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ON THE STRUCTURE OF CONTRACTION OPERATORS WITH DOMINATING SPECTRUM

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INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$.

In this paper we show that an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ whose spectrum is dominating for the unit circle belongs to the class $\mathbf{A}_1(r)$, for some r. In the last years various criteria for membership in the classes $\mathbf{A}_1(r)$ have been obtained (see [1], [2], [3], [4], [8], [10], [11], [15]). Unfortunately, the abstract criterion from [10] is not applicable in the present context. However, our proof relies heavily on the techniques appearing in [10]. The main idea is to apply those methods to some compressions of T corresponding to different parts of its spectrum. Combining the rank-one operators constructed at the first step we obtain another ones close to the given element in the predual Q_T of the dual algebra generated by T in $\mathscr{L}(\mathscr{H})$.

In the first section we recall some useful definitions and results from the theory of dual algebras. We also recall some facts concerning the minimal coisometric extension of a given contraction and list some technical lemmas from [10] for future use.

In the second part we begin by proving some lemmas, treating parts of the spectrum of T. The main intermediate result of this section is Lemma 2.5 which shows how to approximate elements in Q_T by rank-one classes. After that, the proof of the main theorem becomes easier and it is very similar with that appearing in [10, Theorem 4.7].

1. NOTATIONS AND TERMINOLOGY

We recall some definitions and results from the theory of dual algebras (see [4] for basic of dual algebras). If $\mathscr{C}_1(\mathscr{H})$ denotes the space of trace-class operators on $\mathscr{H}$

then it is well-known that $\mathscr{L}(\mathscr{H}) = (\mathscr{C}_1(\mathscr{H}))^*$ via the bilinear map

$$\langle T, L\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{L}(\mathscr{H}),\ L \in \mathscr{C}_1(\mathscr{H}).$$

A dual algebra is, by definition, a weak* closed subalgebra of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$. If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is a dual algebra and $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, where ${}^{\perp}\mathscr{A}$ denotes the preannihilator of $\mathscr{A}$ in $\mathscr{C}_1(\mathscr{H})$, then $\mathscr{A} = (Q_{\mathscr{A}})^*$ via the bilinear map

$$\langle T, [L]\rangle = \mathrm{tr}(TL), \quad T \in \mathscr{A},\ \ [L] \in Q_{\mathscr{A}}.$$

(Here $[L]$ denotes the coset in $Q_{\mathscr{A}}$ containing the trace-class operator L.)

If $T \in \mathscr{L}(\mathscr{H})$ then $\mathscr{A}_T$ denotes the dual algebra generated by T in $\mathscr{L}(\mathscr{H})$. If x and y are vectors from $\mathscr{H}$ then the rank-one operator defined by $(x\otimes y)z = (z, y)x$, $z \in \mathscr{H}$, belongs to $\mathscr{C}_1(\mathscr{H})$ and satisfies $\mathrm{tr}(x \otimes y) = (x, y)$ and $\|x \otimes y\| = \|x \otimes y\|_1 = \|x\|\,\|y\|$. Moreover, if $B \in \mathscr{L}(\mathscr{H})$, then $B(x \otimes y) = Bx \otimes y$.

A dual algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is said to have property $(\mathbf{A}_1(r))$, for some $r \geqslant 1$ if for each $[L]$ in $Q_{\mathscr{A}}$ and $s > r$, there exist vectors x and y in $\mathscr{H}$ satisfying

$$(1) \qquad [L] = [x \otimes y]$$

and

$$(2) \qquad \|x\|\,\|y\| < s\,\|[L]\|.$$

Let $\mathbf{D}$ denote the open unit disc in $\mathbf{C}$ and let $\mathbf{T} = \partial\mathbf{D}$. A set $S \subset \mathbf{D}$ is said to be dominating for $\mathbf{T}$ if almost every point of $\mathbf{T}$ is a nontangential limit of a sequence of points from S. As usual, H^∞ denotes the Banach algebra of all bounded analytic functions on $\mathbf{D}$. It is well-known that $H^\infty = (L^1/H_0^1)^*$, where L^1 and H^1 are the Lebesgue and Hardy spaces on $\mathbf{T}$ and H_0^1 consists of all those f in H^1 satisfying $\int_0^{2\pi} f(e^{it})\,dt = 0$.

Suppose now that $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction (i.e., a contraction whose unitary summand is either absolutely continuous or acts on the space $\{0\}$). For such T, the Sz.-Nagy—Foiaş functional calculus

$$\Phi_T\colon H^\infty \to \mathscr{A}_T$$

is a weak* continuous algebra homomorphism such that $\|\Phi_T\| \leqslant 1$ and $\Phi_T(z) = T$, where z denotes the position function (see [7] and [14]).

The class $\mathbf{A} = \mathbf{A}(\mathscr{H})$ consists of all absolutely continuous contractions in $\mathscr{L}(\mathscr{H})$ for which Φ_T is an isometry. If $T \in \mathbf{A}(\mathscr{H})$ then one knows (cf. [4, Theorem 4.1]) that Φ_T is a weak* homeomorphism between H^∞ and $\mathscr{A}_T$ and there exists an isometry φ_T from Q_T onto (L^1/H_0^1) such that $\Phi_T = \varphi_T^*$. Let $\lambda \in \mathbf{D}$ and let P_λ denote the Poisson kernel

$$P_\lambda(e^{it}) = (1 - |\lambda|^2)|1 - \bar{\lambda}e^{it}|^{-2}, \quad e^{it} \in \mathbf{T}.$$

If $[C_\lambda] = \varphi_T^{-1}([P_\lambda])$, then it is easy to verify that

$$\langle f(T), [C_\lambda]\rangle = f(\lambda), \quad f \in H^\infty.$$

For each $r \geqslant 1$, $\mathbf{A}_1(r)$ denotes the class of all those T in $\mathbf{A}$ for which $\mathscr{A}_T$ has property $(\mathbf{A}_1(r))$.

For any $T \in \mathscr{L}(\mathscr{H})$, $\sigma(T)$ denotes, as usual, the spectrum of T. Furthermore, let $\sigma_e(T)$ denote the essential (Calkin) spectrum of T and let $\sigma_{le}(T)$ denote the left essential spectrum of T. If H is a hole in $\sigma_e(T)$ (i.e., a bounded component of $\mathbf{C} \setminus \sigma_e(T)$) then $i(H)$ denotes the Fredholm index of H.

The following notations from [10] will be used frequently in the paper. If $T \in \mathscr{L}(\mathscr{H})$ then

$$\mathscr{F}_-(T) = \{H \,;\, H \subset \sigma(T),\ H \text{ is a hole in } \sigma_e(T) \text{ and } i(H) \leqslant 0\},$$

$$\mathscr{F}'_+(T) = \{H \,;\, H \subset \sigma(T),\ H \text{ is a hole in } \sigma_e(T) \text{ and } i(H) > 0\}.$$

Let also denote:

$$\sigma_{if}(T) = \{\lambda \,;\, \lambda \in \sigma(T) \setminus (\sigma_e(T) \cup \mathscr{F}_-(T)),\ i(T - \lambda) = 0\}.$$

It follows from spectral theory that for any $T \in \mathscr{L}(\mathscr{H})$, $\|T\| \leqslant 1$ we have

(3) $$\sigma(T) \cap \mathbf{D} = (\sigma_e(T) \cap \mathbf{D}) \cup \mathscr{F}_-(T) \cup \mathscr{F}'_+(T) \cup \sigma_{if}(T).$$

In the following we shall review some useful facts about the minimal coisometric extension of a given contraction. Recall from [14] that if $T \in \mathscr{L}(\mathscr{H})$ and $\|T\| \leqslant 1$, then there exist a Hilbert space $\mathscr{K}$ and a coisometry $B \in \mathscr{L}(\mathscr{K})$ satisfying

(4) $$\mathscr{K} \supset \mathscr{H}$$

(5) $$B\mathscr{H} \subset \mathscr{H}$$

and

(6) $$Bh = Th, \quad \forall h \in \mathscr{H}.$$

We suppose also that B is minimal, which means that

(7) $$\mathscr{K} = \bigvee_{n \geqslant 0} B^{*n}\mathscr{H};$$

then B is unique, up to an isomorphism.

Since $B^* \in \mathscr{L}(\mathscr{K})$ is an isometry, there exists a decomposition

$$B = S^* \oplus R \tag{8}$$

corresponding to the decomposition

$$\mathscr{K} = \mathscr{P} \oplus \mathscr{R} \tag{9}$$

where, if $\mathscr{P} \neq 0$, $S \in \mathscr{L}(\mathscr{P})$ is a unilateral shift and, if $\mathscr{R} \neq 0$, $R \in \mathscr{L}(\mathscr{R})$ is a unitary operator. If T is absolutely continuous, then R is also absolutely continuous (cf. [14, p. 84]).

Suppose now that $T \in \mathscr{L}(\mathscr{H})$, $\|T\| \leqslant 1$ and let $\mathscr{H}_1 \subset \mathscr{H}$ be a semi-invariant subspace for T (i.e., $\mathscr{H}_1 = \mathscr{M} \ominus \mathscr{L}$, where $\mathscr{L} \subset \mathscr{M} \subset \mathscr{H}$ are invariant subspaces for T). If we denote $T_{\mathscr{H}_1} = P_{\mathscr{H}_1} T | \mathscr{H}_1$, then $T_{\mathscr{H}_1}$ satisfies

$$(T_{\mathscr{H}_1})^n = (T^n)_{\mathscr{H}_1} \quad n \geqslant 1. \tag{10}$$

Moreover, if T is absolutely continuous, then $T_{\mathscr{H}_1}$ is also absolutely continuous. If $B \in \mathscr{L}(\mathscr{K})$ is the minimal coisometric extension of T and

$$\mathscr{K}_1 = \bigvee_{n \geqslant 0} B^{*n} \mathscr{H}_1 \tag{11}$$

then $\mathscr{K}_1 \subset \mathscr{K}$ is a semi-invariant subspace for B and

$$B_1 = B_{\mathscr{K}_1} \tag{12}$$

is a minimal coisometric extension for $T_{\mathscr{H}_1}$. Throughout the paper, given a contraction T in $\mathscr{L}(\mathscr{H})$ and a semi-invariant subspace $\mathscr{H}_1 \subset \mathscr{H}$, we assume that the minimal coisometric extension $B_1 \in \mathscr{L}(\mathscr{K}_1)$ of $T_{\mathscr{H}_1}$ is that given by (11) and (12).

If $T \in \mathrm{A}(\mathscr{H})$ and $B \in \mathscr{L}(\mathscr{K})$ is its minimal coisometric extension, then the projection of $\mathscr{K}$ onto $\mathscr{P}$ will be denoted by Q and the projection of $\mathscr{K}$ onto $\mathscr{R}$ will be denoted by A.

The following three lemmas from [10] will be used in the sequel:

LEMMA 1.1. ([10, Lemma 3.5]). *Suppose $T \in \mathrm{A}(\mathscr{H})$ with the minimal coisometric extension $B \in \mathscr{L}(\mathscr{K})$. Then $B \in \mathrm{A}(\mathscr{K})$, $\Phi_T \circ \Phi_B^{-1}$ is an isometry and weak* homomorphism from $\mathscr{A}_B$ onto $\mathscr{A}_T$ and $j = \varphi_B^{-1}\varphi_T$ is a linear isometry of Q_T onto Q_B. Moreover*

$$j([C_\lambda]_T) = [C_\lambda]_B, \quad \lambda \in \mathbf{D}$$

and

$$j([x \otimes y]_T) = [x \otimes y]_B, \quad x, y \in \mathscr{H}.$$

LEMMA 1.2. ([10, Lemma 3.6]). *If $T \in A(\mathscr{H})$ and $B \in A(\mathscr{K})$ is its minimal coisometric extension, then for each $x, y \in \mathscr{H}$ and $w, z \in \mathscr{K}$*

$$\|[x \otimes y]_T\| = \|[x \otimes y]_B\|$$

$$[x \otimes z]_B = [x \otimes P_{\mathscr{H}} z]_B$$

and

$$[w \otimes z]_B = [Qw \otimes Qz]_B + [Aw \otimes Az]_B.$$

LEMMA 1.3. ([10, Lemma 3.7]). *If $T \in A(\mathscr{H})$ with the minimal coisometric extension $B \in A(\mathscr{K})$ and (x_n) is a sequence in $\mathscr{H}$ such that*

$$\|[x_n \otimes y]_T\| \to 0 \quad \forall y \in \mathscr{H}$$

then

$$\|[x_n \otimes z]_B\| \to 0 \quad \forall z \in \mathscr{K}$$

$$\|[Qx_n \otimes z]_B\| \to 0 \quad \forall z \in \mathscr{K}$$

and

$$\|[Ax_n \otimes z]_B\| \to 0 \quad \forall z \in \mathscr{K}.$$

The following lemma will be used in the proof of Lemma 2.2.

LEMMA 1.4. *Suppose $T \in A(\mathscr{H})$ and B in $A(\mathscr{K})$ its minimal coisometric extension. If (z_n) is any sequence in $\mathscr{P}$ that converges weakly to zero, then*

$$\|[y \otimes z_n]_B\| \to 0 \quad \forall y \in \mathscr{K}.$$

Proof. If $\mathscr{P} = 0$ then the result is trivial. If $\mathscr{P} \neq 0$ then, for every y in $\mathscr{K}$

$$\|[y \otimes z_n]_B\| = \sup_{\substack{f \in H^\infty \\ \|f\| = 1}} |(f(B)y, z_n)| =$$

$$= \sup_{\substack{f \in H^\infty \\ \|f\| = 1}} |(f(S^*)Qy, z_n)| = \|[Qy \otimes z_n]_{S^*}\|.$$

This last term tends to zero by [7, Lemma 4.4], since $S^* \in A(\mathscr{P}) \cap C_0.$.

Another result that will be needed is the following.

LEMMA 1.5. *Suppose $T \in A(\mathscr{H})$ with its minimal coisometric extension $B \in A(\mathscr{K})$. Let $\mathscr{K}_1$ be an invariant subspace for T^* and let $B_1 \in \mathscr{L}(\mathscr{K}_1)$ be the minimal coisometric extension of $T_1 = T_{\mathscr{H}_1}$. If $x \in \mathscr{K}$ and $y \in \mathscr{K}_1$ then*

$$[P_{\mathscr{H}_1} x \otimes y]_B = [x \otimes P_{\mathscr{H}_1} y]_B.$$

Proof. Let $f \in H^\infty$ and let $\tilde{f}(z) = \bar{f}(\bar{z})$, $z \in \mathbf{D}$. Since $P_{\mathscr{H}_1}B_1^* = T_1^*P_{\mathscr{H}_1}$, we obtain

$$(f(B)P_{\mathscr{H}_1}x, y) = (f(B_1)P_{\mathscr{H}_1}x, y) = (P_{\mathscr{H}_1}x, \tilde{f}(B_1^*)y) =$$

$$= (x, P_{\mathscr{H}_1}\tilde{f}(B_1^*)y) = (x, \tilde{f}(T_1^*)P_{\mathscr{H}_1}y) = (x, \tilde{f}(T^*)P_{\mathscr{H}_1}y) =$$

$$= (f(T)x, P_{\mathscr{H}_1}y) = (f(B)x, P_{\mathscr{H}_1}y).$$ ∎

The following lemma will be an essential tool in proving Theorem 2.1.

LEMMA 1.6. *Suppose $T \in \mathbf{A}(\mathscr{H})$ with the minimal coisometric extension $B \in \mathscr{L}(\mathscr{P} \oplus \mathscr{R})$. Let $\mathscr{H}_1 \subset \mathscr{H}$ be a semi-invariant subspace for T and let $B_1 \in \mathscr{L}(\mathscr{P}_1 \oplus \mathscr{R}_1)$ be the minimal coisometric extension of $T_1 = T_{\mathscr{H}_1}$. Suppose that $B_1 \in \mathbf{A}(\mathscr{P}_1 \oplus \mathscr{R}_1)$. Suppose also that $0 < \rho < 1$, $\varepsilon > 0$, $\delta > 0$, $a \in \mathscr{H}_1$, $w \in \mathscr{P}_1$, $b \in \mathscr{R}_1$, $z_1 \in \mathscr{P}_1 \oplus \mathscr{R}_1$, $z \in \mathscr{P}$ and $\{\alpha_1, \ldots, \alpha_N\} \subset \mathbf{C}$ are given such that $\sum_{i=1}^{N} |\alpha_i| < \delta$. Let $\{x_n^i\}_{n=1}^\infty \subset \mathscr{H}_1$, $1 \leqslant i \leqslant N$, be given such that $\|x_n^i\| \leqslant 1$ and*

$$\lim_{n\to\infty} \|[x_n^i \otimes y]\| = 0, \quad y \in \mathscr{H},\ 1 \leqslant i \leqslant N. \tag{13}$$

Then there exist a n-tuple $\nu_0 = (n_1^0, \ldots, n_N^0)$, $a_1 \in \mathscr{H}_1$, $w_1 \in \mathscr{P}_1$ and $b_1 \in \mathscr{R}_1$ such that

$$\left\| \sum_{i=1}^{N} \alpha_i [x_{n_i^0}^i \otimes x_{n_i^0}^i]_B + [a \otimes (w + b)]_B - [a_1 \otimes (w_1 + b_1)]_B \right\| < \varepsilon \tag{14}$$

$$\|a_1 - a\| < 3\delta^{1/2} \tag{15}$$

$$\|w_1 - w\| < \delta^{1/2} \tag{16}$$

$$\|b_1\| < \frac{1}{\rho} (\|b\| + \delta^{1/2}) \tag{17}$$

$$\|[(a_1 - a) \otimes z]_B\| < \varepsilon \tag{18}$$

and

$$\|[z_1 \otimes (w_1 - w)]_B\| < \varepsilon. \tag{19}$$

Proof. Most of it is an easy adaptation of the proof of [10, Proposition 4.6]. Only he following modifications are needed:

i) The isometry $j = \varphi_B^{-1} \circ \varphi_T$ must be replaced by $j_1 = \varphi_{B_1}^{-1} \circ \varphi_T$.

ii) Theorem 3.11 from [10] can be made to work for an absolutely continuous contraction. Therefore, it can be applied in the setting T_1, B_1.

iii) In this way one obtains (14) to (17).

iv) To obtain (18) and (19) recall that $a_1 - a = u_\nu + v$, where u_ν is of the form $\sum_{i=1}^{N} \sqrt{\alpha_i} x^i_{n_i}$, $v \in \mathscr{H}_1$ with $\|Q_1 x\|$ small enough and $w_1 - w = \sum_{i=1}^{N} \sqrt{\alpha_i} x^i_{n_i}$. It follows from (13) and Lemma 1.4 that n_i can be chosen to satisfy (14) to (17) and

$$\|[u_\nu \otimes z]_B\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|[z_1 \otimes (w_1 - w)]_B\| < \frac{\varepsilon}{2}.$$

Recall from [14, p. 68] that

$$\|Qx\|^2 = \|x\|^2 - \lim_n \|T^n x\|^2$$

and similarly for Q_1 and T_1.

Since $\mathscr{H}_1$ is semi-invariant for T, it follows that $\|T_1^n v\| \leqslant \|T^n v\|$ for each $n \geqslant 1$, therefore $\|Qv\| \leqslant \|Q_1 v\|$. Since v can be chosen to satisfy $\|Q_1 v\| < \varepsilon/(2\|z\| + 1)$ with $z \in \mathscr{P}$ we get

$$\|[v \otimes z]_B\| \leqslant \|[Qv \otimes z]_B\| \leqslant \|Qv\| \|z\| < \frac{\varepsilon}{2}$$

and the proof of (18) is finished.

2. A SPECTRAL CRITERION FOR MEMBERSHIP IN $\mathbf{A}_1(r)$

The central theorem of the paper is the following:

THEOREM 2.1. *Let $T \in \mathscr{L}(\mathscr{H})$ be any absolutely continuous contraction such that $\sigma(T) \cap \mathbf{D}$ is dominating for $\mathbf{T}$. Then $T \in \mathbf{A}_1(r)$, for some $r < 4^2 6^2$.*

The proof of this theorem will be accomplished by proving a sequence of lemmas.

Our program is the following. Using (3) we cut the spectrum of T into three parts, each of them having different signs in the spectral picture of T (see [12] for the terminology). To each part we associate a set of elements in Q_T that are norm limits of rank-one operators satisfying some vanishing conditions. Once we have established these facts, we use the dominancy of $\sigma(T)$ together with Lemma 1.6 from above to obtain a certain rank-one operator close to a given element in Q_T (see Lemma 2.5 below). As mentioned in the introduction, this will be the crucial step in the proof of Theorem 2.1.

If S is a subset of $\mathbf{D}$ then NTL(S) will denote the set of all nontangential limit points of S. Let Γ be a Borel subset of $\mathbf{T}$ such that $m(\Gamma) > 0$ (here m denotes the normalized Lebesgue measure on $\mathbf{T}$). Then we denote by $\tilde{\chi}_\Gamma = \chi_\Gamma/(m(\Gamma))$ the normalized characteristic function of Γ and let $[\tilde{\chi}_\Gamma]$ be its image in the quotient space L^1/H^1_0. We also denote $[\tilde{\chi}_\Gamma]_T = \varphi_T^{-1}([\tilde{\chi}_\Gamma])$.

The following lemma shows that if $\Gamma \subset \mathrm{NTL}(\sigma_{\mathrm{if}}(T))$ and $m(\Gamma) > 0$, then $[\tilde{\chi}_\Gamma]_T$ belongs to $E_0^l(\mathscr{A}_T)$ (in the terminology of [10]).

LEMMA 2.2. *Let T in $\mathbf{A}(\mathscr{H})$ the with minimal coisometric extension $B \in \mathscr{L}(\mathscr{P} \oplus \mathscr{R})$. Suppose that $m(\mathrm{NTL}(\sigma_{\mathrm{if}}(T))) > 0$. For any Borel subset Γ of $\mathrm{NTL}(\sigma_{\mathrm{if}}(T))$ there exists an orthogonal sequence $\{x_n\}$ in the unit ball of $\mathscr{H} \cap \mathscr{P}$ such that*

$$\|[\tilde{\chi}_\Gamma]_T - [x_n \otimes x_n]_T\| \to 0 \tag{20}$$

and

$$\|[z \otimes x_n]_T\| \to 0 \quad \forall z \in \mathscr{H}. \tag{21}$$

Proof. Let $\mathscr{M} = \bigvee \mathrm{Ker}(T-\lambda)$. Since $R = B|\mathscr{R}$ is unitary one sees that $\mathscr{M} \subset \mathscr{P}$. Let $\lambda \in \sigma_{\mathrm{if}}(T)$, $\Gamma \subset \mathrm{NTL}(\sigma_{\mathrm{if}}(T))$ and $\varepsilon > 0$; it follows from the proof of [5, Lemma 1.2] that there exists $\{\lambda_i\}_{i=1}^N \subset \sigma_{\mathrm{if}}(T)$ and $\{\alpha_i\}_{i=1}^N \subset \mathbf{R}^+$ such that

$$\|\tilde{\chi}_\Gamma - \sum_{i=1} \alpha_i P_{\lambda_i}\|_1 < \varepsilon \quad \text{and} \quad \sum \alpha_i \leqslant 1.$$

Thus

$$\left\|[\tilde{\chi}_\Gamma]_T - \sum_{i=1}^N \alpha_i [C_{\lambda_i}]_T\right\| < \varepsilon.$$

With [15, Theorem 2.2] one gets $x_1 \in \bigvee_{i=1}^N \mathrm{Ker}(T-\lambda_i)$ satisfying $[x_1 \otimes x_1]_T = \sum_{i=1}^N \alpha_i [C_{\lambda_i}]_T$. Therefore $\|x_1\|^2 = \sum \alpha_i \leqslant 1$ and

$$\|[\tilde{\chi}_\Gamma]_T - [x_1 \otimes x_1]_T\| < \varepsilon.$$

Let $\mathscr{N} = \mathscr{M} \ominus \bigvee_{i=1}^N \mathrm{Ker}(T-\lambda_i)$. Then $\mathscr{N}$ is semi-invariant for T and $\sigma_{\mathrm{if}}(T) \setminus \{\lambda_1, \ldots, \lambda_N\} \subset \sigma_{\mathrm{if}}(T_{\mathscr{N}})$. By repeating the above argument one gets $\{\lambda'_1, \ldots, \lambda'_{N'}\} \subset \sigma_{\mathrm{if}}(T) \setminus \{\lambda_1, \ldots, \lambda_N\}$ and $x_2 \in \bigvee_{i=1}^{N'} \mathrm{Ker}(T_{\mathscr{N}} - \lambda'_i)$ such that $\|x_2\| \leqslant 1$ and

$$\|[\tilde{\chi}_\Gamma]_T - [x_2 \otimes x_2]_{..}\| < \frac{\varepsilon}{2}.$$

Using this procedure, we construct, by induction, an orthogonal sequence $\{x_n\}$ in the unit ball of $\mathscr{P} \cap \mathscr{H}$ satisfying (20). Since $\{x_n\}$ converges weakly to 0 it follows from Lemma 1.4 that $\|[y \otimes x_n]_T\| \to 0$ for each $y \in \mathscr{H}$. The proof is finished. ▨

The following lemma deals with the "positive part" of the spectrum.

LEMMA 2.3. *Suppose $T \in \mathbf{A}(\mathscr{H})$ and let $B = S^* \oplus R \in \mathscr{L}(\mathscr{P} \oplus \mathscr{R})$ be its minimal coisometric extension. If $\mu \in \mathscr{F}'_+(T) \cup (\sigma_{le}(T) \cap \mathbf{D})$ then there exists a sequence $\{z_n\}$ in the unit ball of $\mathscr{P}$ such that*

$$\|[C_\mu]_T - [P_{\mathscr{H}} z_n \otimes P_{\mathscr{H}} z_n]_T\| \to 0 \tag{22}$$

and

$$\|[x \otimes P_{\mathscr{H}} z_n]_T\| \to 0 \quad \forall x \in \mathscr{H}. \tag{23}$$

Proof. If $\mu \in \sigma_{le}(T)$ then it follows from [12, Proposition 2.15] that there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that

$$\|(T - \mu)x_n\| \to 0. \tag{24}$$

It is well-known (cf. [9, Theorem 3.1]) that such a sequence satisfies

$$\|[C_\mu]_T - [x_n \otimes x_n]_T\| \to 0$$

and

$$\|[x \otimes x_n]_T\| \to 0 \quad x \in \mathscr{H}.$$

From (24) one easily gets

$$\|Qx_n - x_n\| \to 0$$

and hence $\|P_{\mathscr{H}} Q x_n - x_n\| \to 0$. With $z_n = Qx_n$, (22) and (23) are satisfied.

If $\mu \in \mathscr{F}'_+(T)$ then it follows from elementary Fredholm theory combined with [9, Lemma 2.3] and [10, Lemma 5.2] that there exists an orthonormal sequence $(x_n) \subset \bigvee_{n \geq 1} \operatorname{Ker}(T-\mu)^n \in \mathscr{P}$ satisfying (25) and (26). ▨

Before proving the next lemmas, we introduce some notations. If $T \in \mathbf{A}(\mathscr{H})$ and $B \in \mathbf{A}(\mathscr{K})$ is its minimal coisometric extension, where $\mathscr{K} = \mathscr{P} \oplus \mathscr{R}$ and $B = S^* \oplus R$, then we denote

$$\Lambda_1 = \Lambda_1(T) = \mathscr{F}_-(T) \cup \{\sigma_e(T) \setminus \sigma_{le}(T)\} \tag{24}$$

$$\Lambda_2 = \Lambda_2(T) = \mathscr{F}'_+(T) \cup \{\sigma_{le}(T) \cap \mathbf{D}\} \tag{25}$$

$$\mathscr{H}_1 = \mathscr{H}_1(T) = \bigvee_{\substack{n \geq 0 \\ \lambda \in \Lambda_1}} \operatorname{Ker}(T^* - \bar{\lambda})^n \tag{26}$$

and

$$(27) \qquad \mathcal{H}_2 = \mathcal{H}_2(T) = \{P_{\mathcal{H}} w \,;\; w \in \mathcal{P}\}^-.$$

(Here $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.) Since $T^* P_{\mathcal{H}} w = P_{\mathcal{H}} B^* w$, $w \in \mathcal{P}$, it follows that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant subspaces for T^*. We denote

$$(28) \qquad T_1 = T_{\mathcal{H}_1} \quad \text{and} \quad T_2 = T^* | \mathcal{H}_2 .$$

Let also $B' \in \mathcal{L}(\mathcal{K}')$ denote the minimal coisometric extension of T^* and let $B' = S'^* \oplus R'$ be the canonical decomposition of B', where $S' \in \mathcal{L}(\mathcal{P}')$ and $R' \in \mathcal{L}(\mathcal{R}')$. We also denote by $B_i \in \mathcal{L}(\mathcal{K}_i)$, $i = 1, 2$ the minimal coisometric extensions of T_i. The spaces $\mathcal{P}_i$, $\mathcal{R}_i$ and the projections A', Q', A_i, Q_i, $i = 1, 2$ are defined appropriately. Since $T^* \mathcal{H}_2 \subset \mathcal{H}_2$ one easily sees that $B' \mathcal{H}_2 \subset \mathcal{H}_2$ therefore $\mathcal{H}_2$ reduces B'.

Recall that $C_{0\cdot} = C_{0\cdot}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}),\ \|T\| \leqslant 1 \text{ and } \|T^n x\| \to 0,\ x \in \mathcal{H}\}$ and that $C_{\cdot 0} = (C_{0\cdot})^*$. It is easy to see (cf. [9, Proposition 2.8]) that if $\mathcal{H}_1 \neq \{0\}$, then $T_1 \in C_{\cdot 0}$ hence $\mathcal{P}_1 \neq \{0\}$. Since $S_1^* \in \mathrm{A}(\mathcal{P}_1)$ is a part of B_1, it follows that $B_2 \in \mathrm{A}(\mathcal{K}_2)$ iff $\mathcal{H}_2 \neq \{0\}$.

We treat now the "negative part" of $\sigma(T)$.

LEMMA 2.4. *Suppose $T \in \mathrm{A}(\mathcal{H})$ and let $\mu \in \mathcal{F}_-(T) \cup (\sigma_e(T) \setminus \sigma_{le}(T))$. Then there exists an orthonormal sequence (x_n) in $\mathcal{H}_1$ such that*

$$(29) \qquad [C_\mu]_T = [x_n \otimes x_n]_T, \quad n \in \mathbf{N}$$

and

$$(30) \qquad \|[x_n \otimes z]_T\| \to 0 \quad \forall z \in \mathcal{H}.$$

Proof. If $\mu \in \mathcal{F}_-(T)$ then it follows from [9, Lemmas 2.2 and 2.3] and [10, Lemma 5.2] that there exists an orthonormal sequence $\{x_n\} \subset \bigvee_{n \geqslant 1} \mathrm{Ker}(T^* - \bar{\mu})^n$ satisfying (29) and (30). On the other hand, if $\mu \in \sigma_e(T) \setminus \sigma_{le}(T)$ then by virtue of [12, Proposition 2.15] we have $\dim \mathrm{Ker}(T^* - \bar{\mu}) = \aleph_0$ and accordingly to [13, Corollary 3.5 and Lemma 3.6] each orthonormal sequence $\{x_n\}$ in $\mathrm{Ker}(T^* - \bar{\mu})$ satisfies (29) and (30). ∎

Recall from [4, Proposition 4.6] that every absolutely continuous contraction $T \in \mathcal{L}(\mathcal{H})$ with $\sigma(T) \cap \mathbf{D}$ dominating for $\mathbf{T}$ belongs to $\mathrm{A}(\mathcal{H})$.

We are now prepared to link up all the above results. The main idea is to apply Lemma 1.6 to the compression of T to the subspace $\mathcal{H}_1$ and to the restriction of T^* to $\mathcal{H}_2$. The sequences of rank one operators appearing in the statement of Lemma 1.6 are furnished by the above three lemmas. Using (18) and (19) we shall see that the cross-terms can be made sufficiently small. This in turn implies that a rank-one operator can be constructed to satisfy (32) to (35).

LEMMA 2.5. *Suppose T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ such that $\sigma(T) \cap \mathbf{D}$ is dominating for $\mathbf{T}$. Suppose also that $0 < \rho < 1$, $[L] \in Q_T$, $a \in \mathscr{H}_1$, $w \in \mathscr{P}_1$, $b \in \mathscr{R}_1$, $a' \in \mathscr{H}_2$, $w' \in \mathscr{P}_2$, $b' \in \mathscr{R}_2$, $\delta > 0$ and $\varepsilon > 0$ are given such that*

$$\|[L]_T - [(a + P_{\mathscr{H}}(w' + b')) \otimes (a' + P_{\mathscr{H}_1}(w + b))]_T\| < \delta. \tag{31}$$

Then there exist $a_1 \in \mathscr{H}_1$, $w_1 \in \mathscr{P}_1$, $b_1 \in \mathscr{R}_1$, $a_1' \in \mathscr{H}_2$, $w_1' \in \mathscr{P}_2$ and $b_1' \in \mathscr{R}_2$ such that

$$\|[L]_T - [(a_1 + P_{\mathscr{H}}(w_1' + b_1')) \otimes (a_1' + P_{\mathscr{H}_1}(w_1 + b_1))]_T\| < \varepsilon \tag{32}$$

$$\|a_1 - a\| < 3\delta^{1/2}, \quad \|a_1' - a'\| < 3\delta^{1/2} \tag{33}$$

$$\|w_1 - w\| < \delta^{1/2}, \quad \|w_1' - w'\| < \delta^{1/2} \tag{34}$$

and

$$\|b_1\| < \frac{1}{\rho}(\|b\| + \delta^{1/2}), \quad \|b_1'\| < \frac{1}{\rho}(\|b'\| + \delta^{1/2}). \tag{35}$$

Proof. Let

$$[L_1]_T = [L]_T - [(a + P_{\mathscr{H}}(w' + b')) \otimes (a' + P_{\mathscr{H}_1}(w + b))]_T \tag{36}$$

and set $d = \|[L_1]\|_T$ so $0 \leqslant d < \delta$. If $d = 0$, just set $a_1 = a$, $a_1' = a'$, $w_1 = w$, $w_1' = w'$, $b_1 = b$ and $b_1' = b'$. Thus we may suppose that $d > 0$. Let us recall from (24) and (25) that $\Lambda_1 = \mathscr{F}_-(T) \cup (\sigma_e(T) \setminus \sigma_{le}(T))$ and $\Lambda_2 = \mathscr{F}'_+(T) \cup (\sigma_e(T) \cap \mathbf{D})$. Since $\sigma(T) \cap \mathbf{D}$ is dominating for $\mathbf{T}$ it follows (cf. [4, Proposition 1.21]) that

$$\overline{\mathrm{aco}}\{([C_\lambda]_T\,;\, \lambda \in \Lambda_1 \cup \Lambda_2) \cup ([\tilde{\chi}_\Gamma]_T;\, \Gamma \subset (\mathrm{NTL}(\sigma_{\mathrm{if}}(T)))\}$$

equals the closed unit ball in Q_T. Therefore there exist $\{\lambda_i\}_{i=1}^{N_1} \subset \Lambda_1$, $\{\lambda_i\}_{i=N_1+1}^{N_2'} \subset \Lambda_2$, $\Gamma_i \subset \mathrm{NTL}(\sigma_{\mathrm{if}}(T))$, $N_2' < i \leqslant N_2$ and complex numbers $\{\alpha_i\}_{i=1}^{N_2}$ such that $\sum_{i=1}^{N_2} |\alpha|_i < d$ and

$$\left\| [L_1]_T - \sum_{i=1}^{N_2'} \alpha_i [C_{\lambda_i}] - \sum_{i=N_2'+1}^{N_2} \alpha_i [\tilde{\chi}_{\Gamma_i}]_T \right\| < \frac{\varepsilon}{2}. \tag{37}$$

Thus from Lemmas 2.2, 2.3 and 2.4 there exist sequences $\{x_n^i\}_{n=1}^{\infty}$, $1 \leqslant i \leqslant N_2$ in the unit ball of $\mathscr{H}$ such that

$$(x_n^i) \subset \mathscr{H}_1, \ [C_{\lambda_i}]_T = [x_n^i \otimes x_n^i]_T, \quad n \geqslant 1 \tag{38}$$

and $\forall t \in \mathscr{H}$, $\|[x_n^i \otimes t]_T\| \to 0$, $1 \leqslant i \leqslant N_1$

$$(x_n^i) \subset \mathscr{H}_2, \ \|[C_{\lambda_i}]_T - [x_n^i \otimes x_n^i]_T\| \to 0 \quad \text{as } n \to \infty \tag{39}$$

and $\forall t \in \mathscr{H}$, $\|[t \otimes x_n^i]_T\| \to 0$, $N_1 < i \leqslant N_2'$

$$(x_n^i) \subset \mathscr{H}_2, \ \|[\tilde{\lambda}_{r_i}]_T - [x_n^i \otimes x_n^i]_T\| \to 0 \quad \text{as } n \to \infty \tag{39}$$

and $\forall t \in \mathscr{H}$, $\|[t \otimes x_n^i]_T\| \to 0$, $N_2' < i \leqslant N_2$.

It follows that there exists $n_0 \in \mathbf{N}$ such that for any N_2-tuple $= (n_1, \ldots, n_{N_2})$ with $n_i \geqslant n_0$, $1 \leqslant i \leqslant N_2$, we have

$$\left\| [L_1]_T - \sum_{i=1}^{N_2} \alpha_i [x_{n_i}^i \otimes x_{n_i}^i]_T \right\| < \frac{\varepsilon}{2}. \tag{40}$$

Choose $w_0 \in \mathscr{P}$ so that

$$\|a' - P_{\mathscr{H}} w_0\| < \frac{\varepsilon}{2^4 \cdot 3\delta^{1/2}}. \tag{41}$$

Using Lemma 1.6 in the setting $(\mathscr{H}_1, T_{\mathscr{H}_1})$ one gets $\tilde{a}_1 \in \mathscr{H}_1$, $w_1 \in \mathscr{P}_1$, $b_1 \in \mathscr{A}_1$, and an N_1-tuple $v_0 = (n_1^0, \ldots, n_{N_1}^0)$ such that

$$\begin{aligned} \Bigg\| \sum_{i=1}^{N_1} \alpha_i [x_{n_i^0}^i \otimes x_{n_i^0}^i]_B + [(a + P_{\mathscr{H}_1} P_{\mathscr{H}}(w' + b')) \otimes (w + b)]_B - \\ - [(a + \tilde{a}_1 + P_{\mathscr{H}_1} P_{\mathscr{H}}(w' + b')) \otimes (w_1 + b_1)]_B \Bigg\| < \frac{\varepsilon}{8} \end{aligned} \tag{42}$$

$$\|\tilde{a}_1\| < 3\delta^{1/2} \tag{43}$$

$$\|w_1 - w\| < \delta^{1/2} \tag{44}$$

$$\|b_1\| < \frac{1}{\rho} (\|b\| + \delta^{1/2}) \tag{45}$$

and

$$\|[\tilde{a}_1 \otimes w_0]_B\| < \frac{\varepsilon}{16} \cdot \tag{46}$$

From (41) and (46), we get:

$$\|[\tilde{a}_1 \otimes a']_B\| \leqslant \|[a_1 \otimes (a' - P_{\mathscr{H}} w_0)]_B\| + \|[\tilde{a}_1 \otimes P_{\mathscr{H}} w_0]_B\| < \frac{\varepsilon}{8} \cdot \tag{47}$$

Another application of Lemma 1.6 in the setting $(\mathscr{H}_2, T^*|\mathscr{H}_2)$ yields vectors $a_1' \in \mathscr{H}_2$, $w_1' \in \mathscr{P}_2$, $b_1' \in \mathscr{R}_2$ and an $(N_2 - N_1)$-tuple $v_0' = (n^0_{N_1+1}, \ldots, n^0_{N_2})$ such that

$$\left\| \sum_{N_1 < i \leqslant N_2} \tilde{\alpha}_i [x^i_{n^0_i} \otimes x^i_{n^0_i}]_{B'} + [a' \otimes (w' + b')]_{B'} - [a_1' \otimes (w_1' + b_1')]_{B'} \right\| < \frac{\varepsilon}{8} \tag{48}$$

$$\|a_1' - a'\| < 3\delta^{1/2} \tag{49}$$

$$\|w_1' - w'\| < \delta^{1/2} \tag{50}$$

$$\|b_1'\| < \frac{1}{\rho} (\|b'\| + \delta^{1/2}) \tag{51}$$

$$\|[(a_1' - a') \otimes (a + a_1)]_{B'}\| < \frac{\varepsilon}{16} \tag{52}$$

and

$$[P_{\mathscr{K}_2} P_{\mathscr{H}_1} (w_1 + b_1) \otimes (w_1' - w')]_{B'}\| < \frac{\varepsilon}{8} \cdot \tag{53}$$

Since $B'\mathscr{K}_2 \subset \mathscr{K}_2$, from (53) one gets

$$\|[P_{\mathscr{H}_1}(w_1 + b_1) \otimes (w_1' - w')]_{B'}\| < \frac{\varepsilon}{8} \tag{54}$$

or using Lemma 1.2 and passing to Q_T

$$\|[P_{\mathscr{H}}(w_1' - w') \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T\| < \frac{\varepsilon}{16} \cdot \tag{55}$$

Let us denote $a_1 = a + \tilde{a}_1$. Since $T^*\mathscr{H}_1 \subset \mathscr{H}_1$, from Lemma 1.5 we obtain

$$(56)\qquad \begin{aligned}&[(a + P_{\mathscr{H}_1} P_{\mathscr{H}}(w' + b')) \otimes (w + b)]_B = \\ &= [(a + P_{\mathscr{H}}(w' + b')) \otimes P_{\mathscr{H}_1}(w + b)]_B\end{aligned}$$

and similarly for a_1, w_1 and b_1.

From (42) and the above identities we get by passing to Q_T:

$$(57)\qquad \begin{aligned}&\Big\| \sum_{i=1}^{N_1} \alpha_i [x^i_{n_i^0} \otimes x^i_{n_i^0}]_T + [(a + P_{\mathscr{H}}(w' + b')) \otimes P_{\mathscr{H}_1}(w + b)]_T - \\ &- (a_1 + P_{\mathscr{H}}(w' + b')) \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T \Big\| < \frac{\varepsilon}{8}.\end{aligned}$$

Using Lemma 1.2 and passing to Q_{T^*} we get from (48):

$$\begin{aligned}&\Big\| \sum_{N_1 < i \leq N_2} \bar{\alpha}_i [x^i_{n_i^0} \otimes x^i_{n_i^0}]_{T^*} + [a' \otimes P_{\mathscr{H}}(w' + b')]_{T^*} - \\ &- [a'_1 \otimes P_{\mathscr{H}}(w'_1 + b'_1)]_{T^*} \Big\| < \frac{\varepsilon}{16}.\end{aligned}$$

Therefore

$$(58)\qquad \begin{aligned}&\Big\| \sum_{N_1 < i \leq N_2} \alpha_i [x^i_{n_i^0} \otimes x^i_{n_i^0}]_T + [P_{\mathscr{H}}(w' + b') \otimes a']_T - \\ &- [P_{\mathscr{H}}(w'_1 + b'_1) \otimes a'_1]_T \Big\| < \frac{\varepsilon}{16}.\end{aligned}$$

Let

$$[L_2]_T = [L]_T - [(a_1 + P_{\mathscr{H}}(w'_1 + b'_1)) \otimes (a'_1 + P_{\mathscr{H}_1}(w_1 + b_1))]_T.$$

We estimate the norm of $[L_2]_T$. We have

$$\begin{aligned}\|[L_2]_T\| \leq \Big\| [L]_T - [(a + P_{\mathscr{H}}(w' + b')) \otimes \\ \otimes (a' + P_{\mathscr{H}_1}(w + b))]_T - \sum_{i=1}^{N_2} \alpha_i [x^i_{n_i^0} \otimes x^i_{n_i^0}] \Big\| + \Big\| \sum_{i=1}^{N_1} \alpha_i [x^i_{n_i^0} \otimes x^i_{n_i^0}]_T + \\ + [(a + P_{\mathscr{H}}(w' + b')) \otimes P_{\mathscr{H}_1}(w + b)]_T - [(a_1 + P_{\mathscr{H}}(w' + b')) \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T \Big\| + \\ \Big\| \sum_{N_1 < i \leq N_2} \alpha_i [x^i_{n_i^0} \otimes x^i_{n_i^0}]_T + [P_{\mathscr{H}}(w' + b') \otimes a']_T - \\ - [P_{\mathscr{H}}(w'_1 + b'_1) \otimes a'_1]_T \Big\| + \|[a \otimes a']_T - [a_1 \otimes a'_1]_T\| + \\ + \|[P_{\mathscr{H}}(w' + b' - w'_1 - b'_1) \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T\| = A + B + C + D + E.\end{aligned}$$

It follows from (36) and (40) that $A < \varepsilon/2$. From (57) and (58) we get $B + C < \varepsilon/4$. Let us estimate now the last terms D and E. From (47) and (52) we obtain

$$D = \|[a_1 \otimes a_1']_T - [a \otimes a']_T\| \leqslant \|[(a_1 - a) \otimes a']_T\| +$$

$$+ \|[a_1 \otimes (a_1' - a')]_T\| < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8} \cdot$$

Let us show that

$$(59) \qquad [P_{\mathscr{H}}(b_1' - b') \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T = 0.$$

Indeed, for each $f \in H^\infty$, we have:

$$(f(T)P_{\mathscr{H}}(b_1' - b'),\ P_{\mathscr{H}_1}(w_1 + b_1)) =$$

$$= (b_1' - b',\ \tilde{f}(T^*)P_{\mathscr{H}_1}(w_1 + b_1)) = (b_1' - b',\ \tilde{f}(T_1^*)P_{\mathscr{H}_1}(w_1 + b_1)).$$

Since $\mathscr{H}_1 \subset \bigvee_{\substack{n \geqslant 0 \\ \lambda \in \mathbf{D}}} \operatorname{Ker}(T^* - \bar{\lambda})^n \subset \mathscr{P}'$, and $b_1' - b' \in \mathscr{R}_1' \subset \mathscr{R}'$ it follows that $b_1' - b'$ is orthogonal onto $\mathscr{H}_1$. Finally, from (55) and (59) one obtains

$$\|[P_{\mathscr{H}}(w_1' - w' + b_1' - b') \otimes P_{\mathscr{H}_1}(w_1 + b_1)]_T\| < \frac{\varepsilon}{8} \cdot$$

Therefore $\|[L_2]\| < (\varepsilon/2) + (\varepsilon/4) + (\varepsilon/8) + (\varepsilon/8) = \varepsilon$ and the proof is finished.

Proof of Theorem 2.1. Fix $[L]_T \in Q_T$ such that $0 \neq \|[L]\| < \delta = 1/4$. Let $\{s_n\}_{n=1}^\infty$ be a sequence of positive numbers strictly decreasing to $1/2$ such that $s_1 = 1$ and define $\rho_0 = 1$ and $\rho_n = s_{n+1}/s_n$, $n \in \mathbf{N}$. Set $a_i = a_i' = 0$, $b_i = b_i' = 0$ and $w_i = w_i' = 0$ for $i = 0, 1$. Let $n \geqslant 1$ and suppose that for each k satisfying $0 \leqslant k \leqslant n$, vectors $a_k \in \mathscr{H}_1$, $w_k \in \mathscr{P}_1$, $b_k \in \mathscr{R}_1$, $a_k' \in \mathscr{H}_2$, $w_k' \in \mathscr{P}_2$ and $b_k' \in \mathscr{R}_2$ have been chosen so that for $k = 1, \ldots, n$]

$$(60)_k \qquad \|[L]_T - [(a_k + P_{\mathscr{H}}(w_k' + b_k')) \otimes (a_k' + P_{\mathscr{H}_1}(w_k + b_k))]_T\| < \delta^k$$

$$(61)_k \qquad \|a_k - a_{k-1}\| < 3\delta^{\frac{k-1}{2}}, \quad \|a_k' - a_{k-1}'\| < 3\delta^{\frac{k-1}{2}}$$

$$(62)_k \qquad \|w_k - w_{k-1}\| < \delta^{\frac{k-1}{2}}, \quad \|w_k' - w_{k-1}'\| < \delta^{\frac{k-1}{2}}$$

and

$$(63)_k \quad \|b_k\| < \frac{1}{\rho_{k-1}}\left(\|b_{k-1}\| + \delta^{\frac{k-1}{2}}\right), \quad \|b'_k\| < \frac{1}{\rho_{k-1}}\left(\|b_{k-1}\| + \delta^{\frac{k-1}{2}}\right).$$

Then applying Lemma 2.5, we deduce the existence of vectors $a_{n+1} \in \mathscr{H}_1$, $w_{n+1} \in \mathscr{P}_1$, $b_{n+1} \in \mathscr{R}_1$, $a'_{n+1} \in \mathscr{H}_2$, $w'_{n+1} \in \mathscr{P}_2$ and $b'_{n+1} \in \mathscr{R}_2$ such that inequalities $(60)_{n+1}$ to $(63)_{n+1}$ are fulfilled for $k = n + 1$. Therefore, by induction, one can construct the sequences $(a_n) \subset \mathscr{H}_1$, $(w_n) \subset \mathscr{P}_1$, $(b_n) \subset \mathscr{R}_1$, $(a'_n) \subset \mathscr{H}_2$, $(w'_n) \subset \mathscr{P}_2$ and $(b'_k) \subset \mathscr{R}_2$ satisfying $(60)_n$ to $(63)_n$ for all $n \geqslant 1$. It is clear from (1) and (62) that (a_n), (w_n), (a'_n) and (w'_n) are Cauchy sequences. Define

$$a = \lim a_n, \quad a' = \lim a'_n, \quad w = \lim w_n, \quad w' = \lim w'_n.$$

Using (61) and (62) one easily sees that

$$\|a\| \leqslant \frac{3}{1 - \delta^{1/2}}, \quad \|a'\| \leqslant \frac{3}{1 - \delta^{1/2}}, \quad \|w\| < \frac{1}{1 - \delta^{1/2}}, \quad \|w'\| \leqslant \frac{1}{1 - \delta^{1/2}}.$$

Furthermore, by iterating $(63)_n$ we obtain

$$\frac{1}{2}\|b_n\| \leqslant s_n\|b_n\| \leqslant \sum_{k=1}^{n-1} s_k \delta^{k/2} \leqslant \sum_{k=1}^{\infty} \delta^{k/2}$$

and therefore

$$\|b_n\| \leqslant \frac{2}{1 - \delta^{1/2}}, \quad \|b'_n\| \leqslant \frac{2}{1 - \delta^{1/2}}.$$

Without loss of generality we may suppose that (b_n) converges weakly to b and (b'_n) converges weakly to b'.

It remains to show that

$$\{[(a_n + P_{\mathscr{K}}(w'_n + b'_n)) \otimes (a'_n + P_{\mathscr{K}_1}(w_n + b_n))]_T\}_{n=1}^{\infty}$$

converges weakly to

$$[(a + P_{\mathscr{K}}(w' + b')) \otimes (a' + P_{\mathscr{K}_1}(w + b))]_T.$$

For each $f \in H^{\infty}$, we have

$$|\langle f(T), [a_n \otimes P_{\mathscr{K}_1}(w_n + b_n)]_T - [a \otimes P_{\mathscr{K}_1}(w + b)]\rangle| \leqslant \|a_n - a\| \|P_{\mathscr{K}_1}(w_n + b_n)\| \|f\| +$$

$$+ |(f(T)a, P_{\mathscr{K}_1}(w_n + b_n - w - b))| \to 0 \quad \text{as } n \to \infty.$$

Similarly

$$|\langle f(T), [P_{\mathscr{H}}(w_n' + b_n') \otimes a_n']_T - [P_{\mathscr{H}}(w' + b') \otimes a']_T \rangle|$$

converges to 0 as $n \to \infty$.

Finally, we show that

$$\{[P_{\mathscr{H}}(w_n' + b_n') \otimes P_{\mathscr{H}_1}(w_n + b_n)]_T\}_{n=1}^{\infty}$$

converges weakly to

$$[P_{\mathscr{H}}(w' + b') \otimes P_{\mathscr{H}_1}(w + b)]_T.$$

Indeed, as we have remarked in the proof of Lemma 2.5

$$[P_{\mathscr{H}} b_n' \otimes P_{\mathscr{H}_1}(w_n + b_n)]_T = [P_{\mathscr{H}} b' \otimes P_{\mathscr{H}_1}(w + b)]_T = 0.$$

Therefore

$$|\langle f(T), [P_{\mathscr{H}}(w_n' + b_n') \otimes P_{\mathscr{H}_1}(w_n + b_n)]_T - [P_{\mathscr{H}}(w' + b') \otimes P_{\mathscr{H}_1}(w + b)]_T \rangle| =$$

$$= |\langle f(T), [P_{\mathscr{H}} w_n' \otimes P_{\mathscr{H}_1}(w_n + b_n)]_T - [P_{\mathscr{H}} w' \otimes P_{\mathscr{H}_1}(w + b)]_T \rangle|.$$

Since $\|w_n' - w\| \to 0$ and $\{w_n + b_n\}$ is bounded, the last term converges to zero. It follows that

$$[L]_T = [(a + P_{\mathscr{H}}(w' + b')) \otimes (a' + P_{\mathscr{H}_1}(w + b))]$$

with

$$\|a + P_{\mathscr{H}}(w' + b')\| \, \|a' + P_{\mathscr{H}_1}(w + b)\| \leqslant \frac{6^2}{(1 - \delta^{1/2})^2} = 4^2 6^2.$$

Therefore $T \in \mathbf{A}_1(r)$, with $1 \leqslant r < 4^2 6^2$.

After this paper was completed, I learned that H. Bercovici and B. Chevreau independently proved that $\mathbf{A} = \mathbf{A}_1(r)$ (Bercovici gets the best value $r = 1$), a fact which implies our main result (Theorem 2.1). On the way of proving $\mathbf{A} = \mathbf{A}_1(r)$ our result seems to be a natural one to check, and we hope that our proof shows a small part of the difficulty of this (now solved) problem.

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