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POSITIVE TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES OF BOUNDED SYMMETRIC DOMAINS

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1. INTRODUCTION

Let Ω be a bounded symmetric domain in $\mathbf{C}^n$ with Bergman kernel $K(z, w)$. We assume that Ω is in its standard representation and the volume measure $\mathrm{d}V$ of Ω is normalized so that $K(z, 0) = K(0, w) = 1$ for all z and w in Ω. By 5.7 of [8] and polar coordinates, there exists a positive number ε_Ω such that

$$C_\lambda = \int_\Omega K(z, z)^\lambda \mathrm{d}V(z) < +\infty$$

iff $\lambda < \varepsilon_\Omega$. Let

$$\mathrm{d}V_\lambda(z) = C_\lambda^{-1} K(z, z)^\lambda \mathrm{d}V(z),$$

then $\mathrm{d}V_\lambda$ is a probability measure on Ω for all $\lambda < \varepsilon_\Omega$. We fix a $\lambda < \varepsilon_\Omega$ throughout the paper and consider the weighted Bergman spaces $L^p_a(\Omega, \mathrm{d}V_\lambda)$ $(1 \leqslant p < +\infty)$ consisting of holomorphic functions in $L^p(\Omega, \mathrm{d}V_\lambda)$. When $p = 2$, we have an orthogonal projection P_λ from the Hilbert space $L^2(\Omega, \mathrm{d}V_\lambda)$ onto the closed subspace $L^2_a(\Omega, \mathrm{d}V_\lambda)$. P_λ is given by

$$P_\lambda f(z) = \int_\Omega K_\lambda(z, w) f(w) \mathrm{d}V_\lambda(w),$$

where $K_\lambda(z, w) = K(z, w)^{1-\lambda}$ is the reproducing kernel of $L^2_a(\Omega, \mathrm{d}V_\lambda)$.

Suppose φ is a function on Ω, then the Toeplitz operator T_φ with symbol φ is defined by $T_\varphi f = P_\lambda(\varphi f)$, or

$$T_\varphi f(z) = \int_\Omega K_\lambda(z, w) \varphi(w) f(w) \mathrm{d}V_\lambda(w).$$

We can extend the notion of Toeplitz operators to the case where we allow measures as symbols. To be precise, let μ be a finite complex Borel measure on Ω, then we define the Toeplitz operator T_μ with symbol μ by

$$T_\mu f(z) = \int_\Omega K_\lambda(z, w) f(w) \mathrm{d}\mu(w).$$

Clearly, if $\mathrm{d}\mu(z) = \varphi(z)\mathrm{d}V_\lambda(z)$, then $T_\mu = T_\varphi$.

Our problem here is to obtain and study conditions on μ that will ensure the boundedness, compactness, and membership in S_p (the Schatten ideals) of the Toeplitz operator T_μ. When μ is positive, the conditions we obtain are necessary and sufficient. In order to state the main results of the paper, we need to introduce some notations.

For any a in Ω, let $k_a(z) = K(z, a)/\sqrt{K(a, a)}$. The k_a's are called normalized reproducing kernels of $L^2_a(\Omega, \mathrm{d}V)$. They are unit vectors in $L^2_a(\Omega, \mathrm{d}V)$. It is easy to see that $k_a^{1-\lambda}$ is a unit vector of $L^2_a(\Omega, \mathrm{d}V_\lambda)$ for any a in Ω. Given an (possibly unbounded) operator A on $L^2_a(\Omega, \mathrm{d}V_\lambda)$ with the domain of A containing all the normalized reproducing kernels $k_a^{1-\lambda}$ of $L^2_a(\Omega, \mathrm{d}V_\lambda)$, we can define a function $\tilde{A}(z)$ on Ω by

$$\tilde{A}(z) = \langle A k_z^{1-\lambda}, k_z^{1-\lambda} \rangle_\lambda, \quad z \in \Omega,$$

where $\langle\ ,\ \rangle_\lambda$ is the inner product in $L^2_a(\Omega, \mathrm{d}V_\lambda)$. Since $k_z^{1-\lambda}$ converges to 0 weakly in $L^2_a(\Omega, \mathrm{d}V_\lambda)$ as z approaches $\partial\Omega$ (the topological boundary of Ω), it follows that $\tilde{A}$ is bounded on Ω if A is bounded, and $\tilde{A}(z) \to 0$ $(z \to \partial\Omega)$ if A is compact. When $A = T_\mu$, we will write $\tilde{\mu}_\lambda = \tilde{T}_\mu$ and call $\tilde{\mu}_\lambda$ the Berezin symbol of μ. It is clear that

$$\tilde{\mu}_\lambda(z) = \int_\Omega |k_z(w)|^{2(1-\lambda)} \mathrm{d}\mu(w), \quad z \in \Omega.$$

We will denote by $\beta(z, w)$ the Bergman distance function on Ω. For any z in Ω and $r > 0$, let

$$E(z, r) = \{w \in \Omega : \beta(z, w) < r\}.$$

We denote by $|E(z, r)|$ the normalized volume of $E(z, r)$, that is, $|E(z, r)| = \int_{E(z, r)} \mathrm{d}V(w)$. Given a finite complex Borel measure μ on Ω, we define a function $\hat{\mu}_r$ on Ω by

$$\hat{\mu}_r(z) = \frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}}, \quad z \in \Omega.$$

We will show that $|E(z, r)|^{1-\lambda}$ is equivalent to $V_\lambda(E(z, r))$ for any fixed $r > 0$, thus $\hat{\mu}_r(z)$ is equivalent to $\dfrac{\mu(E(z, r))}{V_\lambda(E(z, r))}$, which is the average over $E(z, r)$ of the function φ with respect to $\mathrm{d}V_\lambda$ when $\mathrm{d}\mu(z) = \varphi(z)\mathrm{d}V_\lambda(z)$. So we can think of $\hat{\mu}$ as an averaging function of μ with respect to $\mathrm{d}V_\lambda$.

We can now state the main results of the paper.

THEOREM A. *For a finite positive Borel measure μ on Ω, the following conditions are all equivalent:*

(1) *T_μ is bounded on $L^2_a(\Omega, \mathrm{d}V_\lambda)$;*

(2) *For all (or some) $p \geq 1$, the mapping i_p defined by $i_p(f) = f$ is a bounded mapping from $L^p_a(\Omega, \mathrm{d}V_\lambda)$ into $L^p(\Omega, \mathrm{d}\mu)$;*

(3) *The Berezin transform $\tilde{\mu}_\lambda$ is bounded on Ω;*

(4) *$\hat{\mu}_r$ is bounded on Ω for all (or some) $r > 0$;*

(5) *$\{\hat{\mu}_r(a_n)\}$ is a bounded sequence, where $\{a_n\}$ is some sequence in Ω independent of μ (see Section 2).*

A positive Borel measure μ satisfying any of the above conditions is called a Carleson measure (on the weighted Bergman space $L^p_a(\Omega, \mathrm{d}V_\lambda)$).

THEOREM B. *For a finite positive Borel measure μ on Ω, the following conditions are all equivalent:*

(1) *T_μ is compact on $L^2_a(\Omega, \mathrm{d}V_\lambda)$;*

(2) *For all (or some) $p \geq 1$, the mapping i_p defined by $i_p(f) = f$ is a compact mapping from $L^p_a(\Omega, \mathrm{d}V_\lambda)$ into $L^p(\Omega, \mathrm{d}\mu)$;*

(3) *The Berezin transform $\tilde{\mu}_\lambda(z) \to 0$ as $z \to \partial\Omega$;*

(4) *$\hat{\mu}_r(z) \to 0$ $(z \to \partial\Omega)$ for all (or some) $r > 0$;*

(5) *$\hat{\mu}_r(a_n) \to 0$ $(n \to +\infty)$.*

A positive Borel measure satisfying any of the above conditions is called a vanishing Carleson measure (on the weighted Bergman space $L^p_a(\Omega, \mathrm{d}V_\lambda)$).

Let S_p $(p \geq 1)$ be the Schatten p-ideal on the Hilbert space $L^2_a(\Omega, \mathrm{d}V_\lambda)$ (see Section 5), then we have:

THEOREM C. *For a finite positive Borel measure μ on Ω and $p \geq 1$, the following conditions are all equivalent:*

(1) *$T_\mu \in S_p$;*

(2) *$\tilde{\mu}_\lambda \in L^p(\Omega, K(z,z)\mathrm{d}V(z))$;*

(3) *$\hat{\mu}_r \in L^p(\Omega, K(z,z)\mathrm{d}V(z))$ for all (or some) $r > 0$;*

(4) *$\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p < +\infty$.*

When $\lambda = 0$ and $p = 2$, Theorems A and B were proved in [4]. If $\Omega = \mathbf{D}$, the open unit disc in $\mathbf{C}$, then Theorems A and B even go back to [12], [7], [14]. For $\Omega = \mathbf{D}$, Theorem C was proved in [11]. Further treatment of Carleson measures on the Bergman spaces can be found in [9], [10], [13]. The readers should be warned that Carleson measures defined here are different from the classical Carleson measures first introduced by L. Carleson in [5]. The classical Carleson measures were designed to work for the Hardy space setting.

2. GEOMETRIC PRELIMINARIES

We collect in this section some fact about the Bergman geometry of Ω that we will need later on.

Recall that $E(a, r)$ is the open Bergman metric ball with center a and radius r. We will need an estimate on the volume of $E(a, r)$.

LEMMA 1. *For any $r > 0$, there exists a constant C (depending on r) such that*

$$C^{-1} \leqslant |E(a,r)|\, |k_a(z)|^2 \leqslant C$$

or all a in Ω and $z \in E(a, r)$.

This is just Lemma 8 of [2]. Note that if we let $z = a$ in the above estimate, then we get

$$C^{-1} \leqslant |E(a, r)|\, K(a, a) \leqslant C$$

for all a in Ω.

LEMMA 2. *For any $r > 0$, $s > 0$, $R > 0$, there exists a constant C (depending on r, s, R) such that*

$$C^{-1} \leqslant \frac{|E(a, r)|}{|E(b, s)|} \leqslant C$$

for all a, b in Ω with $\beta(a, b) \leqslant R$.

This is just Lemma 6 of [2].

LEMMA 3. *For any a, b in Ω with $\beta(a, b) < r$, there exists a point $m_{(a,b)}$ such that*

$$E\left(m_{(a,b)}, \frac{r}{2}\right) \subset E(a, r) \cap E(b, r).$$

Proof. Let $\alpha(t)$ $(0 \leqslant t \leqslant 1)$ be the geodesic (in the Bergman metric) from a to b. Let $m_{(a,b)} = \alpha(1/2)$, then

$$\beta(a, m_{(a,b)}) = \beta(\alpha(0), \alpha(1/2)) = \frac{1}{2}\beta(a, b) < \frac{r}{2}.$$

Similarly, $\beta(b, m_{(a,b)}) < \frac{r}{2}$. Now if $z \in E\left(m_{(a,b)}, \frac{r}{2}\right)$, then

$$\beta(z, a) \leqslant \beta(z, m_{(a,b)}) + \beta(m_{(a,b)}, a) < \frac{r}{2} + \frac{r}{2} = r,$$

$$\beta(z, b) \leqslant \beta(z, m_{(a,b)}) + \beta(m_{(a,b)}, b) < \frac{r}{2} + \frac{r}{2} = r.$$

That is, $z \in E(a, r) \cap E(b, r)$. This completes the proof. ▨

We will need a decomposition of Ω which is similar to the decomposition of **D** into equal-sized squares in the pseudo-hyperbolic metric (see [11], [5]).

LEMMA 4. *For any $r > 0$, there exists a sequence $\{a_n\}$ in Ω satisfying the following two conditions:*

(1) $\Omega = \bigcup_{n=1}^{\infty} E(a_n, r)$;

(2) *There is a positive integer N such that each point z in Ω belongs to at most N of the sets $E(a_n, 2r)$.*

For a proof of Lemma 4, see [4].

LEMMA 5. *For any $r > 0$ and $p \geqslant 1$, there exists a constant C (depending only on r) such that*

$$|f(a)|^p \leqslant \frac{C}{|E(a, r)|} \int_{E(a,r)} |f(z)|^p \,\mathrm{d}V(z)$$

for all f holomorphic and a in Ω.

The proof of Lemma 5 is similar to the proof of Lemma 7 in [4], so we omit the details here.

LEMMA 6. *Suppose $r > 0$ and μ is a positive Borel measure on Ω, then there exists a constant C (depending only on r) such that*

$$\mu(E(a, r)) \leqslant \frac{C}{|E(a, r)|} \int_{E(a,r)} \mu(E(z, r)) \,\mathrm{d}V(z)$$

for all a in Ω.

Proof. For any a in Ω, we have

$$\int_{E(a,r)} \mu(E(z,r))\mathrm{d}V(z) = \int_{E(a,r)} \mathrm{d}V(z) \int_{E(z,r)} \mathrm{d}\mu(w) =$$

$$= \int_{\Omega} \chi_{E(a,r)}(z)\mathrm{d}V(z) \int_{\Omega} \chi_{E(z,r)}(w)\,\mathrm{d}\mu(w) =$$

$$= \int_{\Omega} \mathrm{d}\mu(w) \int_{\Omega} \chi_{E(a,r)}(z)\chi_{E(z,r)}(w)\,\mathrm{d}V(z) = \int_{\Omega} \mathrm{d}\mu(w) \int_{E(a,r)} \chi_{E(z,r)}(w)\,\mathrm{d}V(z).$$

Note that $\chi_{E(z,r)}(w) = \chi_{E(w,r)}(z)$ for all z and w in Ω, thus

$$\int_{E(a,r)} \mu(E(z,r))\mathrm{d}V(z) = \int_{\Omega} \mathrm{d}\mu(w) \int_{E(a,r)} \chi_{E(w,r)}(z)\,\mathrm{d}V(z) =$$

$$= \int_{\Omega} |E(a,r) \cap E(w,r)|\,\mathrm{d}\mu(w) \geqslant \int_{E(a,r)} |E(a,r) \cap E(w,r)|\,\mathrm{d}\mu(w).$$

Applying Lemma 3, we get

$$\int_{E(a,r)} \mu(E(z,r))\mathrm{d}V(z) \geqslant \int_{E(a,r)} \left|E\left(m_{(a,w)}, \frac{r}{2}\right)\right| \mathrm{d}\mu(w)$$

for all a in Ω. By Lemma 2, there exists a constant $C > 0$ depending only on r such that

$$|E(a,r)| \leqslant C \left|E\left(m_{(a,w)}, \frac{r}{2}\right)\right|$$

for all a in Ω and $w \in E(a,r)$. This implies that

$$C \int_{E(a,r)} \mu(E(z,r))\,\mathrm{d}V(z) \geqslant \int_{E(a,r)} |E(a,r)|\,\mathrm{d}\mu(w),$$

or

$$\mu(E(a,r)) \leqslant \frac{C}{|E(a,r)|} \int_{E(a,r)} \mu(E(z,r))\,\mathrm{d}V(z)$$

for all a in Ω. ▨

Note that Lemma 6 says that the function $\mu(E(z,r))$ behaves like a subharmonic function in the Bergman metric.

3. BOUNDED TOEPLITZ OPERATORS

Let $\mu \geqslant 0$ be a finite Borel measure on Ω. We say that μ is a Carleson measure on $L^p_a(\Omega, dV_\lambda)$ if there exists a constant $M > 0$ such that

$$\int_\Omega |f(z)|^p \, d\mu(z) \leqslant M \int_\Omega |f(z)|^p \, dV_\lambda(z)$$

for all f in $L^p_a(\Omega, dV_\lambda)$. The following theorem gives a geometric description o Carleson measures on $L^p_a(\Omega, dV_\lambda)$. In particular, it implies that Carleson measures on $L^p_a(\Omega, dV_\lambda)$ only depend on λ, not on p.

THEOREM 7. *Suppose $\mu \geqslant 0$ is a finite Borel measure on Ω, $p \geqslant 1$, then μ is a Carleson measure on $L^p_a(\Omega, dV_\lambda)$ iff $\mu(E(z, r))/|E(z, r)|^{1-\lambda}$ is bounded on Ω (as a function of z) for all (or some) $r > 0$. Moreover, the following quantities are equivalent for any fixed $r > 0$ and $p \geqslant 1$:*

$$\sup\left\{\frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}} : z \in \Omega\right\}, \quad \sup\left\{\frac{\int_\Omega |f(z)|^p \, d\mu(z)}{\int_\Omega |f(z)|^p \, dV_\lambda(z)} : f \in L^p_a(\Omega, dV_\lambda)\right\}.$$

Proof. Suppose

$$M = \sup\left\{\frac{\int_\Omega |f(z)|^p \, d\mu(z)}{\int_\Omega |f(z)|^p \, dV_\lambda(z)} : f \in L^p_a(\Omega, dV_\lambda)\right\} < +\infty,$$

then

$$\int_\Omega |f(z)|^p \, d\mu(z) \leqslant M \int_\Omega |f(z)|^p \, dV_\lambda(z)$$

for all $f \in L^p_a(\Omega, dV_\lambda)$. In particular, if $a \in \Omega$ and

$$f(z) = k_a(z)^{\frac{2(1-\lambda)}{p}},$$

then we have

$$\int_\Omega |k_a(z)|^{2(1-\lambda)} \, d\mu(z) \leqslant M$$

for all a in Ω. This implies that

$$\int_{E(a,r)} |k_a(z)|^{2(1-\lambda)} \, d\mu(z) \leqslant M$$

for all a in Ω and $r > 0$. By Lemma 1, there exists a constant $C > 0$ (depending on r only) such that

$$|E(a, r)| \, |k_a(z)|^2 \geqslant C^{-1}$$

for all z in $E(a, r)$. Thus we have

$$\int_{E(a,r)} d\mu(z) \leqslant C^{1-\lambda} M |E(a, r)|^{1-\lambda}$$

for

$$\mu(E(a, r)) \leqslant C^{1-\lambda} M |E(a, r)|^{1-\lambda}$$

for all a in Ω. Hence we have

$$\sup\left\{ \frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} : a \in \Omega \right\} \leqslant C^{1-\lambda} \sup\left\{ \frac{\int_\Omega |f(z)|^p \, d\mu(z)}{\int_\Omega |f(z)|^p \, dV_\lambda(z)} : f \in L^p_a(\Omega, dV_\lambda) \right\}.$$

Conversely, suppose

$$M = \sup\left\{ \frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} : a \in \Omega \right\} < +\infty$$

for some $r > 0$. We wish to show that μ is a Carleson measure on $L^p_a(\Omega, dV_\lambda)$ for all $p \geqslant 1$.

Fix $p \geqslant 1$; applying Lemma 5, we get a constant $C > 0$ (depending only on r) such that

$$|f(z)|^p \leqslant \frac{C}{|E(z, r)|} \int_{E(z,r)} |f(w)|^p \, dV(w)$$

for all f holomorphic in Ω and z in Ω. This implies that for any a in Ω, we have

$$\sup\{|f(z)|^p : z \in E(a, r)\} \leqslant \sup\left\{\frac{C}{|E(z, r)|}\int_{E(z,r)} |f(w)|^p \,\mathrm{d}V(w) : z \in E(a, r)\right\} \leqslant$$

$$\leqslant \sup\left\{\frac{C}{|E(z, r)|}\int_{E(a,2r)} |f(w)|^p \,\mathrm{d}V(w) : z \in E(a, r)\right\} =$$

$$= C \sup\left\{\frac{1}{|E(z, r)|} : z \in E(a,r)\right\} \int_{E(a,2r)} |f(w)|^p \,\mathrm{d}V(w).$$

By Lemma 2, there exists a constant $C_1 > 0$ (depending only on r) such that

$$\frac{1}{|E(z, r)|} \leqslant \frac{C_1}{|E(a, r)|}$$

for all $\beta(z, a) \leqslant r$. Thus

$$\sup\{|f(z)|^p : z \in E(a, r)\} \leqslant \frac{CC_1}{|E(a, r)|} \int_{E(a,2r)} |f(w)|^p \mathrm{d}V(w)$$

for all a in Ω. By Lemma 1, there exists another constant $C_2 > 0$ (depending only on r) such that

$$\sup\{|f(z)|^p : z \in E(a, r)\} \leqslant \frac{CC_1C_2}{|E(a, r)|^{1-\lambda}} \int_{E(a,2r)} |f(w)|^p \,\mathrm{d}V_\lambda(w)$$

for all a in Ω. By Lemma 4, there exists a positive integer N and a sequence $\{a_n\}$ in Ω (both depending on r) such that

$$\bigcup_{n=1}^{\infty} E(a_n, r) = \Omega$$

and each point z in Ω belongs to at most N of the sets $E(a_n, 2r)$. Now if f is holomorphic in Ω, then

$$\int_\Omega |f(z)|^p \,d\mu(z) \leqslant \sum_{n=1}^{\infty} \int_{E(a_n,r)} |f(z)|^p \,d\mu(z) \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \mu(E(a_n, r)) \sup\{|f(z)|^p : z \in E(a_n, r)\} \leqslant$$

$$\leqslant CC_1C_2 \sum_{n=1}^{\infty} \frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} \int_{E(a_n,2r)} |f(z)|^p \,dV_\lambda(z) \leqslant$$

$$\leqslant CC_1C_2M \sum_{n=1}^{\infty} \int_{E(a_n,2r)} |f(z)|^p \,dV_\lambda(z) \leqslant CC_1C_2NM \int_\Omega |f(z)|^p \,dV_\lambda(z).$$

This shows that

$$\sup\left\{ \frac{\int_\Omega |f(z)|^p \,d\mu(z)}{\int_\Omega |f(z)|^p \,dV_\lambda(z)} : f \in L^p_a(\Omega, dV_\lambda) \right\} \leqslant CC_1C_2N \sup\left\{ \frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} : a \in \Omega \right\}.$$

We have completed the proof of the theorem.

For a positive Borel measure μ on Ω, let

$$\|\mu\|_p = \sup\left\{ \frac{\int_\Omega |f(z)|^p \,d\mu(z)}{\int_\Omega |f(z)|^p \,dV_\lambda(z)} : f \in L^p_a(\Omega, dV_\lambda) \right\};$$

$$\|\tilde{\mu}_\lambda\|_\infty = \sup\left\{ \int_\Omega |k_a(z)|^{2(1-\lambda)} \,d\mu(z) : a \in \Omega \right\};$$

$$\|\hat{\mu}_r\|_\infty = \sup\left\{ \frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} : a \in \Omega \right\};$$

$$\|\mu\|_d = \sup\left\{ \frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} : n = 1, 2, \dots \right\}.$$

Carefully examining the proof of Theorem 7, we have:

COROLLARY. *For $p \geqslant 1$, $r > 0$, and $\mu \geqslant 0$, the quantities $\|\mu\|_p$, $\|\tilde{\mu}_\lambda\|_\infty$, $\|\hat{\mu}_r\|_\infty$, and $\|\mu\|_d$ are all equivalent.*

Hereafter, we will denote by $\|\mu\|_*$ any of the quantities defined above.

Recall that the Toeplitz operator T_μ is defined by

$$T_\mu f(z) = \int_\Omega K_\lambda(z, w) f(w)\, \mathrm{d}\mu(w).$$

f $\|\mu\|_* < +\infty$, then $T_\mu f$ is well-defined for all f in $L^p_a(\Omega, \mathrm{d}V_\lambda)$ with $p \geqslant 1$.

THEOREM 8. *Suppose $\mu \geqslant 0$ and $\|\mu\|_* < +\infty$, then T_μ is a bounded linear operator on $L^p_a(\Omega, \mathrm{d}V_\lambda)$ for all $1 < p < +\infty$ with norm $\|T_\mu\|_p \leqslant C\|\mu\|_*$, where C is a constant independent of μ (but depending on p and λ).*

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ and g be a polynomial in $L^q_a(\Omega, \mathrm{d}V_\lambda)$, then Fubini's theorem gives

$$\langle T_\mu f, g\rangle_\lambda = \int_\Omega f(z)\overline{g(z)}\mathrm{d}\mu(z).$$

Applying Cauchy-Schwarz inequality, we get

$$|\langle T_\mu f, g\rangle_\lambda| \leqslant \left(\int_\Omega |f(z)|^p \mathrm{d}\mu(z)\right)^{\frac{1}{p}} \left(\int_\Omega |g(z)|^q \mathrm{d}\mu(z)\right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \|\mu\|_p^{\frac{1}{p}} \|\mu\|_q^{\frac{1}{q}} \left(\int_\Omega |f(z)^p| \mathrm{d}V_\lambda(z)\right)^{\frac{1}{p}} \left(\int_\Omega |g(z)|^q \mathrm{d}V_\lambda(z)\right)^{\frac{1}{q}}.$$

By Corollary to Theorem 7, there exists a constant $C > 0$ (depending only on p) such that $\|\mu\|_q \leqslant C\|\mu\|_p$. Thus

$$|\langle T_\mu f, g\rangle_\lambda| \leqslant C^{\frac{1}{q}} \|\mu\|_p \|f\|_p \|g\|$$

for all f in $L^p_a(\Omega, \mathrm{d}V_\lambda)$ and polynomials g. Note that for $\lambda < \varepsilon_\Omega$, the polynomials expand $L^q_a(\Omega, \mathrm{d}V_\lambda)$. Also it is easy to see that $L^p_a(\Omega, \mathrm{d}V_\lambda)^* \cong L^q_a(\Omega, \mathrm{d}V_\lambda)$. Thus the above inequality shows that T_μ is bounded on $L^p_a(\Omega, \mathrm{d}V_\lambda)$ with $\|T_\mu\|_p$ dominated by $\|\mu\|_p$. ▨

When $p = 2$, the converse of the above result is also true.

THEOREM 9. *If* $\mu \geq 0$ *and* T_μ *is bounded on* $L^2_a(\Omega, \mathrm{d}V_\lambda)$, *then* μ *is a Carleson measure on* $L^2_a(\Omega, \mathrm{d}V_\lambda)$ *with* $\|\mu\|_*$ *dominated by* $\|T_\mu\|_2$.

Proof. For any f in $L^2_a(\Omega, \mathrm{d}V_\lambda)$, we have

$$\int_\Omega |f(z)|^2 \mathrm{d}\mu(z) = \langle T_\mu f, f\rangle_\lambda \leqslant \|T_\mu f\|_2 \|f\|_2 \leqslant$$

$$\leqslant \|T_\mu\|_2 \|f\|_2 \|f\|_2 = \|T_\mu\|_2 \int_\Omega |f(z)|^2 \,\mathrm{d}V_\lambda(z).$$

This proves the theorem. ∎

COROLLARY. *For positive Borel measures* μ *on* Ω, $\|\mu\|_*$ *is equivalent to*

$$\|T_\mu\|_2 = \sup\left\{\|T_\mu f\|_2 : \int_\Omega |f(z)|^2 \,\mathrm{d}V_\lambda(z) \leqslant 1\right\}.$$

Let $\mathscr{C}_\lambda$ be the space of all finite complex Borel measures μ on Ω such that $|\mu|$ is a Carleson measure on the weighted Bergman spaces. We extend the definition of $\|\ \|_*$ to $\mathscr{C}_\lambda$ by letting

$$\|\mu\|_* = \||\mu|\|_*$$

for all $\mu \in \mathscr{C}_\lambda$. We have

THEOREM 10. $(\mathscr{C}_\lambda, \|\ \|_*)$ *is a Banach space.*

Proof. Clearly $\mathscr{C}_\lambda$ is a normed space. We only need to show the completeness of $\mathscr{C}_\lambda$. Assume that $\{\mu_n\}$ is a Cauchy sequence, then for any $\varepsilon > 0$, there is a positive integer K with $\|\mu_n - \mu_m\|_* < \varepsilon$ whenever $n, m \geqslant K$. In particular,

$$|\mu_n - \mu_m|(\Omega) = \int_\Omega \mathrm{d}|\mu_n - \mu_m|(z) \leqslant \|\mu_n - \mu_m\|_* < \varepsilon$$

for all $n, m \geqslant K$. This implies that $\{\mu_n\}$ is a Cauchy sequence in $M(\Omega)$ $(= C_0(\Omega)^*)$, the space of all finite complex Borel measures on Ω. By the completeness of $M(\Omega)$, there exists $\mu \in M(\Omega)$ with $|\mu_n - \mu|(\Omega) \to 0$ $(n \to +\infty)$. Next we show that $\mu \in \mathscr{C}_\lambda$ and $\|\mu_n - \mu\|_* \to 0$ $(n \to +\infty)$.

Given $\varepsilon > 0$, choose K such that

$$\int_\Omega |k_a(z)|^{2(1-\lambda)} \mathrm{d}|\mu_n - \mu_m|(z) < \varepsilon$$

for all a in Ω and $n, m \geqslant K$. Let $m \to +\infty$ in the above inequality, then the boundedness of $k_a(z)$ (for fixed a in Ω) implies that

$$\int_\Omega |k_a(z)|^{2(1-\lambda)} \mathrm{d}|\mu_n - \mu|(z) \leqslant \varepsilon \quad (n \geqslant K, a \in \Omega).$$

This shows that $\mu \in \mathscr{C}_\lambda$ and $\|\mu_n - \mu\|_* \leqslant \varepsilon$ for all $n \geqslant K$. Thus $\mu_n \to \mu$ in $(\mathscr{C}_\lambda, \|\ \|_*)$. ▨

4. COMPACT TOEPLITZ OPERATORS

Recall that $\mathscr{C}_\lambda$ is the Banach space of finite complex Borel measures μ on Ω such that $|\mu|$ is a Carleson measure on the Bergman spaces. Let $\mathscr{C}_\lambda^0$ be the subspace of $\mathscr{C}_\lambda$ consisting of measures μ such that

$$\lim_{a \to \partial\Omega} \frac{|\mu|(E(a, r))}{|E(a, r)|^{1-\lambda}} = 0$$

for all (or some) $r > 0$. We have

THEOREM 11. *Given $\mu \in \mathscr{C}_\lambda$, the following conditions are all equivalent:*
(1) $T_{|\mu|}$ *is compact on* $L_a^2(\Omega, \mathrm{d}V_\lambda)$;
(2) $i_p: L_a^p(\Omega, \mathrm{d}V_\lambda) \to L^p(\Omega, \mathrm{d}|\mu|)$ *is compact for all (or some)* $p \geqslant 1$;
(3) $\widetilde{|\mu|}_\lambda(z) \in C_0(\Omega)$, i.e., $\int_\Omega |k_a(z)|^{2(1-\lambda)} \mathrm{d}|\mu|(z) \to 0$ $(a \to \partial\Omega)$;
(4) $\mu \in \mathscr{C}_\lambda^0$, i.e., $\widehat{|\mu|}_r(z) \in C_0(\Omega)$ *for all (or some)* $r > 0$;
(5) $\widehat{|\mu|}_r(a_n) \to 0$ $(n \to +\infty)$;
(6) *There exists a sequence* $\{\mu_n\}$ *in* $\mathscr{C}_\lambda$ *with compact supports such that* $\|\mu_n - \mu\|_* \to 0$ $(n \to \infty)$.

Proof. Since all the statements are about $|\mu|$, we may as well assume that $\mu \geqslant 0$. The proof will follow the order:

$$(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2) \Rightarrow (3)$$

$$(6) \Rightarrow (1) \Rightarrow (4) \Rightarrow (6).$$

(3) ⇒ (4). Suppose

$$\int_{\Omega} |k_a(z)|^{2(1-\lambda)} \mathrm{d}\mu(z) \to 0 \quad (a \to \partial\Omega),$$

then

$$\int_{E(a,r)} |k_a(z)|^{2(1-\lambda)} \mathrm{d}\mu(z) \to 0 \quad (a \to \partial\Omega)$$

for all $r > 0$. Applying Lemma 1, we see that

$$\frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} \to 0 \quad (a \to \partial\Omega)$$

for all $r > 0$.

(4) ⇒ (5). The construction of $\{a_n\}$ implies that $a_n \to \partial\Omega$ as $n \to +\infty$.

(5) ⇒ (2). Assume that

$$\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} \to 0 \quad (n \to +\infty).$$

We wish to show that i_p: $L^p_a(\Omega, \mathrm{d}V_\lambda) \to L^p(\Omega, \mathrm{d}\mu)$ is compact for all $p \geqslant 1$. It suffices to show that $\int_{\Omega} |f_n(z)|^p \, \mathrm{d}\mu(z) \to 0$ if $f_n \to 0$ weakly in $L^p_a(\Omega, \mathrm{d}V_\lambda)$. It is easy to see that $f_n \to 0$ weakly in $L^p_a(\Omega, \mathrm{d}V_\lambda)$ iff $f_n(z) \to 0$ uniformly on compact sets and the norms of f_n in $L^p_a(\Omega, \mathrm{d}V_\lambda)$ are bounded above. Now fix $p \geqslant 1$ and a sequence $\{f_n\}$ in $L^p_a(\Omega, \mathrm{d}V_\lambda)$ which converges to 0 weakly. Let

$$M = \sup\left\{\int |f_n(z)|^p \mathrm{d}V_\lambda(z) : n = 1, 2, \ldots\right\}.$$

Then by the proof of Theorem 7, we have

$$\int_{\Omega} |f_n(z)|^p \mathrm{d}\mu(z) \leqslant \sum_{k=1}^{\infty} \int_{E(a_k, r)} |f_n(z)|^p \mathrm{d}\mu(z) =$$

$$= \sum_{k=1}^{K} \int_{E(a_k, r)} |f_n(z)|^p \mathrm{d}\mu(z) + \sum_{k=K+1}^{\infty} \int_{E(a_k, r)} |f_n(z)|^p \mathrm{d}\mu(z) \leqslant$$

$$\leqslant \sum_{k=1}^{K} \int_{E(a_k, r)} |f_n(z)|^p \mathrm{d}\mu(z) + C \sum_{k=K+1}^{\infty} \frac{\mu(E(a_k, r))}{|E(a_k, r)|^{1-\lambda}} \int_{E(a_k, 2r)} |f(z)|^p \mathrm{d}V_\lambda(z).$$

Given $\varepsilon > 0$, choose a positive integer K such that

$$\frac{\mu(E(a_k, r))}{|E(a_k, r)|^{1-\lambda}} < \varepsilon$$

for all $k \geqslant K + 1$. Then we have

$$\int_{\Omega} |f_n(z)|^p d\mu(z) \leqslant \sum_{k=1}^{K} \int_{E(a_k,r)} |f_n(z)|^p d\mu(z) + C\varepsilon \sum_{k=K+1}^{\infty} \int_{E(a_k,2r)} |f(z)|^p dV_\lambda(z) \leqslant$$

$$\leqslant \sum_{k=1}^{K} \int_{E(a_k,r)} |f(z)|^p d\mu(z) + CNM\varepsilon$$

for all $n = 1, 2, \ldots$. Since $f_n(z) \to 0$ uniformly on compact sets and each $E(a_k, r)$ has compact closure in Ω, thus letting $n \to +\infty$ in the above inequality gives

$$\overline{\lim_{n\to+\infty}} \int_{\Omega} |f_n(z)|^p d\mu(z) \leqslant CNM\varepsilon.$$

It follows that

$$\lim_{n\to\infty} \int_{\Omega} |f_n(z)|^p d\mu(z) = 0$$

since ε is arbitrary and C, N, M are constants independent of ε. Hence i_p is compact.

(2) $\Rightarrow$ (3). Since i_p is compact and $k_a^{\frac{2(1-\lambda)}{p}} \to 0$, weakly in $L^p_a(\Omega, dV_\lambda)$ as $a \to \partial\Omega$, thus $\int_{\Omega} |k_a(z)|^{2(1-\lambda)} d\mu(z) \to 0$ as $a \to \partial\Omega$.

This completes the proof of the equivalence of (2), (3), (4), and (5). Next we prove the equivalence of (1), (4), and (6).

(6) $\Rightarrow$ (1). Suppose $\|\mu_n - \mu\|_* \to 0$, then $\|T_{\mu_n - \mu}\| \to 0$. It is easy to check that $\|T_{|\mu_n|} - T_{|\mu|}\| \leqslant \|T_{|\mu_n - \mu|}\|$. So we have $\|T_{|\mu_n|} - T_{|\mu|}\| \to 0$ $(n \to \infty)$. Since each $T_{|\mu_n|}$ is compact and the set of compact operators on $L^2_a(\Omega, dV_\lambda)$ is closed, $T_{|\mu|}$ must be compact.

(1) $\Rightarrow$ (4). Since T_μ is compact and $k_a^{1-\lambda} \to 0$ weakly in $L^2_a(\Omega, dV_\lambda)$ we must have

$$\langle T_\mu k_a^{1-\lambda}, k_a^{1-\lambda} \rangle_\lambda = \int_{\Omega} |k_a(z)|^{2(1-\lambda)} d\mu(z) \to 0.$$

By the equivalence of (3) and (4), we have $\hat{\mu}_r \in C_0(\Omega)$ for all $r > 0$.

(4) $\Rightarrow$ (6). Suppose $r > 0$ and $\hat{\mu}_r \in C_0(\Omega)$. Given $\varepsilon > 0$, choose $R > 0$ such that $\frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} < \varepsilon$ for all $\beta(0, a) > R$. Let $\mu_n = \chi_{E(0,n)}\mu$ $(n = 1, 2, \ldots)$, then for all $n \geqslant R + r$, we have

$$\|\mu_n - \mu\|_* = \sup\left\{\frac{|\mu_n - \mu|(E(a, r))}{|E(a, r)|^{1-\lambda}} : a \in \Omega\right\} \leqslant$$

$$\leqslant \sup\left\{\frac{|\mu_n - \mu|(E(a, r))}{|E(a, r)|^{1-\lambda}} : \beta(0, a) \leqslant R\right\} + \sup\left\{\frac{|\mu_n - \mu|(E(a, r))}{|E(a, r)|^{1-\lambda}} : \beta(0, a) \geqslant R\right\} =$$

$$= \sup\left\{\frac{|\mu_n - \mu|(E(a, r))}{|E(a, r)|^{1-\lambda}} : \beta(0, a) \geqslant R\right\} \leqslant 2 \sup\left\{\frac{\mu(E(a, r))}{|E(a,r)|^{1-\lambda}} : \beta(0, a) \geqslant R\right\} \leqslant 2\varepsilon.$$

This completes the proof of the theorem. □

COROLLARY. *$\mathscr{C}_\lambda^0$ is a closed subspace of $\mathscr{C}_\lambda$.*

5. TRACE IDEAL TOEPLITZ OPERATORS

Let A be a bounded operator on a separable Hilbert space H, then the s-numbers of A are defined by

$$s_n(A) = \inf\{\|A - B\| : B \in \mathscr{R}_n\}, \quad n = 1, 2, \ldots,$$

where $\mathscr{R}_n$ is the set of all bounded operators on H with rank $\leqslant n$. Since $\mathscr{R}_n \subset \mathscr{R}_{n+1}$, the sequence $\{s_n(A)\}$ is non-increasing: $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A) \geqslant \ldots$. It is well-known that if A is compact on H, then there exist orthonomal vectors $\{e_n\}$ and $\{\sigma_n\}$ in H with

$$A = \sum_{n=1}^{\infty} s_n\langle \cdot\,, e_n\rangle\sigma_n$$

for

$$Ax = \sum_{n=1}^{\infty} s_n \langle x, e_n\rangle\sigma_n.$$

For any $1 \leqslant p < +\infty$, the Schatten ideal $S_p(H)$ is defined to be the set of all compact operators A on H such that $\sum_{n=1}^{\infty} (s_n(A))^p < +\infty$. S_p is a Banach space with the norm $\|A\|_{S_p} = \left[\sum_{n=1}^{\infty} (s_n(A))^p\right]^{\frac{1}{p}}$. S_p is also a two-sided ideal of the full algebra $\mathscr{B}(H)$ of all bounded linear operators on H. For any $A \in S_p$ and $B, C \in \mathscr{B}(H)$, we have,

$$\|BAC\|_{S_p} \leqslant \|B\|\|A\|_{S_p}\|C\|.$$

If $A \in S_1$ and $\{e_n\}$ is an orthonormal basis for H, then

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$$

is convergent and independent of $\{e_n\}$. If $A \in S_1$ and $A \geqslant 0$, then $\|A\|_{S_1} = \operatorname{tr}(A)$. In general, we have

$$\|A\|_{S_p} = [\operatorname{tr}((A^*A)^{\frac{p}{2}})]^{\frac{1}{p}}.$$

For more information on the Schatten ideals, see [6] for example.

In this section, we characterize and study the positive measures μ on Ω such that T_μ is in $S_p(L^2_a(\Omega, \mathrm{d}V_\lambda))$. Hereafter, we will simply write $S_p = S_p(L^2_a(\Omega, \mathrm{d}V_\lambda))$

THEOREM 12. *Suppose $\mu \geqslant 0$ is a finite Borel measure on Ω. If $p \geqslant 1$ and $r > 0$, then the following conditions are all equivalent:*

(1) $T_\mu \in S_p$;

(2) $\tilde{\mu}_\lambda \in L^p(\Omega, K(z,z)\mathrm{d}V(z))$;

(3) $\hat{\mu}_r \in L^p(\Omega, K(z, z)\mathrm{d}V(z))$;

(4) $\sum_{n=1}^{\infty}\left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p = \sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p < +\infty$, *where $\{a_n\}$ is the sequence given by Lemma* 4.

Before proving this theorem, we need two lemmas.

LEMMA 13. *If $A \in S_1$, then*

$$\operatorname{tr}(A) = \int_\Omega \langle AK_\lambda(\cdot, z), K_\lambda(\cdot, z)\rangle_\lambda \mathrm{d}V_\lambda(z).$$

Proof. Let $\{e_n\}$ be an orthonormal basis for $L^2_a(\Omega, \mathrm{d}V_\lambda)$, then $K_\lambda(z, w) = \sum_{n=1}^{\infty} e_n(z)\bar{e}_n(w)$ and so

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle_\lambda = \sum_{n=1}^{\infty}\int_\Omega (Ae_n)(z)\overline{e_n(z)}\mathrm{d}V_\lambda(z) =$$

$$= \sum_{n=1}^{\infty} \int_{\Omega} \langle Ae_n, K_\lambda(\cdot, z)\rangle_\lambda \overline{e_n(z)} \mathrm{d}V_\lambda(z) = \sum_{n=1}^{\infty} \int_{\Omega} \langle e_n, A^* K_\lambda(\cdot, z)\rangle_\lambda \overline{e_n(z)}\, \mathrm{d}V_\lambda(z) =$$

$$= \sum_{n=1}^{\infty} \int_{\Omega} \overline{e_n(z)} \mathrm{d}V_\lambda(z) \int_{\Omega} e_n(w) \overline{A^* K_\lambda(\cdot, z)(w)} \mathrm{d}V_\lambda(w) =$$

$$= \int_{\Omega} \mathrm{d}V_\lambda(z) \int_{\Omega} \left(\sum_{n=1}^{\infty} \overline{e_n(z)} e_n(w) \right) \overline{A^* K_\lambda(\cdot, z)(w)}\, \mathrm{d}V_\lambda(w) =$$

$$= \int_{\Omega} \mathrm{d}V_\lambda(z) \int_{\Omega} K_\lambda(w, z) \overline{A^* K_\lambda(\cdot, z)(w)} \mathrm{d}V_\lambda(w) =$$

$$= \int_{\Omega} \mathrm{d}V_\lambda(z) \overline{\int_{\Omega} A^* K_\lambda(\cdot, z)(w) K_\lambda(z, w) \mathrm{d}V_\lambda(w)} =$$

$$= \int_{\Omega} \overline{A^* K_\lambda(\cdot, z)(z)} \mathrm{d}V_\lambda(z) = \int_{\Omega} \overline{\langle A^* K_\lambda(\cdot, z), K_\lambda(\cdot, z)\rangle_\lambda} \mathrm{d}V_\lambda(z) =$$

$$= \int_{\Omega} \langle AK_\lambda(\cdot, z), K_\lambda(\cdot, z)\rangle_\lambda \mathrm{d}V_\lambda(z). \qquad \blacksquare$$

LEMMA 14. *Suppose A, B are compact and $0 \leqslant A \leqslant B$, then $s_n(A) \leqslant s_n(B)$ for all $n = 1, 2, \ldots$.*

Proof.

$$s_1(A) = \|A\| = \sup\{\langle Ax, x\rangle : \|x\| = 1\} \leqslant$$

$$\leqslant \sup\{\langle Bx, x\rangle : \|x\| = 1\} = \|B\| = s_1(B).$$

For $n \geqslant 1$, we apply the theorem in § 2.1 of [6] to get

$$s_{n+1}(A) = \min\left\{ \sup_{x \in L^\perp} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} : L \text{ is } n\text{-dimensional in } H \right\} \leqslant$$

$$\leqslant \min\left\{ \sup_{x \in L^\perp} \frac{\langle Bx, x\rangle}{\langle x, x\rangle} : L \text{ is } n \text{ dimensional in } H \right\} = s_{n+1}(B). \qquad \blacksquare$$

We now prove Theorem 12. We will follow the order

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1).$$

(1) $\Rightarrow$ (2). Suppose $T_\mu \in S_p$. Since $T_\mu \geqslant 0$, we have

$$\|T_\mu\|_{S_p}^p = \operatorname{tr}(T_\mu^p).$$

Using Lemma 13, we get

$$\|T_\mu\|_{S_p}^p = \int_{\Omega} \langle T_\mu^p K_\lambda(\cdot\,, z),\ K_\lambda(\cdot\,, z)\rangle_\lambda \mathrm{d}V(z) =$$

$$= \int_{\Omega} K(z, z)^{1-\lambda} \langle T_\mu^p k_z^{1-\lambda},\ k_z^{1-\lambda}\rangle_\lambda \,\mathrm{d}V_\lambda(z).$$

Since $p \geqslant 1$ and each $k_z^{1-\lambda}$ is a unit vector in $L_a^2(\Omega, \mathrm{d}V_\lambda)$, we can apply Proposition 6.4 of [1] to get

$$\|T_\mu\|_{S_p}^p \geqslant \int_{\Omega} K(z, z)^{1-\lambda} [\langle T_\mu k_z^{1-\lambda},\ k_z^{1-\lambda}\rangle_\lambda]^p \mathrm{d}V_\lambda(z) =$$

$$= C_\lambda^{-1} \int_{\Omega} (\tilde{\mu}_\lambda(z))^p K(z, z) \mathrm{d}V(z).$$

(2) $\Rightarrow$ (3). By Lemma 1, there is a constant $C > 0$ such that

$$\hat{\mu}_r(a) = \frac{1}{|E(a, r)|^{1-\lambda}} \int_{E(a,r)} \mathrm{d}\mu(z) \leqslant C \int_{E(a,r)} |k_a(z)|^{2(1-\lambda)} \mathrm{d}\mu(z) \leqslant$$

$$\leqslant C \int_{\Omega} |k_a(z)|^{2(1-\lambda)} \mathrm{d}\mu(z) = C\tilde{\mu}_\lambda(a).$$

(3) $\Rightarrow$ (4). By Lemma 6, we have

$$\mu(E(a_n, r)) \leqslant \frac{C}{|E(a_n, r)|} \int_{E(a_n, r)} \mu(E(z, r)) \mathrm{d}V(z), \quad n = 1, 2, \ldots .$$

Cauchy-Schwarz inequality gives

$$[\mu(E(a_n, r))]^p \leqslant \frac{C^p}{|E(a_n, r)|} \int_{E(a_n,r)} (\mu(E(z, r)))^p \mathrm{d}V(z), \quad n = 1, 2, \ldots .$$

By Lemma 2, there exists a constant $C_1 > 0$ such that

$$\left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p \leqslant \frac{C_1}{|E(a_n, r)|} \int_{E(a_n,r)} \left[\frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}}\right]^p \mathrm{d}V(z) \quad n = 1, 2, \ldots .$$

By Lemma 1, there exists a constant $C_2 > 0$ such that

$$\left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p \leqslant C_2 \int_{E(a_n,r)} \left[\frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}}\right]^p K(z, z)\mathrm{d}V(z) \quad n = 1, 2, \ldots .$$

Therefore,

$$\sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p \leqslant C_2 N \int_{\Omega} \left[\frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}}\right]^p K(z, z)\mathrm{d}V(z),$$

or

$$\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p \leqslant C_2 N \int_{\Omega} (\hat{\mu}_r(z))^p K(z, z)\mathrm{d}V(z).$$

(4) $\Rightarrow$ (1). We prove this final implication by complex interpolation. First we prove the implication directly for $p = 1$.

$$\|T_\mu\|_{S_1} = \mathrm{tr}(T_\mu) = \int_{\Omega} \langle T_\mu K_\lambda(\cdot, z), K_\lambda(\cdot, z)\rangle_\lambda \mathrm{d}V_\lambda(z) =$$

$$= \int_{\Omega} \mathrm{d}V_\lambda(z) \int_{\Omega} |K_\lambda(w, z)|^2 \mathrm{d}\mu(w) = \int_{\Omega} \mathrm{d}\mu(w) \int_{\Omega} |K_\lambda(w, z)|^2 \mathrm{d}V_\lambda(z) =$$

$$= \int_{\Omega} K_\lambda(w, w)\mathrm{d}\mu(w) = \int_{\Omega} K(z, z)^{1-\lambda}\mathrm{d}\mu(z) \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \int_{E(a_n,r)} K(z, z)^{1-\lambda}\mathrm{d}\mu(z) \leqslant C \sum_{n=1}^{\infty} \frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}.$$

The last inequality follows from Lemma 1.

Now assume $1 < p < +\infty$ and $\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p < +\infty$; we wish to show that $T_\mu \in S_p$ with $\|T_\mu\|_{S_p}^p \leqslant C \sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p$. For any $\zeta \in \mathbf{C}$ with $0 \leqslant \operatorname{Re}\zeta \leqslant 1$, we define a measure μ_ζ by

$$\mu_\zeta(z) = \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} \right]^{p\zeta-1} \chi_{E(a_n,r)}(z)\mu(z).$$

Since $\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p < +\infty$, we may as well assume that $\mu(E(a_n, r)) < |E(a_n, r)|^{1-\lambda}$ for all $n = 1, 2, \ldots$. Thus

$$|\mu_\zeta|(\Omega) \leqslant \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} \right]^{p\operatorname{Re}\zeta-1} \mu(E(a_n, r)) \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} \right]^{-1} \mu(E(a_n, r)) = \sum_{n=1}^{\infty} |E(a_n, r)|^{1-\lambda} \leqslant$$

$$\leqslant C \sum_{n=1}^{\infty} \int_{E(a_n,r)} K(z, z)^\lambda \mathrm{d}V(z) \leqslant CN \int_\Omega K(z, z)^\lambda \mathrm{d}V(z) = CNC_\lambda < +\infty,$$

where the constant C is from Lemma 1 and N is from Lemma 4. Therefore, each μ is a finite complex Borel measure on Ω. Note that μ_ζ depends holomorphically on ζ.

For any $0 \leqslant \operatorname{Re}\zeta \leqslant 1$, consider the Toeplitz operator T_{μ_ζ} on $L_a^2(\Omega, \mathrm{d}V_\lambda)$ defined by

$$T_{\mu_\zeta} f(z) = \int_\Omega K_\lambda(z, w) f(w) \mathrm{d}\mu_\zeta(w).$$

It is easy to see that

$$T_\mu \leqslant T_{\mu_{\frac{1}{p}}} \leqslant NT_\mu.$$

By Lemma 14, we have

$$\|T_\mu\|_{S_p} \leqslant \|T_{\mu_{\frac{1}{p}}}\|_{S_p} \leqslant N\|T_\mu\|_{S_p}.$$

So in order to prove $T_\mu \in S_p$, it suffices to prove $T_{\mu_{\frac{1}{p}}} \in S_p$. But by complex interpolation (see §3.13 of [6] for example), we have $\|T_{\mu_{\frac{1}{p}}}\|_{S_p} \leqslant M_0^{1-\frac{1}{p}} M_1^{\frac{1}{p}}$, where

$$M_0 = \sup\{\|T_{\mu_\zeta}\| : \mathrm{Re}\,\zeta = 0\},$$

$$M_1 = \sup\{\|T_{\mu_\zeta}\|_{S_1} : \mathrm{Re}\,\zeta = 1\}.$$

Note that for complex interpolation, S_∞ is defined to be the set of all compact operators A with $\|A\|_{S_\infty} = \|A\|$.

Next we show that $M_0 < +\infty$ and M_0 is independent of μ. We also show that

$$M_1 \leqslant C \sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p$$

with C a constant independent of μ. This will finish the proof of the theorem.

For f and g in $L_a^2(\Omega, \mathrm{d}V_\lambda)$, we have

$$\langle T_{\mu_\zeta} f, g \rangle_\lambda = \int_\Omega f(z)\overline{g(z)}\mathrm{d}\mu_\zeta(z).$$

So

$$|\langle T_{\mu_\zeta} f, g\rangle_\lambda|^2 \leqslant \int_\Omega |f(z)|^2 \mathrm{d}|\mu_\zeta|(z) \int_\Omega |g(z)|^2 \mathrm{d}|\mu_\zeta|(z).$$

When $\mathrm{Re}\,\zeta = 0$, we have

$$|\mu_\zeta|(E(a_k, r)) \leqslant \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^{-1} \int_{E(a_k, r)} \chi_{E(a_n, r)}(z)\mathrm{d}\mu(z) =$$

$$= \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^{-1} \mu(E(a_k, r) \cap E(a_n, r)).$$

Note that $E(a_k, r) \cap E(a_n, r) \neq 0$ implies that $a_k \in E(a_n, 2r)$. Thus by (2) of Lemma 4, $E(a_k, r)$ meets at most N of the sets $E(a_n, r)$ $(n = 1, 2, \ldots)$ for any fixed positive integer k. Therefore, for any $k = 1, 2, \ldots,$ there exists $N_k \leqslant N$ such that

$$|\mu_\zeta|(E(a_k, r)) \leqslant \sum_{i=1}^{N_k} \left[\frac{\mu(E(a_{n_i}, r))}{|E(a_{n_i}, r)|^{1-\lambda}}\right]^{-1} \mu(E(a_k, r) \cap E(a_{n_i}, r)) \leqslant$$

$$\leqslant \sum_{i=1}^{N_k} \left[\frac{\mu(E(a_{n_i}, r))}{|E(a_{n_i}, r)|^{1-\lambda}}\right]^{-1} \mu(E(a_{n_i}, r)) = \sum_{i=1}^{N_k} |E(a_{n_i}, r)|^{1-\lambda}.$$

By Lemma 2, $E(a_{n_i}, r) \cap E(a_k, r) \neq 0$ implies that

$$|E(a_{n_i}, r)| \leqslant C|E(a_k, r)|$$

with $C > 0$ depending only on r. Thus for all $k = 1, 2, \dots$, we have

$$|\mu_\zeta|(E(a_k, r)) \leqslant C^{1-\lambda} N_k |E(a_k, r)|^{1-\lambda} \leqslant NC^{1-\lambda}|E(a_k, r)|^{1-\lambda}.$$

By Corollary to Theorem 7, there exists a constant $C > 0$ independent of μ and ζ such that

$$\int_\Omega |f(z)|^2 \mathrm{d}|\mu_\zeta|(z) \leqslant C \int_\Omega |f(z)|^2 \mathrm{d}V_\lambda(z)$$

for all f in $L^2_a(\Omega, \mathrm{d}V_\lambda)$. Therefore,

$$|\langle T_{\mu_\zeta} f, g\rangle_\lambda|^2 \leqslant C^2 \int_\Omega |f(z)|^2 \mathrm{d}V_\lambda(z) \int_\Omega |g(z)|^2 \mathrm{d}V_\lambda(z).$$

This implies that $\|T_{\mu_\zeta}\| \leqslant C$ for all ζ with $\operatorname{Re}\zeta = 0$.

When $\operatorname{Re}\zeta = 1$, we have

$$\int_\Omega K(z, z)^{1-\lambda} \mathrm{d}|\mu_\zeta|(z) \leqslant \sum_{n=1}^\infty \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^{p-1} \int_{E(a_n, r)} K(z, z)^{1-\lambda} \mathrm{d}\mu(z) \leqslant$$

$$\leqslant C \sum_{n=1}^\infty \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^{p-1} \frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}} = C \sum_{n=1}^\infty \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p,$$

where the constant C is from Lemma 1. If $\{f_n\}$ and $\{g_n\}$ are orthonormal bases for $L^2_a(\Omega, \mathrm{d}V_\lambda)$, then

$$\sum_{n=1}^\infty |\langle T_{\mu_\zeta} f_n, g_n\rangle_\lambda| \leqslant \sum_{n=1}^\infty \int_\Omega |f_n(z)||g_n(z)| \mathrm{d}|\mu_\zeta|(z) =$$

$$= \int_\Omega \left(\sum_{n=1}^\infty |f_n(z)||g_n(z)|\right) \mathrm{d}|\mu_\zeta|(z) \leqslant \int_\Omega \left[\sum_{n=1}^\infty |f_n(z)|^2\right]^{\frac{1}{2}} \left[\sum_{n=1}^\infty |g_n(z)|^2\right]^{\frac{1}{2}} \mathrm{d}|\mu_\zeta|(z) =$$

$$= \int_\Omega K_\lambda(z, z) \mathrm{d}|\mu_\zeta|(z) = \int_\Omega K(z, z)^{1-\lambda} \mathrm{d}|\mu_\zeta|(z) \leqslant C \sum_{n=1}^\infty \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p$$

for all $\operatorname{Re}\zeta = 1$. This implies that

$$\|T_{\mu_\zeta}\|_{S_1} \leqslant C \sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{E(a_n, r)^{1-\lambda}} \right]^p$$

for all $\operatorname{Re}\zeta = 1$. Therefore

$$M_1 \leqslant C \sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p.$$

Hence

$$\|T_\mu\|_{S_p} \leqslant M_0^{1-\frac{1}{p}} M_1^{\frac{1}{p}} \leqslant C^{\frac{1}{p}} M_0^{1-\frac{1}{p}} \left(\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p \right)^{\frac{1}{p}}$$

with M_0 and C independent of μ. This completes the proof of Theorem 12. ∎

COROLLARY. *For a positive Borel measure μ on Ω, the following quantities are equivalent:*

(1) $\|T_\mu\|_{S_p}$;

(2) $\left[\int_\Omega (\tilde{\mu}_\lambda(z))^p K(z, z) \mathrm{d}V(z) \right]^{\frac{1}{p}}$;

(3) $\left[\int_\Omega (\hat{\mu}_r(z))^p K(z, z) \mathrm{d}V(z) \right]^{\frac{1}{p}}$;

(4) $\left[\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p \right]^{\frac{1}{p}}$.

6. EXAMPLES OF CARLESON MEASURES AND TRACE IDEAL TOEPLITZ OPERATORS

Consider a special class of measures of the form

$$\mathrm{d}\mu(z) = |f(z)|^s K(z, z)^\alpha \mathrm{d}V(z)$$

with f holomorphic in Ω, $s \geqslant 1$, and α real. We have following:

PROPOSITION 15. *For the measure μ defined above, T_μ is bounded on $L_a^2(\Omega, \mathrm{d}V_\lambda)$ iff $|f(z)|^s K(z, z)^{\alpha-\lambda}$ is bounded in Ω; T_μ is compact on $L_a^2(\Omega, \mathrm{d}V_\lambda)$ iff $|f(z)|^p K(z, z)^{\alpha-\lambda} \to 0$, as $z \to \partial\Omega$.*

Proof. We prove the first statement. The second statement has a similar proof. If $|f(z)|^s K(z, z)^{\alpha-\lambda} \leqslant M$ for all z in Ω, then clearly,

$$\int_\Omega |g(z)|^p \mathrm{d}\mu(z) = \int_\Omega |g(z)|^p |f(z)|^s K(z, z)^\alpha \mathrm{d}V(z) =$$

$$= C_\lambda \int_\Omega |g(z)|^p |f(z)|^p K(z, z)^{\alpha-\lambda} \mathrm{d}V_\lambda(z) \leqslant C_\lambda M \int_\Omega |g(z)|^p \mathrm{d}V_\lambda(z)$$

for all g in $L^p_a(\Omega, \mathrm{d}V_\lambda)$. Thus μ is a Carleson measure on $L^p_a(\Omega, \mathrm{d}V_\lambda)$ and by Corollary to Theorem 9, T_μ is bounded on $L^2_a(\Omega, \mathrm{d}V_\lambda)$.

Conversely, if T_μ is bounded on $L^2_a(\Omega, \mathrm{d}V_\lambda)$, then by Corollary to Theorem 9, there exists a constant $C_1 > 0$ such that

$$\frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} \leqslant C_1$$

for all a in Ω. Choose $C_2 > 0$ such that

$$|f(z)|^s \leqslant \frac{C_2}{|E(z, r)|} \int_{E(z,r)} |f(w)|^s \mathrm{d}V(w)$$

for all z in Ω. By Lemmas 1 and 2, there exists $C_3 > 0$ such that

$$K(z, z)^{\alpha-\lambda} |f(z)|^s \leqslant \frac{C_3}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) =$$

$$= C_3 \frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}} \leqslant C_1 C_3$$

for all z in Ω. Thus $|f(z)|^s K(z, z)^{\alpha-\lambda}$ is bounded in Ω. ▨

PROPOSITION 16. *Let* $\mathrm{d}\mu(z) = |f(z)|^s K(z, z)^\alpha \mathrm{d}V(z)$ *with* f *holomorphic*, $s \geqslant 1$, *and* α *real, then for all* $p \geqslant 1$, *we have* $T_\mu \in S_p$ *iff* $|f(z)|^s K(z, z)^{\alpha-\lambda} \in L^p(\Omega, K(z, z)\mathrm{d}V(z))$.

Proof. Suppose $|f(z)|^s K(z, z)^{\alpha-\lambda} \in L^p(\Omega, K(z, z)\mathrm{d}V(z))$, then

$$M = \int_\Omega |f(z)|^{ps} K(z, z)^{\alpha p - \lambda p + 1} \mathrm{d}V(z) < +\infty.$$

By Lemma 1, there exists $C_1 > 0$ such that

$$\frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}} = \frac{\int_{E(a,r)} |f(z)|^s K(z, z)^\alpha \mathrm{d}V(z)}{|E(a, r)|^{1-\lambda}} \leqslant$$

$$\leqslant \frac{C_1}{|E(a, r)|} \int_{E(a,r)} |f(z)|^s K(z, z)^{\alpha-\lambda} \mathrm{d}V(z)$$

for all a in Ω. Apply Lemma 1 again to get another constant $C_2 > 0$ such that

$$\left(\frac{\mu(E(a, r))}{|E(a, r)|^{1-\lambda}}\right)^p \leqslant \frac{C_1^p}{|E(a, r)|} \int_{E(a,r)} |f(z)|^{sp} K(z, z)^{\alpha p-\lambda p} \mathrm{d}V(z) \leqslant$$

$$\leqslant C_2 \int_{E(a,r)} |f(z)|^{sp} K(z, z)^{\alpha p-\lambda p+1} \mathrm{d}V(z)$$

for all a in Ω. In particular, we have

$$\sum_{n=1}^{\infty} \left[\frac{\mu(E(a_n, r))}{|E(a_n, r)|^{1-\lambda}}\right]^p \leqslant C_2 \sum_{n=1}^{\infty} \int_{E(a_n,r)} |f(z)|^{sp} K(z, z)^{\alpha p-\lambda p+1} \mathrm{d}V(z) \leqslant C_2 NM < +\infty,$$

where N is from Lemma 4. By Theorem 12, we have $T_\mu \in S_p$.

Conversely, suppose $T_\mu \in S_p$, then by Theorem 12,

$$M = \int_\Omega \left[\int_\Omega |k_z(w)|^{2(1-\lambda)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w)\right]^p K(z, z) \mathrm{d}V(z) < +\infty.$$

But by Lemma 1, there exists a constant $\varepsilon_1 > 0$ such that

$$\int_\Omega |k_z(w)|^{2(1-\lambda)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) \geqslant$$

$$\geqslant \int_{E(z,r)} |k_z(w)|^{2(1-\lambda)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) \geqslant$$

$$\geqslant \frac{\varepsilon_1}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w)$$

for all z in Ω. By the proof of Proposition 15, there exists another constant $\varepsilon_2 > 0$ such that

$$\frac{1}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) \geqslant \varepsilon_2 |f(z)|^s K(z, z)^{\alpha-\lambda}$$

for all z in Ω. Therefore,

$$\int_\Omega [|f(z)|^s K(z, z)^{\alpha-\lambda}]^p K(z, z) \mathrm{d}V(z) \leqslant$$

$$\leqslant \frac{1}{\varepsilon_2^p} \int_\Omega \left[\frac{1}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) \right]^p K(z, z) \mathrm{d}V(z) \leqslant$$

$$\leqslant \frac{1}{\varepsilon_1^p \varepsilon_2^p} \int_\Omega \left[\int_\Omega |k_z(w)|^{2(1-\lambda)} |f(w)|^s K(w, w)^\alpha \mathrm{d}V(w) \right]^p K(z, z) \mathrm{d}V(z) = \frac{M}{\varepsilon_1^p \varepsilon_2^p} < +\infty. \quad ▨$$

REMARK. We can replace $|f(z)|^s$ in Propositions 15 and 16 by any positive subharmonic function in Ω.

PROPOSITION 17. *Suppose* φ *is a positive function on* Ω, $p \geqslant 1$, *and* $\varphi \in L^p(\Omega, K(z, z)\mathrm{d}V(z))$, *then* $T_\varphi \in S_p$.

Proof. By definition,

$$T_\varphi f(z) = \int_\Omega K_\lambda(z, w) \varphi(w) f(w) \mathrm{d}V_\lambda(w).$$

Let $\mathrm{d}\mu(z) = \varphi(z)\mathrm{d}V_\lambda(z)$ then $T_\varphi = T_\mu$. By Theorem 12, it suffices to show that $\sum_{n=1}^{\infty} (\hat{\mu}_r(a_n))^p < +\infty$.

$$\hat{\mu}_r(z) = \frac{1}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} \varphi(w) \mathrm{d}V_\lambda(w) =$$

$$= \frac{C_\lambda}{|E(z, r)|^{1-\lambda}} \int_{E(z,r)} \varphi(w) K(w, w)^\lambda \mathrm{d}V(w) \leqslant \frac{C_1}{|E(z, r)|} \int_{E(z,r)} \varphi(w) \mathrm{d}V(w),$$

where C_1 is a constant from Lemma 1. By Cauchy-Schwarz,

$$\hat{\mu}_r(z)^p \leqslant \frac{C_1^p}{|E(z, r)|} \int_{E(z,r)} \varphi(w)^p \mathrm{d}V(w).$$

Apply Lemma 1 again to get another constant $C_2 > 0$ with

$$\hat{\mu}_r(z)^p \leqslant C_2 \int_{E(z,r)} \varphi(w)^p K(w, w) \mathrm{d}V(w)$$

or all z in Ω. In particular,

$$\sum_{n=1}^{\infty} \hat{\mu}_r(a_n)^p \leqslant C_2 \sum_{n=1}^{\infty} \int_{E(a_n,r)} \varphi(w)^p K(w, w) \mathrm{d}V(w) \leqslant$$

$$\leqslant C_2 N \int_{\Omega} \varphi(w)^p K(w, w) \mathrm{d}V(w) < +\infty. \qquad \square$$

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