues lying in the union of the balls* $\{\lambda : |\lambda - \lambda_j| < \varepsilon\}$, $j = 1, 2, \ldots, m$*;*

2) *for each* $j = 1, 2, \ldots, m$ *the finite dimensional spectral projection* $P_j(r)$ *corresponding to the curve* $|\lambda - \lambda_j| = \varepsilon$ *converges to* $P_j(r_0)$ *in norm as* $r \to r_0$*;*

3) $|T^n(r)P(r)| \leqslant K e^{n\alpha\tau}$, *for any positive integer* n, *where* $P(r)$ *is the spectral projection corresponding to the curve* $|\lambda| = e^{\alpha\tau}$.

REMARKS. 1) The assertion about the part of $\sigma(T(r))$ outside the disk of radius $e^{\alpha\tau}$ consisting of isolated eigenvalues with finite multiplicity follows from H_2 and H_4.

2) If $T(r, q)$ depends on two parameters (r, q) in such way that its dependence on q is continuous in the norm of $L(X)$ locally uniformly in r then Theorem I has an obvious extension; in this case assumptions H_2 and H_3 should hold for each q fixed and H_4 for $T_0^n(r, q_0)$.

3) Theorem I also has an obvious extension to the case where instead of operators $T(r)$ and $T_0(r)$ we deal with C_0 semigroups $T(t, r)$ and $T_0(t, r)$ or, more generally, with τ-periodic evolution operators $T(t, s, r)$ and $T_0(t, s, r)$.

Roughly speaking (modulo some uniformity), Theorem I says that if we are able to bound the spectral radius of $T_0(r)$, in a full neighborhood of r_0, by a constant $e^{\alpha_0\tau}$ and $T_0(r)$ is related to $T(r)$ in the sense that assumptions H_2 and H_3 hold, then the part of the spectrum of $T(r)$ which is outside the disk of radius $e^{\alpha_0\tau}$ changes with r in a very nice way.

Before passing to the proof we give two lemmas.

LEMMA 1. *Let* $P_n, n = 1, 2, \ldots,$ *and* P *be finite dimensional projections in a Banach space* X *such that* P_n *converges to* P *in the strong operator topology. If* $\{P_n x, n = 1, 2, \ldots, |x| \leqslant 1, x \in X\}$ *and* $\{P_n^* y, n = 1, 2, \ldots, |y| \leqslant 1, y \in X^*\}$ *are precompact then* P_n *converges to* P *in the norm of* $L(X)$.

Proof. We start by showing that $\dim P_n(X) = \dim P(X)$ if n is large. Let $(e_1, \ldots, e_k)$ be a basis of $P(X)$. Clearly $P_n(e_1), \ldots, P_n(e_k)$ are linearly independent if n is large and then $\dim P_n(X) \geqslant k$; if $\dim P_n(X) > k$ for infinitely many n's then there would be x_n in $P_n(X)$ such that $|x_n| = 1$ and $d(x_n, [P_n(e_1), \ldots, P_n(e_k)]) = 1$;

passing to a convergent subsequence $x_{n_j} \to z$ we would have

$$|x_{n_j} - Pz| = |P_{n_j} x_{n_j} - Pz| \leqslant |P_{n_j}(x_{n_j} - z)| + |P_{n_j} z - Pz|$$

which is a contradiction because this last expression tends to zero and $d(x_{n_j}, [e_1, \ldots, e_k])$ tends to one. As a consequence of that we see the existence and uniqueness of functionals α_i^n, $i = 1, 2, \ldots, k$, in X^*, n large, such that $P_n(x) = \alpha_1^n(x)P_n(e_1) + \ldots + \alpha_k^n(x)P_n(e_k)$; clearly α_i^n belongs to $(\ker P_n)^\perp = P_n^*(X^*)$. Next we claim the sequences α_i^n, $i = 1, \ldots, k$ are bounded; in fact, taking for instance $i = 1$ and defining $d > 0$ by $d = d(e_1, [e_2, \ldots, e_k])$ we see that $d(P_n(e_1), [P_n(e_2), \ldots, P_n(e_k)])$ tends to d and then there is a sequence $\gamma_n \in X^*$ such that $|\gamma_n|$ tends to $1/d$, $\gamma_n(P_n(e_i)) = 0$ for $i \geqslant 2$ and $\gamma_n(P_n(e_1)) = 1$ and, since $\alpha_1^n(x) = \gamma_n P_n(x)$ the claim is proved. In order to show that P_n tends to P uniformly it suffices to show that $\{P_n, n = 1, 2, \ldots\}$ is a precompact subset of $L(X)$ (with respect to the norm topology) and this follows from the precompactness of $\{P_n^* y, |y| \leqslant 1, n = 1, 2 \ldots\}$, the boundedness of α_i^n and from noticing that if $\alpha_i^{n_j}$ converges to β_i in the norm of X^*, $i = 1, 2, \ldots, k$, then P_{n_j} converges in norm to $P(x) = \beta_1(x)e_1 + \ldots + \beta_k(x)e_k$ and the lemma is proved.

REMARK. The projections in $\ell_2(N)$ given by $x \to (x_1, 0, \ldots, 0, x_n, 0, \ldots)$ and $x \to (x_1 + x_n, 0, \ldots, 0, \ldots)$ show that the assumptions in Lemma 1 are sharp.

LEMMA 2. (i) *If S_n is a sequence of compact operators in $L(X)$ which converges to some (compact) operator S in the norm of $L(X)$ then $\{S_n x, n = 1, 2, \ldots, |x| \leqslant 1, x \in X\}$ and $\{S_n^* y, n = 1, 2, \ldots, |y| \leqslant 1, y \in X^*\}$ are precompact.*

(ii) *If S_n and S are as in part* (i) *and $T_n \xrightarrow{s} T$ then $T_n S_n$ converges to TS in the norm of $L(X)$; in particular, if $T_n \xrightarrow{s} T$ and S is a compact operator then $T_n S \to TS$ in the norm of $L(X)$.*

Proof. Clear.

Proof of Theorem I. From assumption H_4 it follows that any complex number λ satisfying $|\lambda| > e^{\alpha_0 \tau}$ belongs to the resolvent set of $T_0(r)$ for $d(r, r_0) < \delta_0$ and $|(T_0(r) - \lambda I)^{-1}|$ is bounded by a constant depending just on M and the distance from λ to the disk of radius $e^{\alpha_0 \tau}$. Defining $S(r, \lambda) = (T_0(r) - \lambda I)^{-1}(T(r) - T_0(r))$ then Lemma 2, part (ii), and assumptions H_1, H_2 and H_3 show that $S(r, \lambda) \to S(r_0, \lambda_0)$ in the norm of $L(X)$ if $|\lambda_0| > e^{\alpha_0 \tau}$ and $(r, \lambda) \to (r_0, \lambda_0)$. Moreover, from the equality $T(r) - \lambda I = (T_0(r) - \lambda I)(I + S(r, \lambda))$ we conclude $T(r) - \lambda I$ is invertible if and only if $I + S(r, \lambda)$ is, and then, from the previous argument, we see that if λ_0 belongs to the resolvent set of $T(r_0)$ and $|\lambda_0| > e^{\alpha_0 \tau}$ all λ's in some neighborhood of λ_0 also belong to the resolvent set of $T(r)$ if $d(r, r_0)$ is small and assertion 1) of the theorem follows from a compactness argument. If Γ_j denotes the circle $|\lambda - \lambda_j| = \varepsilon$, using the fact that the λ's outside the disk of radius $e^{\alpha_0 \tau}$ belong to the resolvent set of $T_0(r)$ we see the spectral projection $P_j(r)$ is given by

$$P_j(r) = \frac{1}{2\pi i}\int_{\Gamma_j} (T(r) - \lambda I)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_j} [(T(r) - \lambda I)^{-1} - (T_0(r) - \lambda I)^{-1}] d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_j} (T(r) - \lambda I)^{-1}(T_0(r) - T(r))(T_0(r) - \lambda I)^{-1} d\lambda.$$

Let r_n be a sequence converging to r_0. Clearly $P_j(r_n)$ converges to $P_j(r_0)$ strongly. Next we claim $\{P_j(r_n)u, |u| \leqslant 1, n = 1, 2, \ldots\}$ is precompact; in fact, if $|u| \leqslant 1$ then $|T_0(r) - \lambda I)^{-1}u| \leqslant a$, for some constant a, any r such that $d(r, r_0) < \delta$ and any λ in Γ_j; defining $S_n = T_0(r_n) - T(r_n)$ and using assumptions H_2 and H_3 and Lemma 2 we conclude $\{S_n(T_0(r) - \lambda I)^{-1}u, |u| \leqslant 1, n = 1, 2, \ldots\}$ is precompact; since the map $(r, \lambda, v) \in B(r_0, \delta) \times \Gamma_j \times X \to (T(r) - \lambda I)^{-1}u \in X$ is continuous the claim is proved. The same argument proves that $\{P_j^*(r_n)\alpha, |\alpha| \leqslant 1, n = 1, 2, \ldots\}$ is precompact and Lemma 1 shows that the convergence of $P_j(r_n)$ to $P_j(r_0)$ takes place in norm. Finally, denoting by Γ the circle $|\lambda| = e^{\alpha\tau}$ we have $T^n(r)P(r) = \frac{1}{2\pi i}\int_{\Gamma} \lambda^n (T(r) - \lambda I)^{-1} d\lambda$ and then $|T^n(r)P(r)| \leqslant N e^{(n+1)\alpha\tau}$ for some constant, N depending just on M and δ, and Theorem I is proved.

Consider now an abstract linear evolution equation in a Banach space X, $\dot{u} = A(t, r)u$, depending on a parameter r in a metric space (E, d); we assume $A(t, r)$ is periodic in t with period $\tau > 0$ and we denote by $T(t, s; r)$, $s \leqslant t$, the corresponding evolution operator. We consider also another linear τ-periodic equation $\dot{u} = A_0(t, r)u$ (which will be called the reduced equation) and we denote by $T_0(t, s; r)$, $s \leqslant t$, the corresponding evolution operator. Next we give conditions under which assumptions H_1, H_2 and H_3 in Theorem I are satisfied for $T(\tau, 0; r)$ and $T_0(\tau, 0; r)$ but, first, we state a lemma which will be useful.

LEMMA 3. *Let $\varphi_n, \alpha_n, \beta_n$ be sequences of nonnegative uniformly bounded measurable real valued functions defined on some interval $[a, b]$ such that α_n tends to zero pointwise and*

$$\varphi_n(t) \leqslant a_n(t) + \int_a^t \beta_n(s)\varphi_n(s) ds, \quad a \leqslant t \leqslant b.$$

Then φ_n tends to zero pointwise on $[a, b]$.

Proof. It follows from the generalized Gronwall's inequality

$$\varphi_n(t) \leqslant \alpha_n(t) + \int_a^b \beta_n(s)\alpha_n(s)\left(\exp\int_s^b \beta_n(u)\mathrm{d}u\right)\mathrm{d}s,$$

and Lebesgue convergence theorem.

THEOREM II. *Assume $A(t, r) = A_0(t, r) + B(t, r)$ where $B(t, r)$ belongs to $L(X)$ and is strongly continuous in t and r. Let $S(t; r)$ and $S_0(t, s; r)$ be defined by*

$$S(t, r) = T(t, 0; r) - T_0(t, 0; r)$$

and

$$S_0(t, s; r) = \int_s^t T_0(t, \sigma; r)\, B(\sigma, r)\, T_0(\sigma, s; r)\mathrm{d}\sigma, \quad 0 \leqslant s \leqslant t \leqslant \tau$$

and suppose there is a $c > 0$ such that for each s and t, $0 \leqslant s \leqslant t \leqslant \tau$, $s \leqslant t \leqslant s + c$, he following conditions are satisfied:

$\bar{\mathrm{H}}_1$) *$T_0(t, s; r)$ and $T_0^*(t, s; r)$ are strongly continuous in r;*

$\bar{\mathrm{H}}_2$) *for any r in E, $S_0(t, s; r)$ is a compact operator;*

$\bar{\mathrm{H}}_3$) *$r \to S_0(t, s; r)$ is continuous in the norm of $L(X)$.*

Then

(i) *$T_0(t, s; r)$ and $T_0^*(t, s; r)$ are strongly continuous in r for $0 \leqslant s \leqslant t \leqslant \tau$;*

(ii) *for any r in E, $\{S(t, r)u,\ 0 \leqslant t \leqslant \tau,\ |u| \leqslant 1\}$ is precompact; in particular, for any r, $S(\tau, r)$ is a compact operator;*

(iii) *for any $t \in [0, \tau]$, $r \to S(t, r)$ is continuous in the norm of $L(X)$; in particular, $r \to S(\tau, 0)$ is continuous in the norm of $L(X)$.*

Proof. For $s + c \leqslant t \leqslant s + 2c$ we have

$$T_0(t, s; r_2) - T_0(t, s; r_1) =$$
$$= T_0(t, s + c; r_2)(T_0(s + c, s; r_2) - T_0(s + c, s; r_1)) -$$
$$- (T_0(t, s + c; r_1) - T_0(t, s + c; r_2))\, T_0(s + c, s; r_1)$$

and

$$T_0^*(t, s; r_2) - T_0^*(t, s; r_1) =$$
$$= T_0^*(s + c, s; r_2)(T_0^*(t, s + c; r_2) - T_0^*(t, s + c; r_1)) +$$
$$+ (T_0^*(s + c, s; r_2) - T_0^*(s + c, s; r_1))T_0^*(t, s + c; r_1)$$

and this proves (i) for $s \leqslant t \leqslant s + 2c$ and a repetition of the same argument shows it holds for any s, t, $0 \leqslant s \leqslant t \leqslant \tau$. Assertion (ii) follows from Lemmas 2 and 3 in [3]. In order to prove (iii) for $c \leqslant t \leqslant 2c$ we start by noticing that $r \to S_0(t, r)$ is continuous in the norm of $L(X)$ for each t in $[c, 2c]$ as a consequence of the equality

$$S_0(t, 0; r) = T_0(t, c; r)\, S_0(c, 0; r) + S_0(t, c; r)\, T_0(c, 0; r),$$

Lemma 2 and assertion (i). Next, if $r_n \to r$, from

$$S(t, r_n) - S(t, r) = S_0(t, 0; r_n) - S_0(t, 0; r_n) + \\ + \int_0^t T_0(t, s; r_n)\,(B(s, r_n) - B(s, r))\, S(s, r)\mathrm{d}s + \\ + \int_0^t T_0(t, s; r_n)\, B(s, r_n)\,(S(s, r_n) - S(s, r))\mathrm{d}s,$$

Lemma 3 and Lebesgue convergence theorem we see $S(t, r_n) \to S(t, r)$ in the norm of $L(X)$ for $c \leqslant t \leqslant 2c$ and a repetition of this argument proves assertion (iii) and the theorem.

As an application of our theory we consider the following time τ-periodic initial-boundary value problem for a hyperbolic system in one space variable in normal form:

$$(1)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} + K(t, x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} + C(t, x)\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = 0, \quad 0 < x < l, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}[v(t, l) - D(t)u(t, l)] = F(t)u(t, l) + G(t)v(t, l) \\ \text{with boundary condition: } u(t, 0) = E(t)v(t, 0), \end{cases}$$

where:

(i) $K(t, x) = \mathrm{diag}\,[\lambda_1(t, x), \ldots, \lambda_N(t, x), \mu_{N+1}(t, x), \ldots, \mu_n(t, x)]$ is a diagonal $n \times n$ matrix whose entries are real valued C^1 functions in t and x for (t, x) in $\mathbf{R} \times [0, l]$, with $\lambda_i(t, x) > 0$, $i = 1, \ldots, N$, and $\mu_j(t, x) < 0$ for $j = N + 1, \ldots, n$;

(ii) $C(t, x)$ is an $n \times n$ matrix whose entries $c_{ij}(t, x)$ are real (or complex) valued continuous functions of (t, x) in $\mathbf{R} \times [0, l]$ and either C^1 in t or C^1 in x;

(iii) $u(t, x) = \mathrm{col}[u_i(t, x)]$, $i = 1, \ldots, N$ is a column vector in $\mathbf{R}^N$ (or $\mathbf{C}^N$) and $v(t, x) = \mathrm{col}[v_j(t, x)]$, $j = N + 1, \ldots, n$ is a column vector in $\mathbf{R}^{n-N}$ (or $\mathbf{C}^{n-N}$);

(iv) $D(t)$, $E(t)$, $F(t)$ and $G(t)$ are matrices of appropriate sizes whose entries are real (or complex) valued C^1 functions defined on $\mathbf{R}$.

We also assume that $K(t, x)$ and $C(t, x)$ satisfy the following condition:

(v) if $k \neq m$ and $\lambda_k(t, x) = \lambda_m(t, x)$ (or $\mu_k(t, x) = \mu_m(t, x)$) somewhere in $\mathbf{R}\times[0, l]$ then $c_{km}(t, x)$ vanishes identically on $\mathbf{R}\times[0, l]$.

For any p, $1 < p < \infty$ we consider the Banach space $X_p = (L_p([0, l]))^n \times \mathbf{R}^{n-N}$ (or $(L_p([0, l]))^n \times \mathbf{C}^{n-N}$), where $L_p([0, l])$ denotes the set of the real (or complex) valued functions whose power p is integrable on $[0, l]$, and we define

$$A(t) : \mathscr{D}(A(t)) \subset X_p \to X_p$$

by

$$A(t)(u, v, d) = \left[-K(t, x) \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} - C(t, x) \begin{pmatrix} u \\ v \end{pmatrix},\ F(t)u(l) + G(t)v(l) \right]$$

where

$$\mathscr{D}(A(t)) = \{(u, v, d) \in X_p : u \in (H_{1,p}([0, l]))^N,\ v \in (H_{1,p}([0, l]))^{n-N},$$

$$u(0) = E(t)v(0), \quad d = v(l) - D(t)u(l)\},$$

$H_{1,p}([0, l])$ denoting the usual Sobolev space. With this setting, the system (1) can be viewed as an abstract equation $\dot{w} = A(t)w$ in X_p and, under assumptions (i) to (iv), $A(t)$ generates an evolution operator $T(t, s)$, $s \leqslant t$, in the sense of Krein [1].

In [3] we have given some properties of the spectrum of the operator $T(\tau, 0)$; in particular, we have shown that the essential spectral radius of $T(\tau, 0)$ is determined by a reduced system. Here we analyse how the spectrum of $T(\tau, 0)$ changes with the coefficients of the system; the complication is due to the fact that as a function of the eigenvalues

$$(\lambda_1(t, x), \ldots, \lambda_N(t, x),\ \mu_{N+1}(t, x), \ldots, \mu_n(t, x)),$$

$T(\tau, 0)$ is continuous with respect to the strong operator topology but not with respect to the uniform operator topology.

First of all we define the spaces of the parameters and their topologies. We denote by r the parameter

$$r = (\lambda_1(t, x), \ldots, \lambda_N(t, x),\ \mu_{N+1}(t, x), \ldots, \mu_n(t, x))$$

and by q the parameter $q = (E(t), D(t), F(t), G(t), C(t, x))$ as elements of spaces of functions. We equip the space of the parameter r with the C^1 topology in (t, x) and the space of the parameter q with the sup topology. We make a distinction between r and q because as a function of q the evolution operator $T(t, s)$ is continuous with respect to the norm of $L(X)$.

In order to apply the theory we have just developed, we have to construct the reduced system and so, together with system (1), we consider the following three reduced systems:

$$(2)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(t,x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_0(t,x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \quad 0 < x < l, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}[v(t,l) - D(t)u(t,l)] = F(t)u(t,l) + G(t)v(t,l) \\ \text{with boundary condition: } u(t,0) = E(t)v(t,0); \end{cases}$$

$$(3)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(t,x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_0(t,x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \quad 0 < x < l, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}[v(t,l) - D(t)u(t,l)] = 0 \\ \text{with boundary condition: } u(t,0) = E(t)v(t,0) \end{cases}$$

and

$$(4)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(t,x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_0(t,x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \quad 0 < x < l, \\ \text{with boundary conditions: } u(t,0) = E(t)v(t,0) \text{ and } v(t,l) = D(t)u(t,l), \end{cases}$$

where $C_0(t,x) = \mathrm{diag}(c_{11}(t,x), \ldots, c_{nn}(t,x))$.

Each system can be viewed as an abstract equation $\dot{w} = A_i(t)w$, $i = 2,3,4$, with $A_i(t)$ and $\mathscr{D}(A_i(t))$ defined in the obvious way and the corresponding evolution operators will be denoted by $T_i(t,s)$, $s \leqslant t$, $i = 2,3,4$. For systems (2) and (3) the phase space is also X_p but for system (4) the phase space is $L_p([0,l])^n$. Letting $X_p^0 = \{(u,v,d) \in X_p : d = 0\}$ it is easy to see that X_p^0 is invariant under $T_3(t,s)$ and $T_3(t,s) = T_4(t,s)$ on X_p^0 (we are identifying $(u,v,0)$ with (u,v)); we denote by $\pi : X_p \to X_p$ the projection onto X_p^0 defined by $\pi(u,v,d) = (u,v,0)$.

The role of T_0 in Theorem I will be played by $T_4(\tau,0)\pi$ but we have to introduce also systems (2) and (3) because they will be important in verifying the assumptions of Theorems I and II. In order to carry out this verification we make an intensive use of the following explicit representation of $T_2(t,s)$, $T_3(t,s)$ and $T_4(t,s)$.

Let $\varphi_i(t,s;x)$, $i = 1, \ldots, N$ and $\psi_j(t,s;x)$, $j = N+1, \ldots, n$ be the solutions of $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(t,x)$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \mu_j(t,x)$ satisfying $\varphi_i(s,s;x) = x$ and $\psi_j(s,s;x) = x$, respectively; for each (t,x) we define $\tau_i(t,x)$, $i = 1, \ldots, N$ and $\tau_j(t,x)$, $j = N + 1, \ldots, n$ by $\varphi_i(\tau_i, t; x) = 0$ and $\psi_j(\tau_j, t; x) = l$, respectively. In [3] it was shown that there is a $c > 0$ such that for $s \leqslant t \leqslant s + c$, $T_2(t,s)$ has the representation:

$$T_2(t,s)(u_0, v_0, d_0) = (u(t,s), v(t,s), d(t,s))$$

where

$$u^i(t, s, x) = \exp - \int_{\tau_i(t,x)}^{t} c_{ii}(\sigma, \varphi_i(\sigma, t; x))\mathrm{d}\sigma \Bigg[\sum_{j=N+1}^{n} e_{ij}(\tau_i(t, x)) \cdot$$

$$\cdot \left(\exp - \int_{s}^{\tau_i(t,x)} c_{jj}(\sigma, \psi_j(\sigma, \tau_i(t, x); 0))\mathrm{d}\sigma\right) v_0^j(\psi_j(s, \tau_i(t, x); 0))\Bigg]$$

if $x \leqslant \varphi_i(t, s; 0)$ and

$$u^i(t, s, x) = \left(\exp - \int_{s}^{t} c_{ii}(\sigma, \varphi_i(\sigma, t; x))\mathrm{d}\sigma\right) u_0^i(\varphi_i(s, t; x))$$

if $\varphi_i(t, s; 0) \leqslant x \leqslant l$;

$$v^j(t, s, x) = \exp - \int_{\tau_j(t,x)}^{t} c_{jj}(\sigma, \psi_j(\sigma, t; x))\mathrm{d}\sigma \Bigg[d^j(\tau_j(t, x), s) +$$

$$+ \sum_{k=1}^{N} d_{jk}(\tau_j(t, x)) \left(\exp - \int_{s}^{\tau_j(t,x)} c_{kk}(\sigma, \varphi_k(\sigma, \tau_j(t, x); l))\mathrm{d}\sigma\right) u_0^k(\varphi_k(s, \tau_j(t, x); l))\Bigg]$$

if $\psi_j(t, s; l) \leqslant x \leqslant l$, and

$$v_j(t, s, x) = \left(\exp - \int_{s}^{t} c_{jj}(\sigma, \psi_j(\sigma, t; x))\mathrm{d}\sigma\right) v_0^j(\psi_j(s, t; x))$$

f $0 \leqslant x \leqslant \psi_j(t, s; l)$;

$$d(t, s) = X(t, s)d_0 + \int_{s}^{t} X(t, \sigma)[G(\sigma)D(\sigma) + F(\sigma)]u(\sigma, s, l)\mathrm{d}\sigma,$$

$X(t, s)$ being the fundamental matrix of the linear system $\dot{d} = G(t)d$.

When necessary we use the notation $T(t, s, r, q)$, $T_2(t, s, r, q)$, $\varphi_i(t, s; x, r)$, etc., to emphasize the dependence with respect to the parameters r and q. We start by showing that $T(t, s, r, q)$ is well behaved as a function of q.

LEMMA 4. *For each $a \geqslant 1$, and $s \leqslant t$, the evolution operator $T(t, s, r, q)$ is continuous in q with respect to the norm of $L(X_p)$ uniformly in r, $|r| \leqslant a$, $\inf(|\lambda_i(t, x)|, |\mu_j(t, x)|) \geqslant 1/a$.*

Proof. From arguments similar to the ones used previously and Gronwall's inequality it suffices to prove the lemma for $s \leqslant t \leqslant s + c$ and $C(t, x) = 0$; in this case,

$$T(t, s, r, q)(u_0, v_0, d_0) = (u(t, s), v(t, s), d(t, s))$$

with

$$u^i(t, s, x) = \sum_{j=N+1}^{n} e_{ij}(\tau_i(t, x))\, v_0^j(\psi_j(s, \tau_i(t, x); 0)) \quad \text{if } 0 \leqslant x \leqslant \varphi_i(t, s; 0)$$

$$u^i(t, s, x) = u_0^i(\varphi_i(s, t; x)) \quad \text{if } \varphi_i(t, s; 0) \leqslant x \leqslant l$$

$$v^j(t, s; x) = d^j(\tau_j(t, x)) + \sum_{k=1}^{N} d_{jk}(\tau_j(t, x)) u_0^k(\varphi_k(s, \tau_j(t, x); l))$$

if $\psi_j(t, s; l) \leqslant x \leqslant l$;

$$v^j(t, s; x) = v_0^j(\psi_j(s, t; x)) \quad \text{if } 0 \leqslant x \leqslant \psi_j(t, s; l);$$

$$d(t, s) = X(t, s)d_0 + \int_s^t X(t, \sigma)[G(\sigma)D(\sigma) + F(\sigma)]\, u(\sigma, s; l)\, d\sigma,$$

where $X(t, s)$ is as before. The conclusion follows from this explicit representation and the lemma is proved.

LEMMA 5. *For each q, s, t, $s \leqslant t$, $T_2(t, s, q, r)$ and $T_2^*(t, s, q, r)$ are strongly continuous in r.*

Proof. Since the maps $x \to \psi_j(s, \tau_i(t, x); 0, r)$, $x \to \varphi_i(s, t; x, r)$, $x \to \tau_j(t, x, r)$, $x \to \varphi_k(s, \tau_j(t, x);\ l, r)$, $x \to \psi_j(s, t; x, r)$ are diffeomorphisms, the lemma follows from the explicit representation of $T_2(t, s, r, q)$ and $T_2^*(t, s, r, q)$.

The proof of the next result can be found in [3].

LEMMA 4. *For each q and r, the sets*

$$\{(T(t, s; r, q) - T_4(t, s; r, q)\pi)(w),\ 0 \leqslant s \leqslant t \leqslant \tau,\ |w| \leqslant 1\}$$

and

$$\{(T_2(t, s; r, q) - T_3(t, s; r, q))(w),\ |w| \leqslant 1,\ 0 \leqslant s \leqslant t \leqslant \tau\}$$

are precompact; in particular, for each q and r, $T(\tau, 0; r, q) - T_4(\tau, 0; r, q)\pi$ is a compact operator.

LEMMA 7. *For each* q, s, t, $0 \leqslant s \leqslant t \leqslant \tau$, *the map* $r \to T(t, s; r, q) - T_4(t, s; r, q)\pi$ *is continuous with respect to the norm of* $L(X_p)$, $1 < p < \infty$; *in particular, for each* $q, r \to T(\tau, 0; r, q) - T_4(\tau, 0; r, q)\pi$ *is continuous with respect to the norm of* $L(X_p)$.

Proof. As before, it suffices to prove the lemma for $s \leqslant t \leqslant s + c$. First we claim that $r \to T_2(t, s, r, q) - T_3(t, s, r, q)$ is continuous with respect to the norm of $L(X_p)$; in fact, $(T_2(t, s, r, q) - T_3(t, s, r, q))(u_0, v_0, d_0) = (0, w(t, s, r), d(t, s, r) - d_0)$, where

$$w^j(t, s, x, r) = \exp - \int_{\tau_j(t,x;r)}^{t} c_{jj}(\sigma, \psi_j(\sigma, t; x, r))\mathrm{d}\sigma\, (d^j(\tau_j(t, x; r), s) - d_0^j)$$

if $\psi_j(t, s; l, r) \leqslant x \leqslant l$

$$w^j(t, s, x, r) = 0 \quad \text{if } 0 \leqslant x \leqslant \psi_j(t, s; l, r)$$

$$d(t, s, r) = X(t, s)d_0 + \int_s^t X(t, \sigma)[G(\sigma)D(\sigma) + F(\sigma)]u(\sigma, s, l, r)\mathrm{d}\sigma$$

with

$$u^i(\sigma, s, l, r) = \exp - \int_s^\sigma c_{ii}(\xi, \varphi_i(\xi, \sigma, l, r))\mathrm{d}\xi\, u_0^i(\varphi_i(s, \sigma, x, r));$$

beginning with the component $d(t, s, r)$, we see each $d^j(t, s, r)$ is equal to $\tilde{d}_0^j(t, s)$ plus the sum of terms of the form

$$\int_s^t \tilde{c}_i(t, \sigma)\left(\exp - \int_s^\sigma c_{ii}(\xi, \varphi_i(\xi, \sigma, l, r))\mathrm{d}\xi\right) u_0^i(\varphi_i(s, \sigma, l, r))\mathrm{d}\sigma;$$

defining a change of variable in σ by $\varphi_i(s, \sigma, l, r) = y$ and denoting by $\sigma = h_i(s, y, l, r)$ its inverse, those terms become

$$\int_l^{\varphi_i(s,t,l,r)} \tilde{C}_1(t, h_i(s, y, l, r)) \frac{\partial h_i}{\partial y}(s, y, l, r)\cdot$$

$$\cdot \exp\left(- \int_s^{h_i(s,y,l,r)} c_{ii}(\xi, \varphi_i(\xi, h_i(s, y, l, r)))\mathrm{d}\xi\right) u_0^i(y)\mathrm{d}y$$

and looking at this expression as a function from the set of the parameters r into the space of the linear operators in some L_p space we see clearly that it is norm continuous. The terms $w_j(t, s, r)$ have a similar treatment (the inequality $\int_a^b |f(t)|dt \leqslant \leqslant (b-a)^{1-(1/p)}|f|_p$ has to be used) and the claim is proved. Next we show $r \to T(t, s, r, q) - T_2(t, s, r, q)$ is continuous with respect to the norm of $L(X_p)$; in fact, according to Theorem II defining $B(\sigma)(u_0, v_0, d_0)(x) = \left(\tilde{C}(\sigma, x)\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, 0\right)$ we have to verify that $r \to \int_s^t T_2(t, \sigma, r, q)\, B(\sigma)\, T_2(\sigma, s, r, q) d\sigma$ is continuous with respect to the norm of $L(X_p)$ but, in view of Lemma 7 and the claim we have just proved, all we have to do is to show $r \to \int_s^t T_3(t, \sigma, r, q) B(\sigma) T_3(\sigma, s, r, q) d\sigma$ has that property. The third component of this operator is zero and so we have to worry about the first two; those components can be written as the sum of terms which have been computed explicitly (see [3]) and a typical one has the form

$$\int_{s_{ikj}(t,s,x,r)}^{\tau_i(t,x,r)} \gamma_{ijk}(t, \sigma, x, r) v_0^k(\beta(s, \sigma, x, r)) d\sigma,$$

where $\frac{\partial \beta}{\partial \sigma} \neq 0$; defining a change of variable $\beta(s, \sigma, x, r) = z$ and denoting by $\sigma = g(s, z, x, r)$ its inverse, the integral above becomes

$$\int_{\tilde{s}_{ikj}(t,s,x,r)}^{\tilde{\tau}_i(t,x,r)} \tilde{\gamma}_{ijk}(t, y, x, r) v_0(y) dy$$

and clearly this expression has the desired property. The same type of argument can be used to show that

$$r \to T_3(t, s, r, q) - T_4(t, s, r, q)\pi = T_3(t, s, r, q)(I - \pi)$$

is continuous with respect to the norm of $L(X_p)$ and the lemma is proved.

Putting together Theorem I, the remarks following it, Theorem II, Lemmas 4, 5, 4 and 7 we can state:

THEOREM III. *Let* $r_0 = (\lambda_1(t, x), \ldots, \lambda_N(t, x), \mu_{N+1}(t, x), \ldots, \mu_n(t, x))$ *and* $q_0 = (E(t), D(t), F(t), G(t), C(t, x))$ *be points in the spaces of the parameters* r *and* q *defined previously and suppose there are a neighborhood* V_0 *of* r_0 *and constants* α_0 *and* $M > 0$ *such that* $|T_4(t, s, r, q_0)| \leqslant M e^{\alpha_0(t-s)}$, $s \leqslant t$, *for any* r *in* V_0. *Suppose also that for some* $\alpha > \alpha_0$, $T(\tau, 0, r_0, q_0)$ *has no spectrum on the circle* $|\lambda| = e^{\alpha\tau}$. *If* $\lambda_1, \ldots, \lambda_m$ *are the eigenvalues of* $T(\tau, 0; r_0, q_0)$ *outside the disk of radius* $e^{\alpha\tau}$ *and* $\varepsilon > 0$ *is given, then there are* $\delta > 0$ *and* $K(\varepsilon, M) > 0$ *such that for* $|r - r_0| < \delta$ *and* $|q - q_0| < \delta$ *the following is true:*

(i) $T(\tau, 0; r, q)$ *has no spectrum on the circle of radius* $e^{\alpha\tau}$ *and its spectrum outside the disk of radius* $e^{\alpha\tau}$ *consists of eigenvalues lying in the union of the balls* $\{\lambda : |\lambda - \lambda_j| < \varepsilon, j = 1, 2, \ldots, m\}$;

(ii) $|P_j(r, q) - P_j(r_0, q_0)| < \varepsilon$, *where* $P_j(r, q)$ *is the spectral projection corresponding to the curve* $|\lambda - \lambda_j| = \varepsilon$;

(iii) $|T(t, s; r, q) P(r, q)| \leqslant K e^{\alpha(t-s)}$, $s \leqslant t$, *where* $P(r, q)$ *is the spectral projection corresponding to the curve* $|\lambda| = e^{\alpha\tau}$.

In order to use Theorem III in concrete situations we have to get for $T_4(t, s)$ an estimate of the type $|T_4(t, s)| \leqslant M e^{\alpha_0(t-s)}$ and, in general, this is not an easy matter. Next we show that, in the autonomous case, to get an estimate of such a type is equivalent to locate the zeros of an entire function which can be written down explicitly; in fact, following [3], we define the $N \times N$ matrix $Y_0^{11}(x, y, \lambda)$ as a diagonal one whose entries are $\exp\left(-\lambda \int_y^x \frac{ds}{\lambda_i(s)} - \int_y^x \frac{c_{ii}(s)}{\lambda_i(s)} ds\right)$, the $(n - N) \times (n - N)$ diagonal matrix whose entries are

$$\exp\left(-\lambda \int_y^x \frac{ds}{\mu_j(as)} - \int_y^x \frac{c_{jj}(s)}{\mu_j(s)} ds\right)$$

and the entire function $h_4(\lambda) = \det(I - D Y_0^{11}(l, 0, \lambda) E (Y_0^{22}(l, 0, \lambda))^{-1})$.

Notice that $h_4(\lambda)$ has the form $1 - \sum_{k=1}^{N_0} b_k e^{\lambda w_k}$, where the w_k's are real and positive and in order to take into account its dependence on the coefficients $(\lambda_1(x), \ldots \ldots, \mu_n(x))$ we will use the notation $h_4(\lambda, r)$. Defining $\gamma_0(r) = \sup\{\operatorname{Re} \lambda : h_4(\lambda, r) = 0\}$ it has been shown in [3] that for any $\alpha_0 > \gamma_0$ there is a constant M such that $|T_4(t, r)| \leqslant M e^{\alpha_0 t}$, $t \geqslant 0$; however, if $\gamma_0(r) \leqslant \alpha_1 < \alpha_0$ for any r in a neighborhood of some r_0 then, showing the constant M can be chosen the same for all r's in that neighborhood requires some work; that will be done next.

We start by discussing a result due to Pitt [4]. We say that a function $h(t) = \sum_{k=1}^{\infty} b_k e^{i w_k t}$, t and w_k real, belongs to the class U if $\sum_{k=1}^{\infty} |b_k| < \infty$. Pitt's result says that if h belongs to U and $|h(t)| \geqslant m > 0$ for all t, then $1/h$ also belongs to U; that is, $\frac{1}{h(t)} = \sum_{k=1}^{\infty} c_k e^{i\gamma_n t}$, with $\sum_{k=1}^{\infty} |c_k| = M < \infty$ and a close look at the proof shows that if h is a finite sum $\sum_{k=1}^{L} a_k e^{iw_k t}$ then M depends just on L, m and $\sup |a_k|$.

The next step is to bound finite exponential sums uniformly from below in a band free of zeros. First let us remark that in some applications we have to work with a class of problems for which the functions $\lambda_1(x), \ldots, \mu_n(x)$ are not independent; to be more specific, taking for instance $n = 4$ and $N = 2$, there are physical problems giving rise to systems for which $\lambda_1(x) = -\mu_1(x)$, $\lambda_2(x) = -\mu_2(x)$, identically on $[0, l]$. In order to take care of those cases we consider entire functions $h_4(\lambda)$ of the form $h_4(\lambda, r_1, \ldots, r_k) = \sum_{i=1}^{L} a_i e^{-\lambda s_i} - 1$, where $s_i = \sum_{k=1}^{k} n_{i,k} r_k$, n_{ik} being fixed non-negative integers such that for each i, $n_{ik} \neq 0$ for some k.

LEMMA 8. *Let $Q \subset \mathbf{R}^k_{++}$ be a bounded set whose closure is still contained in $\mathbf{R}^k_{++}$ and suppose there is a band $a_0 < \operatorname{Re}\lambda < b_0$ where $h_4(\lambda, r_1, \ldots, r_k)$ never vanishes for any $(r_1, \ldots, r_k)$ in Q. Then for $a_0 < a_1 < b_1 < b_0$ there is an $m > 0$ such that $|h_4(\lambda, r_1, \ldots, r_N)| \geqslant m$ for $a_0 < a_1 \leqslant \operatorname{Re}\lambda \leqslant b_1 < b_0$ and any $(r_1, \ldots, r_k)$ in Q.*

Proof. By contradiction assume there are sequences $(r_{1,j}, \ldots, r_{k,j})$ and $\lambda_j = x_j + it_j$ such that $|h_4(\lambda_j; r_{1,j}, \ldots, r_{k,j})| \to 0$ as $j \to +\infty$; passing to a subsequence we can assume $(r_{1,j}, \ldots, r_{k,j})$ converges to $(r_1, \ldots, r_N)$, x_j converges to x, $a_1 \leqslant x \leqslant b_1$, and $t_j r_{i,j} \to \theta_i \pmod{2\pi}$, $i = 1, \ldots, k$. Let $g_j(\lambda)$ be defined by

$$g_j(\lambda) = h_4(\lambda + it_j; r_{i,j}, \ldots, r_{k,j}) = \sum_{i=1}^{L} a_i e^{-(\lambda + it_j)s_{i,j}} - 1;$$

since $s_{i,j} = \sum_{l=1}^{k} n_{i,l} r_{l,j}$ converges to $s_i = \sum_{l=1}^{k} n_{i,l} r_l$ and $t_j s_{i,j}$ converges to some $\beta_i \pmod{2\pi}$ it follows that $g_j(\lambda)$ converges to some $g(\lambda) = \sum_{i=1}^{k} \tilde{a}_i e^{-\lambda s_i} - 1$ uniformly on compact sets of the band $a_0 < \operatorname{Re}\lambda < b_0$; moreover, $g(x) = 0$ because

$$h_4(x_j + it_j; r_{1,j}, \ldots, r_{k,j}) - h_4(x + it_j; r_{1,j}, \ldots, r_{k,j}) \to 0$$

and then by Hurwitz theorem $g(\lambda) \equiv 0$ and this is impossible because $g(\lambda) \to -1$ as $\operatorname{Re}\lambda \to +\infty$ and this proves the lemma.

If we go back to the proof of the main results in [3] and we use Lemma 7 and the remarks preceding it we conclude that the estimate $|T_4(t, r)| \leqslant M e^{\alpha_0 t}$, $t \geqslant 0$, for some constants M and α_0, holds for any r giving rise to $r_1, \ldots, r_k$ in a set Q as above if there is an α_1 such that $\gamma_0(r) \leqslant \alpha_1 < \alpha_0$ for any r in Q. The location of the zeros of finite exponential sums and their dependence with $r_1, \ldots, r_k$ have been studied in [2].

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