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ABSORPTION SEMIGROUPS

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INTRODUCTION

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $1 \leqslant p < \infty$, and let $(U(t);\ t \geqslant 0)$ be a positive C_0-semigroup on $L_p(\mu)$, with generator T. Let $V : \Omega \to \mathbf{R}$ be measurable. The starting point of this paper was the problem to associate a natural C_0-semigroup $(U_V(t);\ t \geqslant 0)$ with the formal expression "$T - V$", for a large class of functions V. The fundamental idea is to approximate V by bounded functions and then use monotonicity arguments.

The "generating" differential equation for the semigroup $U_V(\cdot)$ is, formally,

$$u' = Tu - Vu.$$

If the semigroup $U(\cdot)$ is associated with a transport or diffusion process, then the function V will act as an absorption rate. It is for this reason that we use the term "absorption semigroup" for the perturbed semigroup $(U_V(t);\ t \geqslant 0)$.

The investigation of these problems, for the context indicated above, was begun in [17]. We refer to this paper for more motivation as well as for application to Schrödinger semigroups, i.e., the heat equation with absorption.

The motivation for the present paper is twofold. On the one hand the theory presented in [17] is made more complete. On the other hand, the paper of Lapidus [7] on a dominated convergence theorem for Schrödinger operators motivated us to derive a corresponding result for the general context of absorption semigroups.

In Section 1 we derive monotonicity and convexity properties for absorption semigroups with bounded absorption rates. In this section we put ourselves into a more general context, i.e., we treat absorption semigroups derived from a positive C_0-semigroup on a Banach lattice E. The absorption rates are then elements of the center $\mathscr{Z}(E)$ of $\mathscr{L}(E)$.

In Section 2 we treat unbounded absorption rates for the L_p-context indicated initially. For semibounded V we define $U(\cdot)$-admissibility by the requirement that approximation of V by cut-off yields convergence for the corresponding absorption semigroups. If a general V is such that V^+ and $-V^-$ are $U(\cdot)$-admissible then it is shown that $U_V(\cdot)$ can be defined in a natural way.

Section 3 is devoted to dominated convergence results. It turns out that dominating the positive parts of an a.e. convergent sequence (V_n) of absorption rates by a $U(\cdot)$-admissible absorption rate is not sufficient for the convergence of the corresponding absorption semigroups. For this reason we introduce the notion of $U(\cdot)$-regular absorption rates.

In Section 4 we give examples illustrating the notions of $U(\cdot)$-admissibility and $U(\cdot)$-regularity.

In an appendix we prove a result on positive C_0-semigroups on Banach lattices with order continuous norm.

Concluding this introduction we want to recall some facts concerning convergence of C_0-semigroups. Let E be a Banach space. Let $(U(t);\ t \geqslant 0)$, $(U_n(t);\ t \geqslant 0)$ $(n \in \mathbf{N})$ be C_0-semigroups on E. We write

$$U(\cdot) = \operatorname*{s-lim}_{n \to \infty} U_n(\cdot) \tag{0.1}$$

if $\sup_{0 \leqslant t \leqslant a} \|U(t)x - U_n(t)x\| \to 0$ $(n \to \infty)$ for all $x \in E$, $a > 0$. We refer to [10; Section 3.4] for a discussion. In particular, if T, T_n $(n \in \mathbf{N})$ are the generators of the above semigroups, then (0.1) implies $T_n \to T$ in strong resolvent sense.

1. MONOTONICITY AND CONVEXITY PROPERTIES

In this section let E be a real or complex Banach lattice, and let $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, accordingly. We refer to [13] for the terminology concerning Banach lattices.

The set

$$\mathscr{Z}(E) := \{A \in \mathscr{L}(E)\,;\ \exists\, c \geqslant 0 \text{ such that } \pm \operatorname{Re} A,\ \pm \operatorname{Im} A \leqslant cI\}$$

(where $\operatorname{Re} A = A$, $\operatorname{Im} A = 0$ if E is a real Banach lattice) is called the *center* of $\mathscr{L}(E)$. The elements of $\mathscr{Z}(E)$ are also called (bounded) *multiplication operators*.

1.1. Remarks. (a) For $A \in \mathscr{L}(E)$ the following conditions are equivalent:

(i) $A \in \mathscr{Z}(E)$;

(ii) $\operatorname{Re}(\gamma A) \leqslant \|A\| I$ for all $\gamma \in \mathbf{K}$, $|\gamma| = 1$;

(iii) A is *local*, i.e., $f, g \in E$, $f \perp g$ implies $Af \perp g$;

(iv) $AJ \subset J$ for every ideal J in E.

The implications (ii) ⇒ (i) ⇒ (iv) ⇒ (iii) are obvious. The remaining implication (iii) ⇒ (ii) follows from [18; Theorem 140.4] together with [8; Lemma 1.1]. See also [9; C — I, Section 9, p. 246].

(b) For all $A \in \mathscr{Z}(E)$ there exists $|A| = \sup\{\mathrm{Re}(\gamma A);\ \gamma \in \mathbf{K}, |\gamma| = 1\} \in \mathscr{Z}(E)$, and $|Af| = |A|\,|f|$ holds for all $f \in E$.

In the real case this follows from [18; Theorem 140.4]. In the complex case we note that (a) implies that A is order bounded, and therefore the statement follows from [1; Theorem 2.4].

(c) $\mathscr{Z}(E)$ is isomorphic to a space $C(K)$ (K compact), as a Banach lattice and a Banach algebra.

For the real case this is shown in [8; Proposition 1.5]. The complex case follows from the real case together with (b).

1.2. Lemma. *Let $A \in \mathscr{Z}(E)$.*

(a) *Then $e^A \in \mathscr{Z}(E)$.*

(b) *If A is real then $e^A \geq 0$.*

(c) *If E is a complex Banach lattice and A is real, then $|e^{iA}f| = |f|$ for all $f \in E$.*

Proof. For (a) and (b) cf. [9; C—II, Proposition 5.15, p. 288].

(c) is an easy consequence of Remarks 1.1 (b), (c). ▨

For the remainder of this section let $(U(t);\ t \geq 0)$ be a positive C_0-semigroup on E, and let T denote its generator.

Moreover, for $V \in \mathscr{Z}(E)$ we denote by $U_V(\cdot)$ the C_0-semigroup generated by $T - V$ (symbolically, $U_V(t) = e^{t(T-V)}$ $(t \geq 0)$).

1.3. Proposition. (Monotonicity properties). *Let $(\tilde{U}(t);\ t \geq 0)$ be a (second) positive C_0-semigroup on E, $\tilde{U}(t) \geq U(t)$ $(t \geq 0)$.*

(a) *Let $V \in \mathscr{Z}(E)$, V real. Then, for $t \geq 0$,*

$$\tilde{U}_V(t) \geq U_V(t) \geq 0.$$

(b) *Let $V_1, V_2 \in \mathscr{Z}(E)$, V_1, V_2 real, $V_1 \leq V_2$. Then, for $t \geq 0$,*

$$U_{V_1}(t) \geq U_{V_2}(t),$$

$$\tilde{U}_{V_1}(t) - \tilde{U}_{V_2}(t) \geq U_{V_1}(t) - U_{V_2}(t).$$

Proof. (a) For $n \in \mathbf{N}$ we have $(e^{-(t/n)V}\tilde{U}(t/n))^n \geq (e^{-(t/n)V}U(t/n))^n \geq 0$. For $n \to \infty$, the assertion follows from the Trotter product formula (cf. [4, Theorem 8.12]).

(The Trotter product formula

$$e^{t(\overline{R+S})} = \text{s-}\lim_{n\to\infty} (e^{(t/n)R}\, e^{(t/n)S}),$$

valid for the case that R, S, and $\overline{R+S}$ are generators of contraction semigroups on a Banach space F, is also true if R is just a generator and S is in $\mathscr{L}(F)$: There exists an equivalent norm $|||\cdot|||$ on F for which $|||e^{tR}||| \leqslant e^{\alpha t}$ $(t \geqslant 0)$, for some $\alpha \in \mathbf{R}$. Since also $|||e^{tS}||| \leqslant e^{t|||S|||}$ $(t \geqslant 0)$, the operators $R - \alpha I$, $S - |||S|||I$ satisfy the required properties in $(F, |||\cdot|||)$.)

(b) The first inequality holds since $T - V_1 = (T - V_2) + (V_2 - V_1)$ is a positive perturbation of $T - V_2$. The second inequality is a consequence of (a) and the Duhamel expansion:

$$\tilde{U}_{V_1}(t) - \tilde{U}_{V_2}(t) = \int_0^t \tilde{U}_{V_1}(t-s)(V_2 - V_1)\tilde{U}_{V_2}(s)\mathrm{d}s \geqslant$$

$$\geqslant \int_0^t U_{V_1}(t-s)(V_2 - V_1)U_{V_2}(s)\mathrm{d}s = U_{V_1}(t) - U_{V_2}(t). \qquad \square$$

1.4. Proposition. (Convexity). *Let* $V_1, V_2 \in \mathscr{X}(E)$, V_1, V_2 *real*, $0 \leqslant r \leqslant 1$. *Then, for* $t \geqslant 0$,

$$U_{(1-r)V_1+rV_2}(t) \leqslant (1-r)U_{V_1}(t) + rU_{V_2}(t).$$

Proof. Without restriction we may assume that E is a complex Banach lattice. Then $z \to U_{V_1+z(V_2-V_1)}(t)$ is holomorphic on $\mathbf{C}$; cf. [5; Chapter IX, Theorem 2.1]. Let $S := \{z \in \mathbf{C}\,; 0 \leqslant \operatorname{Re} z \leqslant 1\}$.

Let $f \in E_+$. For $z = \xi + i\eta \in S$, $n \in \mathbf{N}$, we use Lemma 1.2(c) to conclude

$$|(e^{-(t/n)i\eta(V_2-V_1)}U_{V_1+\xi(V_2-V_1)}(t/n))^n f| \leqslant$$

$$\leqslant U_{V_1+\xi(V_2-V_1)}(t)f.$$

For $n \to \infty$ the Trotter product formula implies

$$|U_{V_1+z(V_2-V_1)}(t)f| \leqslant U_{V_1+\xi(V_2-V_1)}(t)f.$$

Let $\varphi \in E'_+$. Then the function $S \ni z \to \langle U_{V_1+z(V_2-V_1)}(t)f, \varphi\rangle$ satisfies the hypotheses of the three lines theorem. Since, by the previous estimate, we have

$$|\langle U_{V_1+(\xi+i\eta)(V_2-V_1)}(t)f, \varphi\rangle| \leqslant \langle U_{V_1+\xi(V_2-V_1)}(t)f, \varphi\rangle$$

for all $0 \leqslant \xi \leqslant 1$, $\eta \in \mathbf{R}$, the three lines theorem implies that

$$[0, 1] \ni r \mapsto \log \langle U_{V_1+r(V_2-V_1)}(t)f, \varphi\rangle$$

is convex (cf. [11; Theorem 12.8]). As a consequence, for $0 \leqslant r \leqslant 1$,

$$\langle U_{V_1+r(V_2-V_1)}(t)f, \varphi\rangle \leqslant$$

$$\leqslant (1-r)\langle U_{V_1}(t)f, \varphi\rangle + r\langle U_{V_2}(t)f, \varphi\rangle =$$

$$= \langle((1-r)U_{V_1}(t) - rU_{V_2}(t))f, \varphi\rangle.$$

Since this holds for all $\varphi \in E'_+$ we obtain

$$U_{V_1+r(V_2-V_1)}(t)f \leqslant ((1-r)U_{V_1}(t) + rU_{V_2}(t))f.$$ ▨

2. ADMISSIBLE ABSORPTION RATES

In this section, let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $1 \leqslant p < \infty$. Further let $(U(t);\ t \geqslant 0)$ be a positive C_0-semigroup on $L_p(\mu)$, with generator T.

The aim of this section is to associate a C_0-semigroup with the formal expression "$T - V$" as generator, for suitable "absorption rates" $V: \Omega \to \mathbf{R}$, and to investigate the properties of the absorption semigroups thus defined. For measurable $V: \Omega \to \mathbf{R}$ we define $V^{\pm} := (\pm V) \vee 0$ and the truncation $V^{(n)} = (V^+ \wedge n) - (V^- \wedge n)$, for $n \in \mathbf{N}$.

If $V \in L_\infty(\mu)$ then $U_V(\cdot)$ was defined in the previous section as the C_0-semigroup with generator $T - V$. We shall investigate a class of absorption rates V for which a C_0-semigroup $U_V(\cdot)$ on $L_p(\mu)$ can be defined by

(2.1) $$U_V(t) := \text{s-}\lim_{n\to\infty} U_{V^{(n)}}(t) \quad (t \geqslant 0).$$

Whenever (2.1) defines a C_0-semigroup its generator will be denoted by T_V.

2.1. Definition. (a) Let V be bounded below. Then, for large $n \in \mathbf{N}$, we have $0 \leqslant U_{V^{(n+1)}}(t) \leqslant U_{V^{(n)}}(t)$, and therefore the limit (2.1) exists for all $t \geqslant 0$. V will be called *$U(\cdot)$-admissible* if $U_V(\cdot)$ is a C_0-semigroup.

(b) If V is bounded above then V will be called *$U(\cdot)$-admissible* if the limit (2.1) exists for all $t \geqslant 0$, and $U_V(\cdot)$ is a C_0-semigroup.

2.2. Proposition. *Let $V \leqslant 0$. Then V is $U(\cdot)$-admissible if and only if* $\sup\{\|U_{V^{(n)}}(t)\|\,;\, 0 \leqslant t \leqslant 1,\ n \in \mathbf{N}\} < \infty$.

Proof. "⇒" is obvious from $U_{V^{(n)}}(t) \leqslant U_V(t)$ $(t \geqslant 0)$.

"⇐". From $U_{V^{(n)}}(t) \leqslant U_{V^{(n+1)}}(t)$ $(t \geqslant 0)$ for all $n \in \mathrm{N}$ and the boundedness assumption we obtain the existence of the limit (2.1), the strong measurability of $U_V(\cdot)f$ for all $f \in E$, and $\sup\{\|U_V(t)\|\,;\ 0 \leqslant t \leqslant 1\} < \infty$. Moreover, $U(t) \leqslant U_V(t)$ $(t \geqslant 0)$, and therefore Proposition A.1 from the appendix implies that $U_V(\cdot)$ is a C_0-semigroup. ∎

2.3. PROPOSITION. (a) *The set*

$$\{V \geqslant 0\,;\ \ V \text{ is } U(\cdot)\text{-admissible}\}$$

is a solid convex cone.

(b) *The set*

$$\{V \leqslant 0\,;\ \ V \text{ is } U(\cdot)\text{-admissible}\}$$

is solid and convex.

Proof. (a) Let $V_1, V_2 \geqslant 0$ be $U(\cdot)$-admissible. For $n \in \mathrm{N}$, Proposition 1.3(b) (with $U(\cdot)$, $\tilde{U}(\cdot)$, V_1, V_2 corresponding to $U_{V_1^{(n)}}(\cdot)$, $U(\cdot)$, 0, $V_2^{(n)}$) yields

$$U(t) - U_{V_2^{(n)}}(t) \geqslant U_{V_1^{(n)}}(t) - U_{V_1^{(n)}+V_2^{(n)}}(t) \geqslant$$

$$\geqslant U_{V_1^{(n)}}(t) - U_{(V_1+V_2)^{(n)}}(t),$$

and for $n \to \infty$ we obtain

$$U(t) \geqslant U_{V_1+V_2}(t) \geqslant U_{V_1}(t) + U_{V_2}(t) - U(t).$$

This shows $\operatorname{s-lim}_{t\to\infty} U_{V_1+V_2}(t) = I$.

If $V_1 \geqslant 0$ is admissible, and $0 \leqslant V_2 \leqslant V_1$, then Proposition 1.3(b) implies $U_{V_1}(t) \leqslant U_{V_2}(t) \leqslant U(t)$ $(t \geqslant 0)$, and therefore $U_{V_2}(\cdot)$ is a C_0-semigroup.

(b) Let $V_1, V_2 \leqslant 0$ be admissible, and let $0 \leqslant r \leqslant 1$. For $n \in \mathrm{N}$, Proposition 1.4 yields

$$U_{((1-r)V_1+rV_2)^{(n)}}(t) \leqslant U_{(1-r)V_1^{(n)}+rV_2^{(n)}}(t) \leqslant$$

$$\leqslant (1-r)U_{V_1^{(n)}}(t) + rU_{V_2^{(n)}}(t) \leqslant (1-r)U_{V_1}(t) + rU_{V_2}(t).$$

This implies the existence of $U_{(1-r)V_1+rV_2}(t)$, and the inequalities

$$U(t) \leqslant U_{(1-r)V_1+rV_2}(t) \leqslant (1-r)U_{V_1}(t) + rU_{V_2}(t)$$

imply $\text{s-}\lim_{t\to 0} U_{(1-r)V_1+rV_2}(t) = I$.

If $V_1 \leqslant 0$ is $U(\cdot)$-admissible and $V_1 \leqslant V_2 \leqslant 0$ then $U(t) \leqslant U_{V_2}(t) \leqslant U_{V_1}(t)$ $(t \geqslant 0)$ implies that $U_{V_2}(\cdot)$ is a C_0-semigroup. ▨

2.4. REMARKS. (a) If $V \geqslant c$ $(\in \mathbf{R})$ is $U(\cdot)$-admissible, and $(V_n)_{n\in\mathbf{N}}$ is a sequence, $c \leqslant V_n \leqslant V$, $V_n \to V$ a.e., then $U_V(\cdot) = \text{s-}\lim_{n\to\infty} U_{V_n}(\cdot)$. This was shown in [17; Proposition A.1].

(b) If $V \leqslant c$ $(\in \mathbf{R})$ is $U(\cdot)$-admissible, and $(V_n)_{n\in\mathbf{N}}$ is such that $V \leqslant V_n \leqslant c$, $V_n \to V$ a.e., then $U_V = \text{s-}\lim_{n\to\infty} U_{V_n}(\cdot)$. This is proved analogously, cf. [17; Proposition A.2].

(c) If V is bounded below or bounded above, and $U(\cdot)$-admissible, and $\tilde{V} \in L_\infty(\mu)$, then $V + \tilde{V}$ is $U(\cdot)$-admissible, and $T_{V+\tilde{V}} = T_V + \tilde{V}$, or expressed differently, $(U_V)_{\tilde{V}}(\cdot) = U_{V+\tilde{V}}(\cdot)$. This follows from (a) and (b) above; cf. [17; Lemma 2.4].

2.5. DEFINITION. Let $V : \Omega \to \mathbf{R}$ be measurable. Then V will be called *$U(\cdot)$-admissible* if V^+ and $-V^-$ are $U(\cdot)$-admissible.

REMARK. The preceding definition is appearingly less restrictive than the one given in [17; Definition 2.5]. The subsequent result, however, will show that the additional requirement of [17] is always fulfilled.

2.6. THEOREM. *Let $V_\pm \geqslant 0$, V_+, $-V_-$ $U(\cdot)$-admissible, $V := V_+ - V_-$. Then*

$$(U_{-V_-})_{V_+}(t) \ (= \text{s-}\lim_{n\to\infty} U_{-V_-+V_+^{(n)}}(t)) =$$

$$= (U_{V_+})_{-V_-}(t) \ (= \text{s-}\lim_{n\to\infty} U_{V_+-V_-^{(n)}}(t)) =$$

$$= \text{s-}\lim_{n\to\infty} U_{V^{(n)}}(t) \ \ (= U_V(t)),$$

and $U_V(\cdot)$ thus defined is a C_0-semigroup.

If (V_n) is a sequence of absorption rates such that $-V_- \leqslant V_n \leqslant V_+$ for all $n \in \mathbf{N}$ (this implies that V_n is $U(\cdot)$-admissible), $V_n \to V$ a.e., then $U_V(\cdot) = \text{s-}\lim_{n\to\infty} U_{V_n}(\cdot)$.

Proof. For $m, n \in \mathbf{N}$, Proposition 1.3(b) implies

$$U_{-V_-}(t) - U_{-V_- + V_+^{(m)}}(t) \geqslant U_{-V_-^{(n)}}(t) - U_{-V_-^{(n)} + V_+^{(m)}}(t),$$

$$U_{-V_-}(t) - U_{-V_-^{(n)}}(t) \geqslant U_{-V_- + V_+^{(m)}}(t) - U_{-V_-^{(n)} + V_+^{(m)}}(t) \geqslant 0.$$

For $m \to \infty$ we obtain

$$U_{-V_-}(t) - U_{-V_-^{(n)}}(t) \geqslant (U_{-V_-})_{V_+}(t) - (U_{V_+})_{-V_-^{(n)}}(t) \geqslant 0,$$

and $n \to \infty$ shows the first equality. The remaining statements follow from [17; Theorem 2.6]. ∎

2.7. Remark. Since, for $U(\cdot)$-admissible V, the C_0-semigroup $U_V(\cdot)$ is always given by (2.1), the monotonicity and convexity properties (Propositions 1.3 and 1.4) carry over to the larger classes of absorption rates which are $U(\cdot)$-admissible.

3. REGULAR ABSORPTION RATES AND DOMINATED CONVERGENCE

Let $(\Omega, \mathscr{A}, \mu)$, $(U(t);\ t \geqslant 0)$ be as in the preceding section.

3.1. Definition. Let $V : \Omega \to [0, \infty)$ be measurable. Then V will be called *$U(\cdot)$-regular* if V is $U(\cdot)$-admissible, and

$$U(\cdot) = (U_V)_{-V}(\cdot) \ (= \operatorname*{s-lim}_{n\to\infty} U_{V - V^{(n)}}(\cdot)).$$

3.2. Remarks. (a) The definition of $U(\cdot)$-regular given above differs from the one in [17; Definition 2.12] (V is $U(\cdot)$-admissible, and $U(\cdot) = \operatorname*{s-lim}_{\eta \downarrow 0+} U_{\eta V}(\cdot)$). If $V \geqslant 0$ is $U(\cdot)$-regular in the above sense then V is also $U(\cdot)$-regular in the sense of [17]. This is an immediate consequence of the dominated convergence theorem (Corollary 3.6) proved below.

We do not know whether the two notions are different at all.

(b) We have introduced the above notion since it allows to obtain the dominated convergence results proved subsequently. In fact, if a dominated convergence theorem of the kind of Corollary 3.6 is to hold then V_+ (in the notation of Corollary 3.6) is necessarily $U(\cdot)$-regular: The sequence $(V_+ - V_+^{(n)})$ is squeezed in between 0 and V_+, also $V_+ - V_+^{(n)} \to 0$ a.e. $(n \to \infty)$, and therefore $\operatorname{s-lim} U_{V_+ - V_+^{(n)}}(\cdot) = U(\cdot)$.

(c) If $V \geqslant 0$ and $D(T) \cap D(V)$ is a core for T then V is $U(\cdot)$-regular. This is shown in [17; Proof of Proposition 2.13, p. 199].

(d) Assume $p = 1$, and assume additionally that $U(\cdot)$ is stochastic (i.e., $\|U(t)f\| = \|f\|$ for all $f \in L_1(\mu)_+$, $t \geqslant 0$). Let $V \geqslant 0$. Then the following conditions are equivalent:

(i) V is $U(\cdot)$-regular,

(ii) $D(T) \cap D(V)$ is a core for T,

(iii) V is $U(\cdot)$-admissible, and $\operatorname{s-lim}_{\eta \downarrow 0+} U_{\eta V}(\cdot) = U(\cdot)$ (i.e., V is $U(\cdot)$-regular in the sense of [17]).

Proof. (i) $\Rightarrow$ (iii) was noted in (a). For (iii) $\Leftrightarrow$ (ii) we refer to [17; Proposition 4.4]. (ii) $\Rightarrow$ (i) was noted in (c).

3.3. PROPOSITION. (a) $\{V \geqslant 0\,;\ V$ *is* $U(\cdot)$*-regular*$\}$ *is a solid convex cone.*
(b) *If* $V \geqslant 0$ *is such that* $-V$ *is* $U(\cdot)$*-admissible then* V *is* $U(\cdot)$*-regular.*

Proof. (a) Let $V_1, V_2 \geqslant 0$ be $U(\cdot)$-regular. Then $V_1 + V_2$ is $U(\cdot)$-admissible by Proposition 2.3(a). For $n \in \mathbf{N}$ we have $V_1 + V_2 - (V_1 + V_2)^{(2n)} \leqslant$ $\leqslant V_1 + V_2 - V_1^{(n)} - V_2^{(n)}$, and as in the proof of Proposition 2.3(a) we obtain

$$U(t) \geqslant U_{V_1+V_2-(V_1+V_2)^{(2n)}}(t) \geqslant U_{V_1-V_1^{(n)}+V_2-V_2^{(n)}}(t) \geqslant$$

$$\geqslant U_{V_1-V_1^{(n)}}(t) + U_{V_2-V_2^{(n)}}(t) - U(t).$$

The $U(\cdot)$-regularity of V_1, V_2 now implies

$$\operatorname{s-lim}_{m\to\infty} U_{V_1+V_2-(V_1+V_2)^{(m)}}(t) = U(t).$$

Let $V_1 \geqslant 0$ be $U(\cdot)$-regular, $0 \leqslant V_2 \leqslant V_1$. Then V_2 is $U(\cdot)$-admissible by Proposition 2.3(a). For $n \in \mathbf{N}$ we have $V_2 - V_2^{(n)} \leqslant V_1 - V_1^{(n)}$, $U_{V_2-V_2^{(n)}}(t) \geqslant$ $\geqslant U_{V_1-V_1^{(n)}}(t)$, and therefore V_2 is $U(\cdot)$-regular.

(b) Let $V \geqslant 0$, $-V$ $U(\cdot)$-admissible. For $n \in \mathbf{N}$, Proposition 1.3(b) implies

$$U_{-V^{(n)}}(t) - U(t) \geqslant U(t) - U_{V^{(n)}}(t),$$

and therefore

$$U(t) \geqslant U_V(t) \geqslant 2U(t) - U_{-V}(t),$$

$\operatorname{s-lim}_{t\to 0} U_V(t) = I$. Thus, V is $U(\cdot)$-admissible. Moreover, for $n \in \mathbf{N}$, Proposition 1.3 together with Remark 2.6 implies

$$U_{-V}(t) - U_{-V^{(n)}}(t) = U_{-V}(t) - (U_{-V})_{V-V^{(n)}}(t) \geqslant$$

$$\geqslant U(t) - U_{V-V^{(n)}}(t) \geqslant 0.$$

For $n \to \infty$, the first term strongly converges to 0, and therefore $\text{s-lim}_{n\to\infty} U_{V-V^{(n)}}(\cdot) = U(\cdot)$. ∎

As a preparation for the dominated converges theorem we show a fact which is also interesting by itself.

3.4. PROPOSITION. *Let $V_\pm \geq 0$, where $-V_-$ is $U(\cdot)$-admissible, and V_+ is $U(\cdot)$-regular. Then V_+ is $U_{-V_-}(\cdot)$-regular*, i.e.,

$$U_{-V_-}(\cdot) = \text{s-lim}_{n\to\infty} U_{-V_-+V_+-V_+^{(n)}}(\cdot).$$

Proof. Let $t \geq 0$. For $k, n \in \mathbf{N}$ we have, by Proposition 1.3(b) and Remark 2.7,

$$U_{-V_-}(t) - U_{-V_-^{(k)}}(t) \geq U_{-V_-+V_+-V_+^{(n)}}(t) - U_{-V_-^{(k)}+V_+-V_+^{(n)}}(t) \geq 0.$$

For all $k \in \mathbf{N}$ we have

$$\text{s-lim}_{n\to\infty} U_{-V_-^{(k)}+V_+-V_+^{(n)}}(\cdot) = U_{-V_-^{(k)}}(\cdot),$$

since $-V_-^{(k)}$ is bounded. (This follows easily from $\text{s-lim}_{n\to\infty} U_{V_+-V_+^{(n)}}(\cdot) = U(\cdot)$ and the Trotter approximation theorem: cf. [10; Theorem 3.4.2].)

Let $f \geq 0$ and choose $\varepsilon > 0$. Then there exists $k \in \mathbf{N}$ such that $\|(U_{-V_-}(t) - U_{-V_-^{(k)}}(t))f\| \leq \varepsilon/3$. The above inequality implies

$$\|U_{-V_-+V_+-V_+^{(n)}}(t)f - U_{-V_-^{(k)}+V_+-V_+^{(n)}}(t)f\| \leq \frac{\varepsilon}{3} \quad \text{for all } n \in \mathbf{N}.$$

By the above convergence, there exists $N \in \mathbf{N}$ such that

$$\|U_{-V_-^{(k)}}(t)f - U_{-V_-^{(k)}+V_+-V_+^{(n)}}(t)f\| \leq \frac{\varepsilon}{3} \quad \text{for all } n \geq N.$$

Putting these inequalities together we obtain

$$\|U_{-V_-}(t)f - U_{-V_-+V_+-V_+^{(n)}}(t)f\| \leq \varepsilon \quad \text{for all } n \in \mathbf{N}.$$ ∎

3.5. THEOREM. (General dominated convergence theorem). *Let $V_\pm \geq 0$, where $-V_-$ is $U(\cdot)$-admissible, and V_+ is $U(\cdot)$-regular. Let V be $U(\cdot)$-admissible. Let (V_n) be a sequence, $-V_- \leq V_n \leq V + V_+$ for all $n \in \mathbf{N}$ (this implies that V_n is $U(\cdot)$-admissible), $V_n \to V$* a.e. *on Ω. Then*

$$U_V(\cdot) = \text{s-lim}\, U_{V_n}(\cdot).$$

Proof. For $k \in \mathbf{N}$ we define $V_{n,k} := V_n \wedge (V^+ + V_+^{(k)})$ $(n \in \mathbf{N})$. Then $V = -(V_- + V_+^{(k)}) + (V + V_- + V_+^{(k)})$, and $-(V_- + V_+^{(k)}) \leqslant 0$, $(V + V_- + V_+^{(k)}) \geqslant 0$ are $U(\cdot)$-admissible. Also, $-(V_- + V_+^{(k)}) \leqslant V_{n,k}$ $(\leqslant V^+ + V_+^{(k)}) \leqslant V + V_- + V_+^{(k)}$ for all n, and $V_{n,k} \to V$ a.e. Therefore, Theorem 2.6 implies $U_V(\cdot) = \text{s-lim}_{n\to\infty} U_{V_{n,k}}(\cdot)$. For all $n \in \mathbf{N}$ we have $0 \leqslant V_n - V_{n,k} \leqslant V_+ - V_+^{(k)}$, and therefore

$$U_{-V_-}(t) - U_{-V_- + (V_+ - V_+^{(k)})}(t) \geqslant$$

$$\geqslant U_{-V_-}(t) - U_{-V_- + (V_n - V_{n,k})}(t) \geqslant U_{V_{n,k}}(t) - U_{V_n}(t) \geqslant 0.$$

Now Proposition 3.4 together with the convergence proved above show the desired conclusion similarly as in the proof of Proposition 3.4. ▨

3.6. Corollary. (Dominated convergence theorem). *Let $V_\pm \geqslant 0$, where $-V_-$ is $U(\cdot)$-admissible, and V_+ is $U(\cdot)$-regular. Let V, V_n $(n \in \mathbf{N})$ be such that $-V_- \leqslant V_n \leqslant V_+$ for all $n \in \mathbf{N}$, and $V_n \to V$* a.e. *Then*

$$U_V(\cdot) = \text{s-lim}_{n\to\infty} U_{V_n}(\cdot).$$

Proof. For all $n \in \mathbf{N}$ we have $V_n \leqslant V_+ \leqslant V + V_- + V_+$, and $V_- + V_+$ is $U(\cdot)$-regular by Proposition 3.3. Now the conclusion follows from Theorem 3.5. ▨

Remark. The attentive reader may be astonished that apparently in the proof of Theorem 3.5 only estimates based on monotonicity are used, and he may miss an argument connecting the convergence of generators with the convergence of semigroups. This connection is in fact used at an early stage in the theory, namely in the proof of Remark 2.4 (a), (b). These facts are used in [17, Theorem 2.6] to prove Theorem 2.6 of the present paper, which in turn enters the proof of Theorem 3.5.

4. EXAMPLES

4.1. Example. This example serves to illustrate the notions of $U(\cdot)$-admissibility and $U(\cdot)$-regularity.

Let $1 \leqslant p < \infty$, and let $(U(t);\ t \geqslant 0)$ be the C_0-semigroup of right translations on $L_p(\mathbf{R})$,

$$U(t)f(x) = f(x - t).$$

If V is measurable and bounded below we obtain

$$U_V(t)f(x) = \exp\left(-\int_{-t}^{0} V(x+s)\mathrm{d}s\right) f(x-t).$$

This implies that $V \geqslant 0$ is $U(\cdot)$-admissible if and only if $\int_{-t}^{0} V(x+s)\mathrm{d}s \to 0$ ($t \to 0+$) a.e., and this condition is equivalent to the existence of a closed null set $N \subset \mathbf{R}$ such that $V \in L_{1,\mathrm{loc}}(\mathbf{R}\setminus N)$.

On the other hand, $V \geqslant 0$ is $U(\cdot)$-regular if and only if $V \in L_{1,\mathrm{loc}}(\mathbf{R})$. Also, $V \leqslant 0$ is $U(\cdot)$-admissible if and only if $V \in L_{1,\mathrm{loc,unif}}(\mathbf{R})$.

Note that all of these notions are p-independent.

4.2. EXAMPLE. This example illustrates that, in Remark 3.2(d) the conclusion becomes false if $U(t)$ is not stochastic.

Let $(U(t);\ t \geqslant 0)$ be the C_0-semigroup of right translations on $L_1(-\infty, 0)$,

$$U(t)f(x) = f(x-t).$$

Let $V(x) := \dfrac{1}{|x|}$. Then it is easy to see that V is $U(\cdot)$-regular. On the other hand,

$$D(T) = \{f \in L_1(\mathbf{R}) : f \text{ absolutely continuous, } Tf = -f' \in L_1(\mathbf{R})\}$$

$$(= W_1^1(-\infty, 0)),$$

$$D(T) \cap D(V) \subset \{f \in D(T);\ f(0) = 0\}.$$

(Note that evaluation $W_1^1(-\infty, 0) \ni f \to f(0)$ is a continuous linear functional.) This shows that $D(T) \cap D(V)$ is not dense in $D(T)$ with respect to the graph norm.

4.3. EXAMPLE. Let $1 \leqslant p < \infty$, and denote by $(U_p(t);\ t \geqslant 0)$ the C_0-semigroup on $L_p(\mathbf{R}^\nu)$ associated with the (unperturbed) heat equation $u_t = (1/2)\Delta u$. If $V : \mathbf{R}^\nu \to \mathbf{R}$ is $U_p(\cdot)$-admissible, then $U_{p,V}(\cdot)$ is given by the Feynman-Kac formula; cf. [17; Section 6].

In this example, precise relations between $U_p(\cdot)$-admissibility, $U_p(\cdot)$-regularity and integrability properties of certain functions can be obtained; cf. [17; Proposition 6.1]. As a consequence, the general dominated convergence theorem (Theorem 3.5) can be concluded from the Feynman-Kac formula together with Lebesgue's dominated convergence theorem.

We recall that, for $V \geqslant 0$, the notions of $U_p(\cdot)$-admissibility and $U_p(\cdot)$-regularity are p-independent; cf. [17]. A characterization of these properties in terms of capacities can be found in [15]. Moreover, we mention that $0 \leqslant V \in L_{1,\mathrm{loc}}(\mathbf{R}^\nu)$ implies that V is $U_p(\cdot)$-regular (by Remark 3.2(c)), but that there exists a $U_p(\cdot)$-regular $V \geqslant 0$ which is nowhere in L_1; cf. [16].

For $V \leqslant 0$, however, the notion of $U_p(\cdot)$-admissibility is p-dependent. For $p = 1$, $0 \geqslant V \in \hat{K}_\nu$, $c_\nu(V) < 1$ implies that V is $U_1(\cdot)$-admissible. On the other hand, if $V \leqslant 0$ is $U_1(\cdot)$-admissible, then $V \in \hat{K}_\nu$ (cf. [17; Section 5]). For $p = 2$, $0 \geqslant V \in L_q + L_\infty$ (where $q = \nu/2$ if $\nu \geqslant 3$, $q > 1$ if $\nu = 2$, $q = 1$ if $\nu = 1$) implies that V is $U_2(\cdot)$-admissible. (This follows from [2; Lemma 2.1] together with [17; Proposition 5.7].) We note that in fact a slightly more general condition introduced in [14] suffices; cf. [2; Remark 2.1], [7; Section 4.2].

Concluding this example we note that, in view of the preceding remarks, our dominated convergence theorem (Corollary 3.6) yields a generalization of [7, Theorem 4.1].

APPENDIX

In this appendix we prove a result which is used in Section 2.

A.1. PROPOSITION. *Let E be a Banach lattice with order continuous norm. Let $(U(t);\ t \geqslant 0)$ be a positive C_0-semigroup on E. Let $(\tilde{U}(t);\ t \geqslant 0)$ be a semigroup on E such that $\tilde{U}(\cdot)f$ is strongly measurable for all $f \in E$, and*

$$U(t) \leqslant \tilde{U}(t) \quad (t \geqslant 0), \tag{A.1}$$

$$\sup_{0 \leqslant t \leqslant 1} \|\tilde{U}(t)\| < \infty. \tag{A.2}$$

Then $(\tilde{U}(t);\ t \geqslant 0)$ is a (positive) C_0-semigroup.

Proof. (i) By a theorem of Dunford, the measurability assumption for $\tilde{U}(\cdot)$ is equivalent to the strong continuity of $(0,\infty) \ni t \mapsto \tilde{U}(t)$; cf. [3; Lemma VIII 1.3].

(ii) In (iii) below we are going to show that, for all $f \in E$,

$$\{\tilde{U}(t)f;\ 0 < t \leqslant 1\}$$

is relatively weakly compact. A result of R. Sato [12] (cf. [6; § 7.1, Theorem 1.11]) then implies that $P := \operatorname{s-}\lim_{t \to 0+} \tilde{U}(t)$ exists, and obviously is a projection. Assumption (A.1) implies $P \geqslant I$, and therefore $P = I$. (For $f \geqslant 0$ we obtain $0 \leqslant Pf - f \leqslant P(Pf - f) = 0$.)

Thus, $(\tilde{U}(t);\ t \geqslant 0)$ is a C_0-semigroup.

(iii) (The idea of this part of the proof is taken from [6; § 7.1, Corollary 1.12].) Let $f \geqslant 0$. We define $g := \int_0^1 \tilde{U}(s) f \mathrm{d}s$ ($\geqslant 0$). For $n \in \mathbf{N}$, $0 < t \leqslant 1$ we obtain

$$\tilde{U}(t)(f \wedge (ng)) \leqslant n\tilde{U}(t) \int_0^1 \tilde{U}(s) f \mathrm{d}s \leqslant n \int_0^2 \tilde{U}(s) f \mathrm{d}s.$$

The order continuity of the norm of E implies that the order intervals of E are weakly compact (cf. [13; Chapter II, Theorem 5.10]), and therefore

$$\{\tilde{U}(t)(f \wedge ng);\ 0 < t \leqslant 1\}$$

is relatively weakly compact for all $n \in \mathbf{N}$. If, for $n \in \mathbf{N}$, we define $f_n = n \int_0^{1/n} U(s) f \mathrm{d}s$ then $0 \leqslant f_n \leqslant ng$ and $f_n \to f$. $f \geqslant f \wedge (ng) \geqslant f \wedge f_n \to f$, $f \wedge (ng) \to f$. Now assumption (A.2) implies that $\{\tilde{U}(t)f;\ 0 < t \leqslant 1\}$ is relatively weakly compact. ∎

REMARK. For reflexive E the proof simplifies since then the relative weak compactness of the sets $\{\tilde{U}(t)f;\ 0 < t \leqslant 1\}$ follows from the boundedness.

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