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TYPE III_0 TRANSFORMATIONS OF MEASURE SPACE AND OUTER CONJUGACY OF COUNTABLE AMENABLE GROUPS OF AUTOMORPHISMS

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INTRODUCTION

The present paper is a study of the classification of dynamical systems (groups of automorphisms of a measure space). In contrast to classical problems of ergodic theory primary among which is the classification of automorphisms up to isomorphism (conjugacy), we are here interested in a weaker relation of equivalence, the so called outer conjugacy of groups of automorphisms. Such problems have recently become attractive in connexion with the developing trajectory theory of dynamical systems and also with the study of automorphisms of von Neumann algebras. A. Connes and W. Krieger [6] suggested necessary and sufficient conditions for outer conjugacy for automorphisms from the normalizer of the approximately finite group Γ of transformations of a Lebesgue space preserving a measure (type II). We solved a similar problem for the case where for the group Γ there is no invariant measure (type III) [2], [3]. Thus, a complete system of invariants of outer conjugacy of actions of the group $\mathbf{Z}$ lying in the normalizer $N[\Gamma]$ of Γ, has been found.

There naturally arises the problem of the study of outer conjugacy of the actions of groups having a more complex structure than the group $\mathbf{Z}$ has. This became possible after the problem of description of abstract groups which are ranges of cocycles of approximately finite groups of automorphisms was solved [9]. First, we found a complete system of invariants for outer conjugacy of actions of countable amenable groups from the normalizer of the group Γ preserving a finite or infinite measure [4]. By the methods developed in [4], necessary and sufficient conditions for outer conjugacy of actions of countable amenable groups were then proved for the case where Γ is of type III_λ $(0 < \lambda \leqslant 1)$ [1].

In this paper, the most complicated case of the type III_0 group Γ of automorphisms is considered. The main result of the paper is that necessary and sufficient conditions for outer conjugacy of the actions ρ_1 and ρ_2 of the countable amenable group G, such that $\rho_i(g) \in N[\Gamma]$, $i = 1,2$, are found.

Section 1 presents certain preliminary discussion required to proceed to subsequent presentation. The next three sections contain the proof of the principal result of this paper. Sections 2 and 3 consider special cases of actions of a countable amenable group lying in the normalizer of a type III_0 group. The concluding section proves the central theorem in the general situation. Section 4 deals with the problem of existence of an action ρ of a countable amenable group G, such that $\rho(g) \in N[\Gamma]$, $g \in G$, where Γ is an a.f. type III_0 group of automorphisms. This problem is naturally subdivided into three cases: (A), (B) and (C) (see Subsection 2.2). In this paper we fully study the case (B), which finds a direct application, and present an example showing possible realization of the cases (A) and (C). These cases cannot be considered without using some new ideas and the solution of the problem of their realization will be supplied in a subsequent paper.

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1. PRELIMINARIES

1.1. In this section we shall present some preliminaries on ergodic theory which we shall need below. Definitions and more detailed results may be found in [16], [12], [20], [10], [19], [8].

The set of all automorphisms of a Lebesgue space $(X, \mathscr{B}, \mu)$ will be denoted by $\mathrm{Aut}(X, \mathscr{B}, \mu)$. Let Γ be a countable subgroup from $\mathrm{Aut}(X, \mathscr{B}, \mu)$*). The set of the automorphisms from $\mathrm{Aut}(X, \mathscr{B}, \mu)$, whose trajectories are contained in the trajectories of the group Γ, is called the full group of automorphisms $[\Gamma]$ of the group Γ, i.e.

$$[\Gamma] = \{\gamma \in \mathrm{Aut}(X, \mathscr{B}, \mu) : \gamma x \in \Gamma x \quad \text{for} \quad \mu\text{-a.e. } x \in X\}$$

where $\Gamma x = \{gx : g \in \Gamma\}$ is the trajectory of x.

The set $N[\Gamma] = \{R \in \mathrm{Aut}(X, \mathscr{B}, \mu) : R[\Gamma]R^{-1} = [\Gamma]\}$ which is clearly a subgroup in $\mathrm{Aut}(X, \mathscr{B}, \mu)$ is called the normalizer of the group of automorphisms $[\Gamma]$.

The group of automorphisms Γ is called approximately finite (a.f.) if there exists an automorphism T such that $[\Gamma] = [T]$.

Two groups of automorphisms $\Gamma_1 \subset \mathrm{Aut}(X_1, \mathscr{B}_1, \mu_1)$ and $\Gamma_2 \subset \mathrm{Aut}(X_2, \mathscr{B}_2, \mu_2)$ are called weakly equivalent, if there exist a one-to-one measurable map $\varphi: X_1 \to X_2$ such that $[\Gamma_1] = \varphi^{-1}[\Gamma_2]\varphi$ and the measures μ_2 and $\varphi \circ \mu_1$ are equivalent (denoted by $\mu_2 \sim \varphi \circ \mu_1$).

The ergodic group of automorphisms $\Gamma \subset \mathrm{Aut}(X, \mathscr{B}, \mu)$ is called a type II_1 (II_∞) group, if there exists a measure $\nu \sim \mu$ such that $g \circ \nu = \nu$ for any $g \in \Gamma$ and the measure $\nu(X)$ is finite (infinite). If there is no Γ-invariant measure equivalent to the measure μ, then the group Γ is called a type III group. Type III groups admit a

*) This paper deals only with countable groups of automorphisms.

further classification by introducing a set $r(\Gamma)$. By definition, a number $r \in r(\Gamma)$, if for any $\varepsilon > 0$ and any set A of positive measure there exists a set $B \subset A$ ($\mu B > 0$) and an automorphism $g \in \Gamma$ such that $gB \subset A$ and for μ-a.e. $x \in B$

$$\left| \frac{dg^{-1} \circ \mu}{d\mu}(x) - r \right| < \varepsilon.$$

The set $r(\Gamma)$ is closed in $[0, \infty)$ and $r(\Gamma) \setminus \{0\}$ is a subgroup of $\mathbf{R}_+^* = (0, \infty)$. Consequently, $r(\Gamma) \setminus \{0\}$ may only be one of the following groups: $\{1\}$, $\{\lambda^n : n \in \mathbf{Z}\}$ ($0 < \lambda < 1$), $\mathbf{R}_+^*$. Respectively, the group Γ is type III_0, or III_λ ($0 < \lambda < 1$), or III_1.

1.2. Let us describe in more detail the a.f. ergodic groups of type III_0 automorphisms as was done in [18].

Let $(X, \mathscr{B}, \mu)$, $(Y, \mathscr{F}, \nu)$ be Lebesgue spaces with $\mu(X) = 1$, $\nu(Y) = \infty$ and $Q \in \mathrm{Aut}(X, \mathscr{B}, \mu)$, $S \in \mathrm{Aut}(Y, \mathscr{B}, \nu)$ ergodic automorphisms, where $S \circ \nu = \nu$ (i.e., S is type II_∞).

For any automorphism $V \in \mathrm{N}[S]$ the measure $V^{-1} \circ \nu$ is also S-invariant and S-ergodic; therefore, there exists a number $\Phi(V) \in \mathbf{R}$ such that

$$V^{-1} \circ \nu = e^{\Phi(V)} \nu. \tag{1.1}$$

Let $\varphi(x)$ be a measurable real-valued function on X such that $\varphi(x) > \delta > 0$ and let $x \to U_x$ ($x \in X$) be a measurable field of automorphisms of $(Y, \mathscr{F}, \nu)$ (i.e. the map $(x, y) \to (x, U_x y)$, where $U_x \in \mathrm{Aut}(Y, \mathscr{F}, \nu)$, is measurable in $X \times Y$). Suppose that $U_x \in \mathrm{N}[S]$ and

$$\varphi(x) \doteq \Phi(U_x) + \log \frac{dQ^{-1} \circ \mu}{d\mu}(x) \tag{1.2}$$

for μ-a.e. $x \in X$. Consider the following automorphisms from $\mathrm{Aut}(X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu) = \mathrm{Aut}(X_0, \mathscr{B}_0, \mu_0)$:

$$Q_0(x, y) = (Qx, U_x y), \quad S_0(x, y) = (x, Sy). \tag{1.3}$$

Evidently, $Q_0 \in \mathrm{N}[S_0]$ and therefore the group of automorphisms $\mathscr{G}$ with Q_0 and S_0 as generators will be a.f.

In [18] the following statements are proved:

(1) the group $\mathscr{G}$ is an ergodic a.f. group of type III_0 automorphisms;

(2) any ergodic type III_0 automorphism T is weakly equivalent to a certain group $\mathscr{G}(Q, \varphi)$;

(3) two type III_0 groups of automorphisms $\mathscr{G}(Q_1, \varphi_1)$ and $\mathscr{G}(Q_2, \varphi_2)$ are weakly equivalent to each other, if and only if the corresponding special flows $\{W(Q_1, \varphi_1)(t)\}$ and $\{W(Q_2, \varphi_2)(t)\}$ constructed from the basis automorphisms Q_1 and Q_2 and the ceiling functions φ_1 and φ_2, are isomorphic (the definition of the special flows may by found, e.g. in [16], [20]).

1.3. Recall the definition of the modulus of an automorphism (see [2], [3], [11]) and the properties of automorphisms from $N[\mathcal{G}(Q, \varphi)]$.

If $\Gamma \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ is a certain group of automorphisms, then the group $\Gamma_d \subset \mathrm{Aut}(X \times \mathbf{R}, \mathcal{B} \times \mathcal{E}, \mu \times \tau)$ — where τ is the usual Lebesgue measure on $\mathbf{R}$ — dual to Γ is generated by the automorphisms

$$(1.4) \qquad g_d(x, u) = \left(gx,\ u + \log \frac{dg^{-1} \circ \mu}{d\mu}(x)\right), \quad g \in \Gamma.$$

The flow of the automorphisms $\{T_s\}_{s\in\mathbf{R}}$, $T_s(x, u) = (x, u + s)$ commutes with the group Γ_d. Therefore, if $\xi(\Gamma_d)$ is the measurable hull of the partition of the group Γ_d into trajectories, then on the quotient space $(X\times\mathbf{R})/\xi(\Gamma_d)$ the flow $\{T_s\}$ generates a quotient flow $\{W_\Gamma(s)\}_{s\in\mathbf{R}}$ which is called the flow associated with the group Γ.

If $R \in N[\Gamma]$, then $R_d \in N[\Gamma_d]$ and the automorphism R_d defines on $(X\times\mathbf{R})/\xi(\Gamma_d)$ a certain automorphism denoted by $\mathrm{mod}\, R$ which is called the modulus of the automorphism R. Consider the centralizer of the associated flow:

$$C\{W_\Gamma\} = \{\gamma \in \mathrm{Aut}((X\times\mathbf{R})/\xi(\Gamma_d), (\mu\times\tau)/\xi(\Gamma_d)) : \gamma W_\Gamma(t) = W_\Gamma(t)\gamma\}.$$

Then, we easily see that $\mathrm{mod}\, R \in C\{W_\Gamma\}$.

If the group Γ of automorphisms of the space $(X, \mathcal{B}, \mu)$ preserves the measure μ and $R \in N[\Gamma]$, then the number

$$\frac{dR^{-1} \circ \mu}{d\mu}(x) = \mathrm{mod}\, R$$

is called the modulus of the automorphism R. This definition is quite consistent with the above definition of the modulus.

Let now $\Gamma = \mathcal{G}(Q, \varphi)$. The flow associated with Γ will be a special flow constructed by the base automorphism Q^{-1} and the ceiling function $\varphi(Q^{-1}x)$.

The following main properties of automorphisms from the normalizer of the group $\mathcal{G} = \mathcal{G}(Q, \varphi)$ have been found in [3]:

(a) the map $R \to \mathrm{mod}\, R : N[\Gamma] \to C\{W_\Gamma\}$ is a surjective homomorphism;

(b) for any automorphism $R \in N[\mathcal{G}]$, there exists an automorphim $g \in [\mathcal{G}]$ such that

$$(1.5) \qquad gR(x, y) = (ax, V_x y),$$

where the automorphism a belongs to $N[Q]$ and $x \to V_x$ $(x \in X)$ is a measurable field of automorphisms taking values in $N[S]$. Automorphisms of the form (1.5) will be called *skew products*.

We shall need the following result presented in [17] (see also [16]): for any flow of automorphisms there exists a special flow isomorphic to it, whose ceiling function takes only two values. Because an a.f. group of type III_0 automorphisms is entirely defined by the flow associated with it, and (see (1.2)) the equality

$$\frac{\mathrm{d}(\mu \times \nu)(Q_0(x, y))}{\mathrm{d}(\mu \times \nu)(x, y)} = \varphi(x)$$

is true for μ-a.e. $x \in X$, then the above result suggests that for any a.f. group of type III_0 automorphisms there exists a group weakly equivalent to it, all its elements having the Radon-Nikodym derivative taking values in a countable set, e.g. in the group $\Lambda(r_1, r_2) = \{r_1^n r_2^m : n, m \in \mathbf{Z}\}$, where $r_1 > 0$, $r_2 > 0$ and $\log r_1$, $\log r_2$ are rationally independent.

1.4. In the group of all automorphisms of the Lebesgue space $(X, \mathscr{B}, \mu)$ with the probability measure μ one may introduce the concept of the distance between automorphisms. The formula

$$d_{\mathrm{u}}(T, S) = \mu(\{x \in X: Tx \neq Sx\}), \quad T, S \in \mathrm{Aut}(X, \mathscr{B}, \mu)$$

defines a metric that is called the uniform metric. Weak convergence in $\mathrm{Aut}(X, \mathscr{B}, \mu)$ is defined as follows. If $T \in \mathrm{Aut}(X, \mathscr{B}, \mu)$, then on $L^1(X, \mathscr{B}, \mu)$ there is a linear operator $U(T)$ defined by

$$(U(T)f)(x) = f(T^{-1}x)\frac{\mathrm{d}\mu(T^{-1}x)}{\mathrm{d}\mu(x)}, \quad f \in L^1(X, \mathscr{B}, \mu).$$

Put

$$d_{\mathrm{w}}(T, S) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\|(U(T) - U(S))\xi_k\|_{L^1} + \|(U(T^{-1}) - U(S^{-1}))\xi_k\|_{L^1}}{1 + \|(U(T) - U(S))\xi_k\|_{L^1} + \|(U(T^{-1}) - U(S^{-1}))\xi_k\|_{L^1}},$$

where $\{\xi_k\}_{k=1}^{\infty}$ is a countable sequence of functions dense in $L^1(X, \mathscr{B}, \mu)$. The metrics d_{u} and d_{w} allow to introduce a topology on the group $\mathrm{N}[\Gamma]$, where $\Gamma \subset \mathrm{Aut}(X, \mathscr{B}, \mu)$. We shall say that a sequence $\{R_n\}_{n=1}^{\infty}$ of elements of $\mathrm{N}[\Gamma]$ converges to $R \in \mathrm{N}[\Gamma]$, if $d_{\mathrm{w}}(R_n, R) \to 0$ and $d_{\mathrm{u}}(R_n g R_n^{-1}, RgR^{-1}) \to 0$ as $n \to \infty$ for any $g \in \Gamma$. Then, $\mathrm{N}(\Gamma)$ becomes a topological group and the topology introduced is generated by the metric

$$d(R_1, R_2) = d_{\mathrm{w}}(R_1, R_2) + \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_{\mathrm{u}}(R_1 g_k R_1^{-1}, R_2 g_k R_2^{-1})}{1 + d_{\mathrm{u}}(R_1 g_k R_1^{-1}, R_2 g_k R_2^{-1})}$$

where $R_1, R_2 \in \mathrm{N}[\Gamma]$ and $\Gamma = \{g_k : k \in \mathrm{N}\}$.

In [12] it was proved that $N[\Gamma]$ is a complete separable group. Here is also a result of [12] which will be used below. If $S \in \mathrm{Aut}(X, \mathscr{B}, \mu)$ and $S \circ \mu = \mu$, $\mu(X) = \infty$, then the closure of the full group $[S]$ in the metric d coincides with the set $\{R \in N[S] : \mathrm{mod}\, R = 1\}$.

1.5. In this work, we use the concept of an array as it was introduced in [18]. The following notation will be used: $\mathbf{1} = \mathbf{1}_X$ for the identity automorphism of the space $(X, \mathscr{B}, \mu)$ and e for the unit of the group.

Let Γ be an ergodic group of automorphisms of $(X, \mathscr{B}, \mu)$. The expressiou

$$\xi = (A, \Xi, A(\cdot), \gamma(\cdot\,,\cdot)) \tag{1.6}$$

will be called a *Γ-array* of the set $A \subset X$, $\mu(A) > 0$, if the following conditions are fulfilled:

(i) Ξ is a finite set of indices;

(ii) $\bigcup_{i \in \Xi} A(i) = A$, $A(i) \cap A(j) = \emptyset$ $(i \neq j)$, $\mu(A(i)) > 0$;

(iii) $\gamma(j, i)A(i) = A(j)$, $\gamma(i, i) = \mathbf{1}$, $\gamma(i_2, i_1)\gamma(i_1, i_0) = \gamma(i_2, i_0)$, $\gamma(j, i)x \in \Gamma_x$ for a.e. $x \in A(i)$, $i, j \in \Xi$.

Denote by $\mathscr{G}(\xi)$ the finite group of automorphisms of the set A generated by $\gamma(i, j)$, $i, j \in \Xi$ and by $\mathscr{P}(\xi)$ the collection of sets of the form $\bigcup_{i \in \Delta} A(i)$, where $\Delta \subset \Xi$.

We will refer to pairs $(A(i), \gamma(j, i))$, $i, j \in \Xi$ as elements of the array ξ.

Let two Γ-arrays be defined: $\xi_1 = (A, \Xi, A(\cdot), \gamma(\cdot\,,\cdot))$ and $\xi_2 = (A(i_0), \Omega, B(\cdot), \delta(\cdot\,,\cdot))$, where $i_0 \in \Xi$. Define a new Γ-array $\xi_1 \times \xi_2$, which we shall call the refinement of the array ξ_1 with respect to ξ_2, according to the equality

$$\xi_1 \times \xi_2 = (A, \Xi \times \Omega, C(\cdot\,,\cdot), \tau(\cdot\,,\cdot\,;\,\cdot,\cdot)),$$

where $C(i, n) = \gamma(i, i_0)B(n)$, $\tau(i_1, n_1; i, n) = \gamma(i_1, i_0)\delta(n_1, n)\gamma(i_0, i)$, $i_0, i, i_1 \in \Xi$, $n, n_1 \in \Omega$.

We shall also call the Γ-array (1.6) transitive, because for any two sets $A(i)$ and $A(j)$ property (iii) in the definition of the array is true. If this property is not true for all sets, then such an array will be called non-transitive. A typical example of non-transitive array is the Γ-array $\xi = \bigsqcup_{i=1}^{n} \xi(i)$ consisting of a disjoint union of the transitive arrays $\xi(i)$. Namely, let a set A be represented as a disjoint union of the sets $A(i)$, $i = 1, 2, \ldots, n$ and let the Γ-array $\xi(i) = (A(i), [1, M_i], A_i(\cdot), \gamma_i(\cdot\,,\cdot))$ be defined on the set $A(i)$. Then the restriction of the Γ-array $\xi = \bigsqcup_{i=1}^{n} \xi(i)$ to the set $A(i)$ coincides with the array $\xi(i)$. In this case, the transitivity components (i.e. the transitive subarrays) are the arrays $\xi(i)$ defined on the sets $A(i)$, $i = 1, 2, \ldots, n$. Below we shall use the notation $\mathscr{G}(\xi) = \bigsqcup_{i=1}^{n} \mathscr{G}(\xi(i))$, $P(\xi) = \bigsqcup_{i=1}^{n} P(\xi(i))$.

If the Γ-array ξ given in (1.6) exists, it will be said to be defined over the partition $(A, \Xi, A(\cdot))$.

1.6. Let $\Gamma \subset \mathrm{Aut}(X, \mathscr{B}, \mu)$ and H be a locally compact separable group. A measurab e map $\alpha: X \times \Gamma \to H$ is called a cocycle with the values in the group H, if for any g_1, g_2 from Γ the equality

$$\alpha(x, g_1 g_2) = \alpha(g_2 x, g_1)\,\alpha(x, g_2) \tag{1.7}$$

is fulfilled for μ-a.e. $x \in X$.

An example of a cocycle is the Radon-Nikodym cocycle $r: X \times \Gamma \to \mathbf{R}_+^*$

$$r(x, \gamma) = \frac{d\mu(\gamma x)}{d\mu(x)}.$$

If $\alpha: X \times \Gamma \to H$ is a cocycle and the group Γ acts freely, then α admits natural expansion to a map $\tilde{\alpha}: X \times [\Gamma] \to H$ which also satisfies (1.7). We shall assume all cocycle which will be considered to be expanded up to cocycles of full groups of automorphisms.

1.7. Let G be an abstract countable group. An *action* of the group G on $(X, \mathscr{B}, \mu)$ is an injective homomorphism ρ of G in $\mathrm{Aut}(X, \mathscr{B}, \mu)$.

Let $\Gamma \subset \mathrm{Aut}(X, \mathscr{B}, \mu)$, ρ_1 and ρ_2 be actions of the countable group G on $(X, \mathscr{B}, \mu)$, where $\rho_i(g) \in \mathrm{N}[\Gamma]$, $g \in G$, $i = 1, 2$. The actions ρ_1 and ρ_2 are called outer conjugate, if there exists an automorphism $R \in \mathrm{N}[\Gamma]$ such that for any $g \in G$

$$\rho_1(g) = R^{-1}\rho_2(g)\gamma R,$$

where $\gamma = \gamma(g) \in [\Gamma]$.

The problem of the outer conjugacy is significant only if the actions ρ_1 and ρ_2 of the group G are outer to Γ, i.e. there is an element g of the group G such that $\rho_i(g) \notin [\Gamma]$, $i = 1, 2$.

The results of [5] directly lead to a statement important to subsequent presentation.

THEOREM 1.1. *Let $T \in \mathrm{Aut}(X, \mathscr{B}, \mu)$ be an ergodic automorphism and ρ an action of an amenable countable group G on $(X, \mathscr{B}, \mu)$ such that $\rho(g) \in \mathrm{N}[T]$, $g \in G$. Then the group of automorphisms generated by $\rho(G)$ and $[T]$ is* a.f.

For a countable group G and an ergodic group of automorphisms $\Gamma \subset \mathrm{Aut}(X, \mathscr{B}, \mu)$ define the automorphisms $\lambda(g)$, $g \in G$ such that

$$\begin{gathered} \lambda(g) \in \mathrm{N}[\Gamma], \\ \lambda(g_1)\lambda(g_2) = \gamma(g_1, g_2)\lambda(g_1 g_2), \\ \lambda(g)\lambda(g^{-1}) \in [\Gamma], \end{gathered} \tag{1.8}$$

where $\gamma(g_1, g_2) \in [\Gamma]$ for g_1 and g_2 from G. The map $\lambda : G \to \mathrm{N}[\Gamma]$ satisfying (1.8) will be called a p-action of the group G. This term is analogous to the projective group representations (see, e.g. [15]). It follows from (1.8) that the automorphisms $\gamma(\cdot\,,\,\cdot)$ satisfy the following relation for a 2-cocycle:

$$\gamma(g_2, g_3)^{\lambda(g_1)}\gamma(g_1, g_2g_3) = \gamma(g_1, g_2)\gamma(g_1g_2, g_3), \tag{1.9}$$

where $\gamma(g_2, g_3)^{\lambda(g_1)} = \lambda(g_1)\gamma(g_2, g_3)\lambda(g_1)^{-1}$, $g_1, g_2, g_3 \in G$.

THEOREM 1.2. *Let Γ be an* a.f. *ergodic group of automorphisms of $(X, \mathscr{B}, \mu)$ and let $\lambda: G \to \mathrm{N}[\Gamma]$ be a p-action of a countable amenable group G on $(X, \mathscr{B}, \mu)$. Then, the group of automorphisms generated by $\lambda(G)$ and $[\Gamma]$ is* a.f.

Proof. Consider the Lebesgue space $(X\times G, \mu\times\chi)$, where χ is the Haar measure on G and the group of automorphisms $\tilde{\Gamma} \subset \mathrm{Aut}(X\times G, \mu\times\chi)$ with the generators

$$\tilde{\gamma}(x, k) = (\gamma x, k), \quad \gamma \in \Gamma,$$

$$\tau(h)(x, k) = (x, hk), \quad h \in G, (x, k) \in X\times G.$$

Obviously, $\tilde{\Gamma}$ is an ergodic a.f. group of automorphisms. Define for each $h \in G$ the automorphism $v(h) \in [\tilde{\Gamma}]$ by the formula

$$v(h)(x, k) = (\gamma(h, h^{-1}k)x, h^{-1}k),$$

where $\gamma(h, h^{-1}k)$ satisfies (1.8). Then,

$$v(h)^{-1}(x, k) = (\gamma(h, k)^{-1}x, hk).$$

Put $\lambda_0(g) = \lambda(g)\times\mathbf{1}$, $g \in G$; then $\lambda_0: G \to \mathrm{N}[\tilde{\Gamma}]$ is a p-action the group G, and therefore for any $g, h \in G$

$$\lambda_0(g)\lambda_0(h)\lambda_0(gh)^{-1}(x, k) = (\gamma(g, h)x, k). \tag{1.10}$$

By (1.9):

$$\begin{aligned} v(h)^{\lambda_0(g)}v(g)v(gh)^{-1}(x, k) &= (\gamma(h, k)^{\lambda_0(g)}\gamma(g, hk)\gamma(gh, h)^{-1}x, k) = \\ &= (\gamma(g, h)x, k). \end{aligned} \tag{1.11}$$

From (1.10) and (1.11) it follows that

$$\lambda_0(g)v(h)\lambda_0(g)^{-1}v(g)v(gh)^{-1} = \lambda_0(g)\lambda_0(h)\lambda_0(gh)^{-1},$$

whence

$$v(g)^{-1}\lambda_0(g)v(h)^{-1}\lambda_0(h) = v(gh)^{-1}\lambda_0(gh). \tag{1.12}$$

Put $\lambda_0'(g) = v(g)^{-1}\lambda_0(g)$, $g \in G$. Relation (1.12) shows that $\lambda_0'\colon G \to \mathrm{N}[\tilde{\Gamma}]$ is an action of the group G. By Theorem 1.1, the group of automorphisms generated by $[\tilde{\Gamma}]$ and $\lambda_0'(G)$ is a.f. Thus we obtain the statement of the theorem. ▨

THEOREM 1.3. *Let G be a countable group, Γ a type II_∞ or III ergodic group of automorphisms of $(X, \mathscr{B}, \mu)$ and let $\lambda\colon G \to \mathrm{N}[\Gamma]$ be a p-action. Then, there exist automorphisms $\gamma(g) \in [\Gamma]$, $g \in G$ such that $g \to \gamma(g)\lambda(g)$ defines an action of the group G.*

Proof. We can assume $\mu(X) = \infty$ and the set X to be represented as a disjoint union $X = \bigcup_{g \in G} X_g$, where $\mu(X_g) = \infty$, $g \in G$. Let $\sigma(g)$, $g \in G$ be automorphisms from $[\Gamma]$ such that $\sigma(g)X_h = X_{gh}$, $h \in G$ and $\sigma(g_1)\sigma(g_2) = \sigma(g_1g_2)$, $g_1, g_2 \in G$. Choose for any $g \in G$ an automorphism $u(g) \in [\Gamma]$ having the property: $u(g)\lambda(g)X_e = X_e$, where e is the unit of G. Put, for $g \in G$,

$$\gamma'(g)x = \sigma(h)u(g)\lambda(g)\sigma(h)^{-1}\lambda(g)^{-1}x, \quad x \in \lambda(g)X_h,\ h \in G.$$

Then $\gamma'(g) \in [\Gamma]$ and $\gamma'(g)\lambda(g)\colon X_h \to X_h$ for all $h \in G$. Check that $\lambda'(g) = \gamma'(g)\lambda(g)$ commutes with $\sigma(h_1)$, $h_1 \in G$. Indeed, for $x \in X_g$, $h \in G$

$$\sigma(h_1)\lambda'(g)x = \sigma(h_1h)u(g)\lambda(g)\sigma(h)^{-1}x =$$

$$= \sigma(h_1h)u(g)\lambda(g)\sigma(h_1h)^{-1}\lambda(g)^{-1}\lambda(g)\sigma(h_1)x = \lambda'(g)\sigma(h_1)x.$$

Thus, we can apply to the p-action $\lambda'\colon G \to \mathrm{N}[\Gamma]$ the method that was used to prove Theorem 1.2. ▨

1.8. In this paper, we shall use the terminology and the facts of the measurable groupoid theory (see [10], [19], [8]). Note that the results and the proofs of the paper can be expressed in the terms of the groupoid theory. However, we use the characteristic approach of the ergodic theory to study the countable groups of automorphisms of the measure space, because we intend to essentially use definitions and facts of [6], [2], [3], [4], [12], [18].

We remind the reader of terminology. By $(\mathscr{H}, C)$ or simply by $\mathscr{H}$ we denote a measurable groupoid, where C is a class of Borel measures containing a quasi-invariant symmetric measure on $\mathscr{H}$. For an element $x \in \mathscr{H}$, $r(x)$ and $s(x)$ denote the left and the right units of x respectively, and xy and x^{-1} denote the product of x and y, and the inverse element of x, respectively. The space of units of the groupoid $\mathscr{H}$ will be denoted by $\mathscr{H}^{(0)}$. A Borel function $\beta\colon \mathscr{H} \to \mathscr{G}$ ($\mathscr{H}$ and $\mathscr{G}$ are measurable groupoids) is called a homomorphism, if for a.e. (x, y) for which the product xy is defined, the equality $\beta(xy) = \beta(x)\beta(y)$ is true. In particular, $\mathscr{G}$ may be a group.

The groupoid $\mathscr{H}$ is called approximately finite (a.f.), if $\mathscr{H} = \bigcup_{k=1}^{\infty} \mathscr{H}_k$, $\mathscr{H}_k \subset \subset \mathscr{H}_{k+1}$ and every groupoid $\mathscr{H}_k$ defines on the space of units $\mathscr{H}^{(0)} = \mathscr{H}_k^{(0)}$ a finite relation of equivalence.

Later on an important role will play the cohomologic Bures-Connes-Krieger-Sutherland theorem presented in [13].

THEOREM 1.4. *Let $\mathscr{G}$ be an* a.f. *groupoid, G a Polish group, H a normal Borel subgroup in G, and $\bar{\bar{H}}$ the closure of H. Let β_1 and β_2 be Borel homomorphisms from $\mathscr{G}$ in G such that $\beta_1 = \beta_2 \,(\mathrm{mod}\, \bar{H})$. Then, there exist Borel maps $h: \mathscr{G} \to H$ and $P: X = \mathscr{G}^{(0)} \to \ddot{H}$, such that*

$$\beta_2(\gamma) = h(\gamma)P(r(\gamma))\beta_1(\gamma)P(s(\gamma))^{-1}, \quad \gamma \in \mathscr{G}. \tag{1.13}$$

Since we shall need formulae from the proof of Theorem 1.4, we shall present in the Appendix the highlights of the proof, according to [13].

2. OUTER CONJUGACY. I

2.1. G will denote everywhere in this paper an arbitrary countable amenable group. By $\mathscr{G} = \mathscr{G}(Q, \varphi)$ we shall denote an a.f. ergodic group of type III_0 automorphisms defined on $(X_0, \mathscr{B}_0, \mu_0) = (X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu)$ according to (1.2) and (1.3) by the automorphisms Q_0, S_0 and the function $\varphi(x)$. By ρ we shall denote an action of the group G from $\mathrm{N}[\mathscr{G}(Q, \varphi)]$, i.e. $\rho(g) \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$, $g \in G$. Recall that we exclude the trivial case where the action ρ is such that $\rho(g) \in [\mathscr{G}]$, $g \in G$.

In view of the results of [2], [3] (see (1.5)), there exists for every $g \in G$ an automorphism $t(g) \in [\mathscr{G}]$ such that

$$t(g)\rho(g)(x, y) = (a(g)x,\ V_x(g)y), \quad g \in G, \tag{2.1}$$

where $a(g) \in \mathrm{N}[Q]$, $x \to V_x(g)$ $(x \in X)$ is a measurable field of automorphisms taking values in $\mathrm{N}[S]$. Put

$$\tilde{\rho}(g) = t(g)\rho(g), \quad g \in G.$$

Then,

$$\begin{aligned} \tilde{\rho}(g_1)\tilde{\rho}(g_2)(x, y) &= t(g_1)\rho(g_1)t(g_2)\rho(g_2)(x, y) = \\ &= t\rho(g_1g_2)(x, y) = tt(g_1g_2)^{-1}\tilde{\rho}(g_1g_2)(x, y) = \\ &= t'(a(g_1g_2)x,\ V_x(g_1g_2)y), \end{aligned} \tag{2.2}$$

where t, $t' \in [\mathscr{G}]$. On the other hand,

$$\tilde{\rho}(g_1)\tilde{\rho}(g_2)(x, y) = (a(g_1)a(g_2)x,\ V_{a(g_2)x}(g_1)V_x(g_2)y). \tag{2.3}$$

Let us denote:

$$Z(n, U, x) = \begin{cases} U_{Q^{n-1}x} U_{Q^{n-2}x} \cdots U_x, & n > 0 \\ \mathbf{1}_Y, & n = 0 \\ U^{-1}_{Q^{-1}x} \cdots U^{-1}_{Q^n x}, & n < 0. \end{cases}$$

Comparing (2.2) and (2.3), we conclude that the automorphism $t' \in [\mathscr{G}]$ is a skew product, i.e.

$$t'(x, y) = (Q^{n(x)}x,\ Z(n(x), U, x)s_x y),$$

where $x \to s_x \in [S]$ is a measurable field of automorphisms. Consequently, for a.e. $x \in X$

$$Q^{n_1(x)}a(g_1g_2)x = a(g_1)a(g_2)x, \tag{2.4}$$

$$s'_x Z(n,\ U,\ a(g_1g_2)x)V_x(g_1g_2)y = V_{a(g_2)x}(g_1)V_x(g_2)y, \tag{2.5}$$

where $n_1(x) = n(a(g_1g_2)x)$, $s'_x \in [S]$ with the automorphism $x \to Q^{n_1(x)}x$ belonging to $[Q]$.

Similarly,

$$\tilde{\rho}(g)^{-1}(x, y) = (a(g)^{-1}x,\ V_{a(g)^{-1}x}(g)^{-1}y)$$

and

$$\tilde{\rho}(g)^{-1}(x, y) = t_1\tilde{\rho}(g)^{-1}(x, y) = t_1(a(g^{-1})x,\ V_x(g^{-1})y).$$

Therefore, there is a measurable function $x \to m(x)$ such that

$$Q^{m(x)}a(g^{-1})x = a(g)^{-1}x, \tag{2.6}$$

$$\sigma_x Z(m,\ U,\ a(g^{-1})x)V_x(g^{-1}) = V_{a(g^{-1})x}(g)^{-1}, \tag{2.7}$$

where $\sigma_x \in [S]$, $x \in X$.

REMARK 2.1. (1) It follows from (2.2) – (2.7) that the automorphisms $\tilde{\rho}(g) \in \mathrm{N}[\mathscr{G}]$ and $a(g) \in \mathrm{N}[Q]$, $g \in G$ define the p-actions $\tilde{\rho}(\cdot)$ and $a(\cdot)$ of G. Thus, the group generated by $a(g)$, $g \in G$ and $[Q]$ is a.f. (Theorem 1.2).

(2) Any automorphism $q \in [Q]$ may be extended to an automorphism $q_0 \in [\mathcal{G}]$ as follows: if $qx = Q^{n(x)}x$, then we shall set $q_0(x, y) = (Q^{n(x)}x, Z(n, U, x)y)$. Thus, it follows that if in relation (2.1) for a certain $g_0 \in G$ it is true that $a(g_0) \in [Q]$, then the automorphism $t(g_0) \in [\mathcal{G}]$ may evidently be chosen to be such that $\tilde{\rho}(g_0)(x, y) = (x, V_x(g_0)y)$, i.e. $a(g_0) = \mathbf{1}_X$. Hence, by an appropriate choice of the automorphism $t(g) \in [\mathcal{G}]$, one can arrive either at $a(g) \notin [Q]$ or at $a(g) = \mathbf{1}_X$.

LEMMA 2.2. *Let*

$$H = \{g \in G : a(g) = \mathbf{1}\}. \tag{2.8}$$

Then, H is a normal subgroup of the group G.

Proof. The cases of $H = \{e\}$ and $H = G$ are trivial. Let $h \in H$ and for any $g \in G$ we shall consider ghg^{-1}. In Subsection 1.3, the surjective homomorphism mod: $N[\mathcal{G}] \to C\{W_{\mathcal{G}}(\cdot)\}$ was defined. Then,

$$\operatorname{mod} \rho(ghg^{-1}) = \operatorname{mod} \rho(g) \cdot \operatorname{mod} \rho(h) \cdot \operatorname{mod} \rho(g)^{-1}.$$

Thus $h \in H$ if and only if $\operatorname{mod} \rho(h) \in \{W_{\mathcal{G}}(\cdot)\}$. Therefore, $\operatorname{mod} \rho(ghg^{-1}) \in \{W_{\mathcal{G}}(\cdot)\}$, whence $ghg^{-1} \in H$. ▨

We shall now present another formula which we shall need later. Since for $h \in H$ and $g \in G$, according to Lemma 2.2, $ghg^{-1} = h' \in H$, then $\rho(g)\rho(h) = \rho(h')\rho(g)$ or $t\tilde{\rho}(g)\tilde{\rho}(h) = \tilde{\rho}(h')\tilde{\rho}(g)$, where $t \in [\mathcal{G}]$. We have

$$\tilde{\rho}(h')\tilde{\rho}(g)(x, y) = (a(g)x, V_{a(g)x}(h')V_x(g)y),$$

$$\tilde{\rho}(g)\tilde{\rho}(h)(x, y) = (a(g)x, V_x(g)V_x(h)y),$$

whence we see that $t \in [S_0]$ and therefore,

$$s_x V_x(g) V_x(h) = V_{a(g)x}(h') V_x(g), \tag{2.9}$$

where $x \to s_x \in [S]$.

Note that from (2.1) and (2.4)—(2.8), the set of automorphisms $\tilde{\rho}(G)$ forms a group modulo $[\mathcal{G}]$, the set $\tilde{\rho}(H)$ forms a group modulo $[S]$, and the set $a(G)$ forms a group modulo $[Q]$.

2.2. Consider two actions ρ_1 and ρ_2 of the group G which belong to $N[\mathcal{G}(Q, \varphi)]$. Our principal result is the following criterion for outer conjugacy of the actions ρ_1 and ρ_2 (the required definitions may be found in Subsections 1.2, 1.3 and 1.7).

THEOREM 2.3. *In order that the actions ρ_1 and ρ_2 of a countable amenable group G such that $\rho_i(g) \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$, $i = 1, 2$, $g \in G$, where $\mathscr{G} = \mathscr{G}(Q, \varphi)$ is an ergodic group of type III_0 automorphisms, should be outer conjugate, it is necessary and sufficient that the following conditions should be fulfilled:*

(2.10) $$\operatorname{mod} \rho_1(g) = h^{-1} \operatorname{mod} \rho_2(g) h, \quad g \in G,$$

(2.11) $$\{g \in G : \rho_1(g) \in [\mathscr{G}]\} = \{g \in G : \rho_2(g) \in [\mathscr{G}]\},$$

where the automorphism $h \in C\{W_{\mathscr{G}}(\cdot)\}$ and $C\{W_{\mathscr{G}}(\cdot)\}$ is the centralizer of the flow associated with $\mathscr{G}$.

We will divide the proof of Theorem 2.3 into three parts, representing three distinct cases. In each part, we shall formulate and prove a theorem analogous to Theorem 2.3. Note also that the necessity of conditions (2.10) and (2.11) follows immediately from the definition of outer conjugacy; thus it is only their sufficiency that is to be proved.

LEMMA 2.4. *Let $\mathscr{G}(Q, \varphi)$, G, ρ_1, ρ_2 be the same as in Theorem 2.3 satisfying conditions (2.10) and (2.11). Then, there exists an action ρ_2' isomorphic to the action ρ_2 and such that*

$$\operatorname{mod} \rho_1(g) = \operatorname{mod} \rho_2'(g), \quad g \in G.$$

The proof follows from the results of [3] (see Subsection 1.3): it is sufficient to choose in $\mathrm{N}[\mathscr{G}(Q, \varphi)]$ an automorphism R, for which $\operatorname{mod} R = h$ (the automorphism h being the same as in Theorem 2.3), and to use the fact that the map mod is a homomorphism. Then, put $\rho_2'(g) = R^{-1}\rho_2(g)R$, $g \in G$.

Thus, in proving Theorem 2.3, we may replace (2.10) by the condition

(2.12) $$\operatorname{mod} \rho_1(g) = \operatorname{mod} \rho_2(g), \quad g \in G.$$

Now, if we apply the results of Subsection 2.1 to the actions ρ_1 and ρ_2, we shall have

(2.13) $$t_i(g)\rho_i(g)(x, y) = \tilde{\rho}_i(g)(x, y) = (a_i(g)x, V_x^i(g)y), \quad i = 1, 2.$$

Applying [3, Proposition 1.5] to $\tilde{\rho}_1(g)$, $\tilde{\rho}_2(g)$, $g \in G$, we may assume that

2.14) $$\Phi(V_x^1(g)) = \Phi(V_x^2(g)),$$

(2.15) $$a_1(g) = a_2(g) = a(g), \quad g \in G.$$

Therefore, it follows from (2.8) and (2.15) that $H_1 = H_2 = H$.

Thus, for the actions ρ_1 and ρ_2 from $N[\mathscr{G}]$ satisfying (2.12), there are only the following three possibilities:

(A) the group H is trivial: $H = \{e\}$;

(B) the group H coincides with the group G;

(C) the group H is a proper subgroup of G.

The case (C) is general. In the concluding section we shall discuss the question of possibility of the realization of the cases (A), (B) and (C). The situations (A) and (C), generally speaking, are possible only for some groups $\mathscr{G}(Q, \varphi)$. In Subsection 4.1, we construct an example of a type III_0 group $\mathscr{G}$ for which the cases (A) and (C) are realized. Lemma 4.3 (see below) shows that the case (B) is always realized.

In this section we consider the case (A), while (B) and (C) will be studied in two subsequent sections.

2.3. Theorem 2.3 becomes, in case (A), the following:

THEOREM 2.5. *Let* $\mathscr{G} = \mathscr{G}(Q, \varphi) \subset \mathrm{Aut}(X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu)$, $\rho_i(g) \in N[\mathscr{G}]$, $i = 1, 2$, $g \in G$ *and* $\mathrm{mod}\,\rho_1(g) = \mathrm{mod}\,\rho_2(g)$, $g \in G$. *Let also* $\tilde{\rho}_i(g)$ *be defined according to* (2.13) *and the automorphisms* $a(g)$, $g \in G$ *satisfy the condition* $a(g) \neq \mathbf{1}_X$, $g \neq e$. *Then, the actions* ρ_1 *and* ρ_2 *of the group* G *are outer conjugate.*

(Note that the condition of Theorem 2.5 lead to the equality (2.11).)

Proof. Let $\mathscr{H}$ be the equivalence relation generated by the automorphisms $a(g)$, $g \in G$ and Q. By Theorem 1.2 and Remark 2.1, $\mathscr{H}$ is a.f. Any element $(x, \gamma) \in \mathscr{H}$ uniquely defines the quantities $n = n(x, \gamma)$ and $g = g(x, \gamma)$ by means of the equality $\gamma x = a(g)Q^n x$.

Let us define the homomorphisms β_1 and β_2 from $\mathscr{H}$ in $N[S]$ in the following way. Approximately finiteness of $\mathscr{H}$ implies that there exists an automorphism $R \in \mathrm{Aut}(X, \mathscr{B}, \mu)$ such that its full group $[R]$ defines the groupoid $\mathscr{H}$. Then $Rx = Q^n a(g)x$, where $n = n(x)$, $g = g(x)$. Put in this case for $i = 1, 2$:

(2.16) $$\beta_i(x, R) = W_x^i = Z(n, U, a(g)x)V_x^i(g), \quad x \in X$$

and define for any $n \in \mathbf{N}$

(2.17) $$\beta_i(x, R^n) = W^i_{R^{n-1}x} \ldots W^i_{Rx} W^i_x,$$

(2.18) $$\beta_i(x, R^{-n}) = (W^i_{R^{-1}x})^{-1} \ldots (W^i_{R^{-n}x})^{-1}.$$

Thus, by means of (2.17) and (2.18) the maps β_i ($i = 1, 2$) are homomorphisms of the groupoid $\mathscr{H}$ in $N[S]$.

It follows from (2.14) and (2.16) that $\beta_1 = \beta_2$ (mod $[\bar{S}]$), where the closure $[\bar{S}]$ is considered with respect to the topology in $N[S]$ generated by the metric d (see

Subsection 1.4). Therefore, applying Theorem 1.4, we obtain the maps $P : X \to [\tilde{S}]$ and $\sigma: \mathscr{H} \to [S]$ such that

$$\beta_2(\gamma) = \sigma(\gamma)P(r(\gamma))^{-1}\beta_1(\gamma)P(s(\gamma)), \quad \gamma \in \mathscr{H}. \tag{2.19}$$

For the element $\gamma = (x, Q) \in \mathscr{H}$ it follows from (2.19) that

$$U_x = \sigma_1(x)P(Qx)^{-1}U_xP(x) \tag{2.20}$$

and for $\gamma = (x, a(g)) \in \mathscr{H}$ we have

$$V_x^2(g) = \sigma_g(x)P(a(g)x)^{-1}V_x^1(g)P(x), \tag{2.21}$$

where $\sigma_1(x)$, $\sigma_g(x) \in [S]$.

Put $P(x, y) = (x, P(x)y)$, then $P \in \mathrm{Aut}(X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu)$. It follows from (2.20) that $P^{-1}Q_0P = Q_0\sigma_1$ $(\sigma_1 \in [S_0])$, i.e. $P \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$; from (2.21) it follows that for any $g \in G$

$$P^{-1}\tilde{\rho}_1(g)P(x, y) = \sigma_g\tilde{\rho}_2(g)(x, y), \quad \sigma_g \in [S_0]. \tag{2.22}$$

Equation (2.22) shows that the actions ρ_1 and ρ_2 are outer conjugate via the automorphism P. ▨

3. OUTER CONJUGACY. II

3.1. In this section we shall study the case (B) (see Subsection 2.2), i.e. the situation where the automorphisms $\tilde{\rho}_i(g)$ are of the form

$$\tilde{\rho}_i(g)(x, y) = (x, V_x^i(g)y), \quad g \in G,\ i = 1, 2. \tag{3.1}$$

Recall that in this case the group $H = \{g \in G : a(g) = 1\}$ coincides with the whole group G. Theorem 2.3 becomes as follows.

THEOREM 3.1. *Let ρ_1 and ρ_2 be actions of a countable amenable group G on $(X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu) = (X_0, \mathscr{B}_0, \mu_0)$ such that $\rho_i(g) \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$, $i = 1, 2$ and $\tilde{\rho}_i(g)$, $g \in G$ satisfy* (3.1). *Then, ρ_1 and ρ_2 are outer conjugate if and only if*

$$\Phi(V_x^1(g)) = \Phi(V_x^2(g)), \quad g \in G \tag{3.2}$$

and

$$G_0 = \{g \in G : V_x^1(g) \in [S]\} = \{g \in G : V_x^2(g) \in [S]\}. \tag{3.3}$$

Clearly, conditions (3.2) and (3.3) are necessary for outer conjugacy. We proceed now to prove their sufficiency.

Note that the group G_0 is a normal subgroup of the group G. If $G_0 = G$, then the statement of Theorem 3.1 becomes trivial. Therefore, we shall assume G_0 either to be a proper subgroup of G, or $G_0 = \{e\}$. The two cases are considered simultaneously.

Let Γ_i be an a.f. group of automorphisms of $(X_0, \mathscr{B}_0, \mu_0)$ constructed by $\rho_i(G)$ and $\mathscr{G}(Q, \varphi)$, $i = 1, 2$. According to the results presented in Subsection 1.3, the function $\varphi(x)$ may be assumed to take on only two values: $\log k_1$ and $\log k_2$. Therefore, the Radon-Nikodym cocycle r of the group $\mathscr{G}$ takes on values in the group $\Lambda(k_1, k_2) = \{k_1^n k_2^m : n, m \in \mathbf{Z}\}$.

Since $\tilde{\rho}_i(g) \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$, $g \in G$ we have

$$V^i_{Qx}(g)U_x V^i_x(g) = U_x s^i_x, \quad i = 1, 2, \tag{3.4}$$

where $x \to s^i_x \in [S]$. Then, from ergodicity of Q and (3.4) it follows that $\Phi(V^i_x(g)) = \text{const}$ for μ-a.e. $x \in X$. Since (3.2) is valid, put for $i = 1, 2$

$$\Phi(V^i_x(g)) = \log \tau(g).$$

Thus, for any $\gamma \in \Gamma_i$ and a.e. $x_0 \in X_0$

$$r(x_0, \gamma) \in \Sigma,$$

where Σ is a countable subgroup of $\mathbf{R}^*_+$ with k_1, k_2 and $\tau(g)$, $g \in G$ as generators.

Every element γ of Γ_i $(i = 1, 2)$ may be represented as

$$\gamma x_0 = \tilde{\rho}_i(g) t x_0, \quad x_0 = (x, y) \in X_0, \tag{3.5}$$

where $t \in [\mathscr{G}]$ and $g = g(x_0) \in G$. If the same element γ is represented as $\gamma x_0 = \tilde{\rho}_i(g_1) t_1 x_0$, then evidently $g_1^{-1} g \in G_0$. Hence we may define a map $\alpha_i \colon X_0 \times \Gamma_i \to \hat{G} = G/G_0$ by setting

$$\alpha_i(x_0, \gamma) = \hat{g}, \tag{3.6}$$

where $\hat{g} \in \hat{G}$ and contains the element $g \in G$ defined in (3.5). It is easy to see that α_i $(i = 1, 2)$ is a measurable cocycle.

3.2. For the sake of simplicity, we omit for the time being the subscript i and assume that the group of automorphisms $\Gamma \subset \mathrm{Aut}(X_0, \mathscr{B}_0, \mu_0)$ is generated by the groups $\rho(G)$ and $\mathscr{G}(Q, \varphi)$, where $\rho(G) \subset \mathrm{N}[\mathscr{G}(Q, \varphi)]$. Besides, on $X_0 \times \Gamma$ there are two cocycles defined: $\alpha : X_0 \times \Gamma \to \hat{G}$ and $r : X_0 \times \Gamma \to \Sigma$.

If $A \subset X_0$, $\mu_0(A) > 0$, then denote by $[\Gamma]_A$ the group $\{\gamma \in [\Gamma] : \gamma x_0 = x_0$ for μ_0-a.e. $x_0 \in X_0 \setminus A\}$.

THEOREM 3.2. *Let $\Gamma \subset \mathrm{Aut}(X_0, \mathscr{B}_0, \mu_0)$ be as above and let A an arbitrary subset in X_0 of finite measure. Then, there exists a sequence of Γ_A-arrays $\xi_k = \bigsqcup_{i=1}^{M_k} \xi_k(i)$ such that*

1) $\Gamma_A x_0 = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{M_k} \mathscr{G}(\xi_k(i)) x_0$ *for* a.e. $x_0 \in A$, *where* $[\Gamma_A] = [\Gamma]_A$;

2) $\mathscr{B} \cap A = \sigma\left(\bigcup_{k=1}^{\infty} \bigsqcup_{i=1}^{M_k} \mathscr{P}(\xi_k(i))\right)$;

3) *on any element of the array* $\xi_k(i)$ $(i = 1, 2, \ldots, M_k,\ k = 1, 2, \ldots)$ *the cocycles* α *and* r *are constant;*

4) *for any fixed* $k \in \mathbf{N}$ *the sets on which the arrays are defined form a partition of the set* A.

We shall premise the proof of Theorem 3.2 with the following statement.

LEMMA 3.3. *Let Γ and A as in Theorem 3.2; let $\varepsilon > 0$, $D \subset A$ be given with $\mu_0(D) > 0$; let T be an automorphism of the set A such that $[T] = [\Gamma]_A$. Then, there exists a Γ_A-array $\zeta = \bigsqcup_{i=1}^{m} \zeta_i$ of A, for which*

(a) $\mu_0(\{x_0 \in A : Tx_0 \in \mathscr{G}(\zeta)x_0\}) > (1 - \varepsilon)\mu_0(A)$;

(b) *there exists a set* $D' \in \mathscr{P}(\zeta)$ *such that*

$$\mu_0(D \triangle D') < \varepsilon\mu_0(A);$$

(c) *on any element of the array* ζ, *the cocycles* α *and* r *are constant.*

Proof. Let $\xi = (A, [1, N], C(\cdot), \gamma(\cdot, \cdot))$ be a Γ_A-array of A such that

$$\mu_0(\{x_0 \in A : Tx_0 \in \mathscr{G}(\xi)x_0\}) > \left(1 - \frac{\varepsilon}{4}\right)\mu_0(A) \tag{3.7}$$

and for a certain $D'' \in \mathscr{P}(\xi)$

$$\mu_0(D \triangle D'') < \frac{\varepsilon}{4}\mu_0(A). \tag{3.8}$$

Let us take in $C(1)$ a subset C' such that for any automorphism $\gamma(j, 1)$ $(j = 1, 2, \ldots, N)$ the function $r(x_0, \gamma(j, 1))$ takes only a finite number of values from the group Σ, when $x_0 \in C'$. It will be also required that

$$\mu_0(C(1) - C') < \delta, \tag{3.9}$$

where the number $\delta > 0$ will be chosen later. Divide the set C' into subset $C'(i)$,

$i = 1, 2, \ldots, p$ such that for a.e. $x_0 \in C'(i)$

(3.10) $$r(x_0, \gamma(j, 1)) = \sigma_i(j) = \text{const}, \quad j = 1, 2, \ldots, N.$$

Then,

$$A = \left(\bigcup_{j=1}^{N} \bigcup_{i=1}^{p} \gamma(j, 1)C'(i) \right) \cup \left(\bigcup_{j=1}^{N} \gamma(j, 1)(C(1) - C') \right).$$

This union consists of disjoint sets.

By $\zeta'(i)$ we denote a Γ_A-array which results from restriction of the array ξ to the set $\bigcup_{j=1}^{N} \gamma(j, 1)C'(i)$, $i = 1, 2, \ldots, p$. On the set

(3.11) $$A' = \bigcup_{j=1}^{N} \gamma(j, 1)(C(1) - C')$$

take the trivial array $\zeta'(0)$, i.e. put $\mathscr{G}(\zeta'(0)) = \{\mathbf{1}\}$, $\mathscr{P}(\zeta'(0)) = A'$.

Remind that in the group $\mathscr{G}(Q, \varphi)$ (and therefore in Γ) there is an automorphism S_0 preserving the infinite measure μ_0. Introduce the concept of S_0-expansion for the Γ-arrays $\zeta'(i)$, $i = 0, 1, \ldots, p$. Suppose that for the arrays $\zeta'(i)$ and $\zeta'(i_1)$ there exists an automorphism $s \in [S_0]$ such that for some $j, j_1 \in [1, N]$

(3.12) $$s\gamma(j, 1)C'(i) = \gamma(j_1, 1)C'(i_1)$$

or (for $i = 0$)

$$sA' = \gamma(j_1, 1)C'(i_1).$$

Then, from the two Γ-arrays $\zeta'(i)$ and $\zeta'(i_1)$, one can straightforwardly construct a Γ-array $\tilde{\zeta}'$ taking into account (3.12) which will be called the S_0-expansion of the arrays $\zeta'(i)$ and $\zeta'(i_1)$. By applying the operation of S_0-expansion to the set $\{\zeta'(i) : i = 0, 1, \ldots, p\}$, we obtain the set of the Γ-arrays $\{\tilde{\zeta}'(i) : i = 1, 2, \ldots, p_1\}$, $p_1 \leqslant p$.

Choose now a number $\delta > 0$ such that (3.9) and (3.11) should lead to the inequality

(3.13) $$\mu_0(A') < \frac{\varepsilon}{4} \mu_0(A).$$

Then, there obviously exists a set $\tilde{D}' \in \bigsqcup_{i=1}^{p_1} \mathscr{P}(\tilde{\zeta}'(i))$ such that

(3.14) $$\mu_0(\tilde{D}' \triangle D'') < \frac{\varepsilon}{4} \mu_0(A)$$

and, therefore, (3.8) and (3.14) lead to the inequality

$$\mu_0(D \triangle \tilde{D}') < \frac{\varepsilon}{2} \mu_0(A). \tag{3.15}$$

Now, since

$$\bigsqcup_{i=1}^{p} \mathscr{G}(\zeta'(i))x_0 \subset \bigsqcup_{i=1}^{p_1} \mathscr{G}(\tilde{\zeta}'(i))x_0$$

and, remembering (3.13),

$$\mu_0\left(\left\{x_0 \in A : \mathscr{G}(\xi)x_0 \subset \bigsqcup_{i=1}^{p_1} \mathscr{G}(\tilde{\zeta}'(i))x_0\right\}\right) > \left(1 - \frac{\varepsilon}{4}\right)\mu_0(A),$$

then (3.7) becomes

$$\mu_0\left(\left\{x_0 \in A : Tx_0 \in \bigsqcup_{i=1}^{p_1} \mathscr{G}(\tilde{\zeta}'(i))x_0\right)\right\} > \left(1 - \frac{\varepsilon}{2}\right)\mu_0(A). \tag{3.16}$$

Note also that on any element of the array $\tilde{\zeta}' = \bigsqcup_{i=1}^{p_1} \tilde{\zeta}'(i)$ the cocycle r is constant by construction (see (3.10)).

Then, let us do the following. Repeat for every array $\tilde{\zeta}'(i)$ all the constructions made previously for the array ξ, so as to reach the constant cocycle α. Namely, every array $\tilde{\zeta}'(i)$ generates a finite collection of Γ-arrays $\{\eta_n(i) : n = 1, 2, \ldots, n_i\}$ such that on any element of the array $\eta_n(i)$, $n = 1, 2, \ldots, n_i$ the cocycle α is constant. After that, enlarge each of the Γ-arrays $\{\eta_n(i) : n = 1, 2, \ldots, n_i, i = 1, 2, \ldots, p_1\}$ with respect to S_0-expansion. Denote the resulting set by $\{\zeta(i) : i = 1, 2, \ldots, m\}$. Construct the arrays $\eta_n(i)$ so that the approximation of the automorphism T and the set D should become worse by no more than $\frac{\varepsilon}{2} \mu_0(A)$. Then, it follows from (3.10), (3.15) and (3.16) that the array $\zeta = \bigsqcup_{i=1}^{m} \zeta(i)$ satisfies the conditions of the lemma.*) ▨

Proof of Theorem 3.2. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers monotonically converging to zero and $\{D_k\}_{k=1}^{\infty}$ a sequence of sets, dense in $\mathscr{B} \cap A$ and containing each element an infinite number of times. Apply Lemma 3.3 for $\varepsilon = \varepsilon_1$, $D = D_1$. The resulting Γ_A-array $\xi_1 = \bigsqcup_{i=1}^{M_1} \xi_1(i)$ satisfies the conditions (a) to (c)

*) This lemma could be proved in the same way for the cocycle $\alpha \times r$.

of Lemma 3.3, where $\xi_1(i)$ is an array over the partition $(A_1(i), [1, L_i], B_i(\cdot))$ and $\bigcup_{i=1}^{M_1} A_1(i) = A$. Consider the set $B = \bigcup_{i=1}^{M_1} B_i(1)$ and choose on it a Γ_A-array η such that $B_i(1) \in \mathscr{P}(\eta)$, $i \in [1, M_1]$ and the refinement of the array ξ_1 by η, i.e. the array $\xi_1 \times \eta$ should have the following properties:

$$\mu_0(\{x_0 \in A : Tx_0 \in \mathscr{G}(\xi_1 \times \eta)x_0\}) > \left(1 - \frac{\varepsilon_2}{2}\right)\mu_0(A), \tag{3.17}$$

$$\mu_0(D_2 \triangle D_2'') < \frac{1}{2}\varepsilon_2, \tag{3.18}$$

where $D_2'' \in \mathscr{P}(\xi_1 \times \eta)$. Then, we perform to the array η what we did to ξ in the proof of Lemma 3.3. Namely, we distinguish in B $\mathscr{G}(\eta)$-invariant subsets on which the cocycles α and r are constant. By the operation of S_0-expansion, we obtain the finite set of the arrays $\{\xi_2(i) : i = 1, 2, \ldots, M_2\}$, such that $\mathscr{P}(\xi_1) \subset \subset \bigsqcup_{i=1}^{M_2} \mathscr{P}(\xi_2(i))$. Then, the array $\xi_2 = \bigsqcup_{i=1}^{M_2} \xi_2(i)$ has the following properties: the group $\mathscr{G}(\xi_2)$ approximates the automorphism T accurate to ε_2, and the set D_2 is approximated by a set $D_2' \in \mathscr{P}(\xi_2)$ also accurate to ε_2. In constructing the arrays $\eta_i(j)$, the approximation worsened by $(1/2)\varepsilon_2$ as against (3.17), (3.18) (see the proof of Lemma 3.3). Note that any set from $\mathscr{P}(\xi_1)$ is in $\mathscr{P}(\xi_2)$; thus, the array ξ_2 is a refinement of ξ_1.

By continuing this procedure a countable number of times, we arrive at a sequence of Γ-arrays $\left\{\xi_k = \bigsqcup_{i=1}^{M_k} \xi_k(i)\right\}_{k=1}^{\infty}$ satisfying conditions (1)—(4) of the theorem. ▨

3.3. THEOREM 3.4. *Let Γ_i be the group of automorphisms of $(X_0, \mathscr{B}_0, \mu_0) = (X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu)$ generated by $\rho_i(G)$ and $\mathscr{G}(Q, \varphi)$ as above in Subsection* 3.1*; let α_i be the cocycle from $X_0 \times \Gamma_i$ with values in $\hat{G} = G/G_0$ defined according to* (3.6), $i = 1, 2$. *Suppose that conditions* (3.2) *and* (3.3) *of Theorem* 3.1 *are fulfilled. Then, there exists an automorphism $\theta \in \mathrm{Aut}(X_0, \mathscr{B}_0, \mu_0)$ such that $\theta[\Gamma_1]\theta^{-1} = [\Gamma_2]$ and $\theta \circ \mu_0 = \mu_0$,*

$$\alpha_1(x_0, \gamma) = \alpha_2(\theta x_0, \theta\gamma\theta^{-1}), \quad \gamma \in [\Gamma_1]$$

for μ_0-a.e. $x_0 \in X_0$.

Proof. Choose a measurable set $A \subset X \times Y$ having the following property for μ-a.e. $x \in X$

$$\nu(A_x) = \nu(\{y \in Y : (x, y) \in A\}) = 1, \tag{3.19}$$

so that $\mu_0(A) = 1$. Fix also a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ monotonically converging to zero and a sequence of sets $\{D_n\}_{n=1}^{\infty}$ dense in $\mathscr{B}_0 \cap A$ and such that each of its elements D_n appears in it an infinite number of times.

Lemma 3.3 leads to a Γ_1-array $\zeta_1 = \bigsqcup_{i=1}^{N_1} \zeta_1(i)$ of the set A such that

$$\zeta_1(i) = (A_1(i), [1, N_1(i)], A_1(i)(\cdot), \gamma_1(i)(\cdot\,, \cdot))$$

and the following inequalities are fulfilled

$$\mu_0(\{x_0 \in A : T_1 x_0 \in \mathscr{G}(\zeta_1)x_0\}) > 1 - \varepsilon_1, \tag{3.20}$$

$$\mu_0(D_1 \triangle D_1') < \varepsilon_1, \tag{3.21}$$

where $[T_1] = [\Gamma_1]$, $D_1' \in \mathscr{P}(\zeta_1)$ and $\bigcup_{i=1}^{N_1} A_1(i) = A$. Besides, for μ_0-a.e. $x_0 \in A_1(i)(j)$

$$\alpha_1(x_0, \gamma_1(i)(j_1, j)) = \text{const}, \tag{3.22}$$

$$r_1(x_0, \gamma_1(i)(j_1, j)) = \text{const} \tag{3.23}$$

for all $j, j_1 \in [1, N_1(i)]$.

Let us consider a partition of the set A into $\sum_{i=1}^{N_1} N_1(i)$ parts such that

$$A = \bigcup_{i=1}^{N_1} A_1'(i), \quad A_1'(i) = \bigcup_{j=1}^{N_1(i)} A_1'(i)(j), \tag{3.24}$$

all these unions consisting of disjoint sets. The sets $A_1'(i)(j)$ are to be chosen so that for μ-a.e. $x \in X$ and all $j \in [1, N_1(i)]$, $i \in [1, N_1]$

$$\nu(\{y \in Y : (x, y) \in A_1'(i)(j)\}) = \nu(\{y \in Y : (x, y) \in A_1(i)(j)\}). \tag{3.25}$$

In particular, one may take in (3.24) $A_1'(i)(j) = A_1(i)(j)$.

Let us show that over partition (3.24) of A, a Γ-array $\zeta_1' = \bigsqcup_{i=1}^{N_1} \zeta_1'(i)$ may be defined. Indeed, from (3.22) and (3.23) it follows that if $\alpha_1(x_0, \gamma_1(i)(j_1, j)) = \hat{g} \in \hat{G}$, $r_1(x_0, \gamma_1(i)(j_1, j)) = \sigma \in \Sigma$ for $x_0 \in A_1(i)(j)$, then relations (3.5) and (3.6) give

$$\gamma_1(i)(j_1, j)\, x_0 = \tilde{\rho}_1(g)\, t_1 x_0, \tag{3.26}$$

where $g \in \hat{g}$, $t_1 \in [\mathscr{G}]$ and

$$r_1(x_0, t_1) = \sigma\tau(g)^{-1}. \tag{3.27}$$

Note that if also $\gamma_1(i)(j_1, j)x_0 = \tilde{\rho}_1(g')t_1'(x_0)$ with $g' \in \hat{g}$, $t_1' \in [\mathscr{G}]$, then, as $\tau(G_0) = 1$, $r_1(x_0, t_1) = r_1(x_0, t_1')$. Put now for $x_0 \in A_1'(i)(j)$,

$$\tilde{\gamma}_1'(i)(j_1, j)x_0 = \tilde{\rho}_2(g')t_1 x_0, \tag{3.28}$$

where g' is an element from $\hat{g}$. Since (see (3.2)) $\Phi(V_x^1(g')) = \Phi(V_x^2(g')) = \Phi(V_x^1(g))$ for μ-a.e. $x \in X$, then, by using (3.25), we have

$$\begin{aligned} &\nu(\{y \in Y : (x, y) \in \tilde{\gamma}_1'(i)(j_1, j)A_1'(i)(j)\}) = \\ &= \nu(\{y \in Y : (x, y) \in \gamma_1(i)(j_1, j)A_1(i)(j)\}) = \\ &= \nu(\{y \in Y : (x, y) \in A_1'(i)(j_1)\}). \end{aligned} \tag{3.29}$$

Therefore there exists an automorphism $s \in [S_0]$ such that

$$s\tilde{\gamma}_1'(i)(j_1, j)A_1'(i)(j) = A_1'(i)(j_1) \tag{3.30}$$

(the automorphism s evidently depends on i, j and j_1). Put

$$\gamma_1'(i)(j, 1) = s(i, j)\tilde{\gamma}_1'(i)(j, 1), \quad j = 1, 2, \ldots, N(i), \quad i = 1, 2, \ldots, N_1,$$

where $s(i, j)$ satisfies (3.30). The elements $\gamma_1'(i)(j, 1)$ generate over the partition $(A_1'(i), [1, N_1(i)], A_1'(i)(\cdot))$ a Γ_2-array

$$\zeta_1'(i) = (A_1'(i), [1, N_1(i)], A_1'(i)(\cdot), \gamma_1'(i)(\cdot, \cdot))$$

for $i = 1, 2, \ldots, N_1$. Put $\zeta_1 = \bigsqcup_{i=1}^{N_1} \zeta_1'(i)$. We see from the construction of the Γ_2-array ζ_1' that

$$\alpha_1(x_0, \gamma_1(i)(j_1, j)) = \alpha_2(x_0', \gamma_1'(i)(j_1, j)), \tag{3.31}$$

$$r_1(x_0, \gamma_1(i)(j_1, j)) = r_2(x_0', \gamma_1'(i)(j_1, j)) \tag{3.32}$$

for $x_0 \in A_1(i)(j)$, $x_0' \in A_1'(i)(j)$, $j, j_1 \in [1, N_1(i)]$.

The second step of the proof is as follows. From Theorem 3.2, there exists a Γ_2-array $\zeta_2' = \bigsqcup_{i=1}^{N_2} \zeta_2'(i)$ of the set A refining the Γ_2-array ζ_1' and such that $\mathscr{P}(\zeta_2')$ approximates the set D_1 accurate to ε_2 and $\mathscr{G}(\zeta_2')$ approximates the automorphism T_2 ($[T_2] = [\Gamma_2]_A$) on a set with a measure larger than $1 - \varepsilon_2$. Otherwise, for ζ_2', inequalities similar to (3.20) and (3.21) are valid. In addition, if $\zeta_2'(i) =$

$= (A_2'(i), [1, N_2(i)], A_2'(i)(\cdot), \gamma_2'(i)(\cdot\,,\,\cdot))$, then

$$\alpha_2(x_0', \gamma_2'(i)(j_1, j)) = \text{const},$$

$$r_2(x_0', \gamma_2'(i)(j_1, j)) = \text{const}$$

for $x_0' \in A_2'(i)(j)$, $j, j_1 \in [1, N_2(i)]$.

By repeating the arguments used at the first step of the proof, construct a Γ_1-array $\zeta_2 = \bigcup_{i=1}^{N_2} \zeta_2(i)$, $\zeta_2(i) = (A_2(i), [1, N_2(i)], A_2(i)(\cdot), \gamma_2(i)(\cdot\,,\,\cdot))$ of the set A such that

$$\nu(\{y \in Y : (x, y) \in A_2(i)(j)\}) = \nu(\{y \in Y : (x, y) \in A_2'(i)(j)\}),$$

$$\alpha_1(x_0, \gamma_2(i)(j_1, j)) = \alpha_2(x_0', \gamma_2'(i)(j_1, j)), \tag{3.33}$$

$$r_1(x_0, \gamma_2(i)(j_1, j)) = r_2(x_0', \gamma_2'(i)(j_1, j)) \tag{3.34}$$

for $x_0 \in A_2(i)(j)$, $x_0' \in A_2'(i)(j)$, $j, j_1 \in [1, N_2(i)]$, $i \in [1, N_2]$.

In this case, we make the Γ_1-array so that it should be a refinement of the Γ_1-array ζ_1. Note that even if at the first step we took $A_1'(i)(j) = A_1(i)(j)$, then at the second step $A_2'(i)(j)$ is not generally speaking equal to $A_2(i)(j)$, because the groups of automorphisms $\mathscr{G}(\zeta_1)$ and $\mathscr{G}(\zeta_2)$ are different.

By repeating the above steps of the proof an countable number of times, we obtain finally two sequences $\{\zeta_n\}_{n=1}^{\infty}$ and $\{\zeta_n'\}_{n=1}^{\infty}$ of Γ_1- and Γ_2-arrays, respectively, which satisfy all conditions of Theorem 3.2. Moreover, it follows from (3.31)—(3.34) that on the elements of the arrays with identical indices the cocycles α_1 and α_2 (and r_1 and r_2 as well) take the same values. Then, it is obvious from these facts (for details see [4, Theorem 2.3]) that there exists an automorphism θ_1 mapping the set A onto itself, such that $\theta_1[\Gamma_1]_A\theta_1^{-1} = [\Gamma_2]_A$ and

$$\alpha_1(x_0, \gamma) = \alpha_2(\theta_1 x_0, \theta_1\gamma\theta_1^{-1}), \quad \gamma \in [\Gamma_1]_A. \tag{3.35}$$

Since, according to the construction, the automorphism θ_1 conserves the partition $\{x\} \times A_x$ of the set A (see (3.25), (3.28)—(3.30)), then it is of the form

$$\theta_1(x, y) = (x, \theta_{1x} y), \tag{3.36}$$

where $x \to \theta_{1x}$ $(x \in X)$ is a measurable field of automorphisms. It also follows from the construction that $\theta_1 \circ \mu_0 = \mu_0$.

Let us conclude the proof of the theorem. By virtue of ergodicity of S and condition (3.19), there exists an automorphism $s_0 \in [S_0]$ such that

$$X \times Y = \bigcup_{n \in \mathbf{Z}} s_0^n A, \quad s_0^n A \cap A = \emptyset, \quad n \in \mathbf{N}. \tag{3.37}$$

The automorphism s_0 and the group $[\Gamma_i]_A$ generate the full group of automorphisms $[\Gamma_i]$, $i = 1, 2$ (see [7]).

Define an automorphism $\theta \in \mathrm{Aut}(X \times Y, \mathscr{B} \times \mathscr{F}, \mu \times \nu)$ by

(3.38) $$\theta x_0 = s_0^n \theta_1 s_0^{-n} x_0, \quad x_0 \in s_0^n A,\ n \in \mathbf{Z}.$$

Obviously, $\theta[\Gamma_1]\theta^{-1} = [\Gamma_2]$ because $\theta s_0 \theta^{-1} = s_0$.

Any automorphism $\gamma_1 \in [\Gamma_1]$ may be represented as

(3.39) $$\gamma_1 x_0 = s_0^m \gamma_1' s_0^{-n} x_0,$$

where $x_0 \in s_0^n A$, $\gamma_1' \in [\Gamma_1]_A$ and $m = m(x_0)$ is defined from the condition $\gamma_1 x_0 = s_0^m A$.

From (3.35) and (3.39), we have for $\gamma_1 \in [\Gamma_1]$

(3.40) $$\begin{aligned} \alpha_1(x_0, \gamma_1) &= \alpha_1(x_0, s_0^m \gamma_1' s_0^{-n}) = \\ &= \alpha_1(s_0^{-n} x_0, \gamma_1') = \alpha_2(\theta_1 s_0^{-n} x_0, \theta_1 \gamma_1' \theta_1^{-1}), \end{aligned}$$

where $x_0 \in s_0^n A$. It is included in (3.40) that the cocycles α_i $(i = 1, 2)$ are equal to e on the elements from $[S_0]$.

On the other hand, it follows from (3.38) and (3.39) that

$$\theta \gamma_1 \theta^{-1} \theta x_0 = \theta s_0^m \gamma_1' s_0^{-n} x_0 = s_0^m \theta_1 \gamma_1' \theta_1^{-1} \theta_1 s_0^{-n} x_0$$

for $x_0 \in s_0^n A$. Therefore, for $x_0 \in s_0^n A$, $n \in \mathbf{Z}$

(3.41) $$\begin{aligned} \alpha_2(\theta x_0, \theta \gamma_1 \theta^{-1}) &= \alpha_2(\theta_1 s_0^{-n} x_0, s_0^m \theta_1 \gamma_1' \theta_1^{-1}) = \\ &= \alpha_2(\theta_1 s_0^{-n} x_0, \theta_1 \gamma_1' \theta_1^{-1}). \end{aligned}$$

By comparing (3.40) and (3.41), we find that

(3.42) $$\alpha_1(x_0, \gamma_1) = \alpha_2(\theta x_0, \theta \gamma_1 \theta^{-1}), \quad \gamma_1 \in [\Gamma_1].$$ ▨

LEMMA 3.5. *Any automorphism θ from* $\mathrm{Aut}(X_0, \mathscr{B}_0, \mu_0)$, *satisfying the conclusions of Theorem* 3.4, *is in* $\mathrm{N}[\mathscr{G}(Q, \varphi)]$.

Proof. Proceeding from the definition of the cocycles α_i, $i = 1, 2$ we have that

(3.43) $$[\mathscr{G}] = \{\gamma \in [\Gamma_i] : \alpha_i(x_0, \gamma) = e\}.$$

Therefore, if in (3.42) we take $\gamma_1 \in [\mathscr{G}]$, then in virtue of (3.43), we shall have

$$\theta \gamma_1 \theta^{-1} \in [\mathscr{G}],$$

i.e. $\theta \in \mathrm{N}[\mathscr{G}]$. ▨

Proof of Theorem 3.1. For any element $\hat{g} \in \hat{G}$ there exists in $X_0 = X \times Y$ a set of positive measure $E(\hat{g})$ such that for $x_0 \in E(\hat{g})$ and a certain $\gamma_1(\hat{g}) \in [\Gamma_1]$

$$\alpha_1(x_0, \gamma_1(\hat{g})) = \alpha_2(\theta x_0, \theta\gamma_1(\hat{g})\theta^{-1}) = \hat{g}.$$

Using relations (3.5) and (3.6) ,we find for $x_0 \in E(\hat{g})$

$$\gamma_1(\hat{g})x_0 = \tilde{\rho}_1(g)tx_0,$$

$$\theta\gamma_1(\hat{g})\theta^{-1}\theta x_0 = \tilde{\rho}_2(g')t'\theta x_0,$$

where $g,\ g' \in \hat{g}$ and $t, t' \in [\mathcal{G}]$. Therefore,

(3.44) $$\tilde{\rho}_1(g)tx_0 = \theta^{-1}\tilde{\rho}_2(g')t'\theta x_0, \quad x_0 \in E(\hat{g}).$$

From the definition of the group G_0, we have $\tilde{\rho}_2(g)t_1 = \tilde{\rho}_2(g')$, $t_1 \in [\mathcal{G}]$. Therefore, (3.44) is for $x_0 \in E(\hat{g})$ as follows:

(3.45) $$\tilde{\rho}_1(g)x_0 = \theta^{-1}\tilde{\rho}_2(g)t_2\theta x_0,$$

where $t_2 \in [\mathcal{G}]$. However, equality (3.45) is indeed true for μ_0-a.e. $x_0 \in X_0$, which follows from ergodicity of the group $\mathcal{G}$ and Lemma 3.5. It follows from (3.40) that $\theta \in \mathrm{N}[S_0]$. This proves outer conjugacy of the p-actions $\tilde{\rho}_1$ and $\tilde{\rho}_2$ and thus of the actions ρ_1, ρ_2 as well. ▨

REMARK 3.6. From the construction of the automorphisms θ_1 and s_0 (see (3.36) and (3.37)), the automorphism θ providing outer conjugacy is as follows:

(3.46) $$\theta(x, y) = (x, \theta_x y),$$

where $x \to \theta_x$ $(x \in X)$ is a measurable field of automorphisms. As well, the automorphism θ preserves the measure $\mu \times \nu = \mu_0$, i.e. the field of automorphisms $x \to \theta_x$ preserves the measure ν.

4. OUTER CONJUGACY. III

4.1. Let us study the case (C), where there exists a proper subgroup $H \subset G$ (H is a normal subgroup) such that (see (2.13)), for $i = 1, 2$

(4.1) $$\tilde{\rho}_i(g)(x, y) = (x,\ V_x^i(g)y), \quad g \in H,$$

(4.2) $$\tilde{\rho}_i(g)(x, y) = (a(g)x,\ V_x^i(g)y), \quad a(g) \neq \mathbf{1}_X,\ g \in G - H.$$

Let

$$G_0^i = \{g \in G : \rho_i(g) \in [\mathscr{G}(Q, \varphi)]\}, \quad i = 1, 2.$$

LEMMA 4.1. *Let ρ_1 and ρ_2 be actions of a countable amenable group G lying in $N[\mathscr{G}(Q, \varphi)]$ such that for $\tilde{\rho}_i(g)$, $g \in G$, $i = 1, 2$ relations* (4.1) and (4.2) *are fulfilled. Let also*

$$\text{mod}\, \rho_1(h) = \text{mod}\, \rho_2(h), \quad h \in H, \tag{4.3}$$

$$G_0^1 = G_0^2, \tag{4.4}$$

where H is defined in (4.1). *Then the action ρ_1 is isomorphic to an action ρ_1' such that $\rho_1'(h) = \rho_2(h)$, $h \in H$.*

Proof. It is easy to see that the group G_0^1 $(= G_0^2)$ lies in H and is a normal subgroup. Then relations (4.3) and (4.4) show that for the actions ρ_1 and ρ_2 of H all the conditions of Theorem 3.1 are fulfilled. Therefore, according to Remark 3.6, there exists an automorphism $\theta \in N[\mathscr{G}]$ of the form of (3.46) for which

$$\theta\rho_1(h)\theta^{-1} = \rho_2(h)t, \quad h \in H,$$

where $t = t(h) \in [\mathscr{G}]$. Thus, the action $\rho_1'(h) = \theta\rho_1(h)\theta^{-1}$, $h \in H$ satisfies the condition of the lemma. ▨

REMARK 4.2. (1) In the group $\mathscr{G}(Q, \varphi)$ constructed from the automorphisms Q_0 and S_0 according to Subsection 1.2, the automorphism S_0 may be replaced by a weakly equivalent automorphism S_0', the group $\mathscr{G}(Q, \varphi)$ being thus replaced by a weakly equivalent one. Moreover, the measurable field of automorphisms $x \to U_x \in N[S]$ may also be replaced by any other measurable field of automorphisms $x \to U_x' \in N[S']$ provided that $\Phi(U_x) = \Phi(U_x')$ [18].

(2) We may choose an ergodic automorphism $S' \in \text{Aut}(Y', \mathscr{F}', \nu')$ such that $S' \circ \nu' = \nu'$, $\nu'(Y') = \infty$ and such that there exists a flow $\{T_s\}_{s \in \mathbf{R}}$ in the normalizer $N[S']$ for which $T_s \circ \nu' = e^s \nu'$. To do so, it is sufficient to take as $[S']$ the group of automorphisms dual to an a.f. ergodic type III_1 group (see (1.4)).

(3) The function $\Phi(U_x) = \varphi(x) + \log \dfrac{dQ^{-1} \circ \mu}{d\mu}(x)$ may be chosen to take a countable number of values. Indeed, by Subsection 1.3, $\varphi(x) \in A(k_1, k_2)$ and the measure μ may be replaced by an equivalent measure μ' such that $\log \dfrac{dQ^{-1} \circ \mu'}{d\mu'}$ is a piecewise constant function. As a result of such replacements, the group $\mathscr{G}(Q, \varphi)$ transforms into a weakly equivalent group.

The following statement shows how the case (B) can be realized for an arbitrary a.f. type III_0 group $\mathcal{G}$.

LEMMA 4.3. *Let* $\mathcal{G} = \mathcal{G}(Q, \varphi)$ *be an* a.f. *ergodic type* III_0 *group,* H *a countable amenable group,* H_0 *a normal subgroup of* H, G_0 *a normal subgroup of* H_0 *and of* H, $\lambda : H \to \mathbf{R}_+^*$ *a homomorphism such that* $\lambda(H_0) = 1$. *Then there exists an action* ρ_0 *of the group* H *on* $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ *such that*

$$\rho_0(h)(x, y) = (x, V^0(h)y), \quad h \in H, \tag{4.5}$$

$$\rho_0(H) \subset \mathrm{N}[\mathcal{G}], \quad \rho_0(G_0) \subset [\mathcal{G}], \tag{4.6}$$

$$\operatorname{mod} V^0(h) = \lambda(h), \quad h \in H. \tag{4.7}$$

Proof. Consider the amenable group $\hat{H} = H/G_0$ which is finite or countable. Let T be an automorphism of a space $(X_1, \mathcal{B}_1, \mu_1)$ preserving the finite measure μ_1, $\varkappa$ the Haar measure on the group $\hat{H}$. According to the results of [9], there exists a cocycle for the action of T on X_1, $\psi : X_1 \times \mathbf{Z} \to \hat{H}$ such that the automorphism

$$\tilde{T}(x_1, \hat{g}) = (Tx_1, \hat{g}\psi(x_1, 1)) \tag{4.8}$$

of the space $(X_1 \times \hat{H}, \mu_1 \times \varkappa)$ is ergodic. Define an action of the group H on $X_1 \times \hat{H}$ by the formula

$$\rho'(g)(x_1, \hat{g}_1) = (x_1, \widehat{gg_1}), \quad g \in H, (x_1, \hat{g}_1) \in X_1 \times \hat{H}. \tag{4.9}$$

Then, for any $g \in H$ the automorphism $\rho'(g)$ has the following properties:

$$\rho'(g) \circ (\mu_1 \times \varkappa) = \mu_1 \times \varkappa, \tag{4.10}$$

$$\rho'(g)\tilde{T} = \tilde{T}\rho'(g), \tag{4.11}$$

and

$$\rho'(g) = \mathbf{1} \Leftrightarrow g \in G_0. \tag{4.12}$$

Now, let us apply Remark 4.2 and consider in $(Y', \mathcal{F}', \nu')$ an automorphism S' preserving the infinite measure ν' and such that in $\mathrm{N}[S']$ there is a flow $\{R_s\}$ having the property $R_s \circ \nu' = e^s \nu'$. Define now an action ρ'' of the group $\tilde{H} = H/H_0$ by

$$\rho''(\tilde{h})y' = R_{\log \lambda(h)} y', \quad \tilde{h} \in \tilde{H}, \tag{4.13}$$

where $h \in \tilde{h}$. Because $\lambda(h_1) = \lambda(h_2)$, if $h_1 = h_0 h_2$ $(h_0 \in H_0)$, then (4.13) defines indeed an action of $\tilde{H}$.

Let us use again Remark 4.2, and replace the group $[S_0]$ by a weakly equivalent group of type II_∞ automorphisms with the following automorphisms of the space $X \times X_1 \times \hat{H} \times Y'$ as generators (see (4.8)):

$$(4.14)\qquad (\mathbf{1} \times \tilde{T} \times \mathbf{1})(x, x_1, \hat{g}, y') = (x, Tx_1, \hat{g}\psi(x_1, 1), y'),$$

$$(4.15)\qquad (\mathbf{1} \times \mathbf{1} \times S')(x, x_1, \hat{g}, y') = (x, x_1, \hat{g}, S'y).$$

Define also an automorphism Q_0' by

$$(4.16)\qquad Q_0'(x, x_1, \hat{g}, y') = (Qx, x_1, \hat{g}, U_x'y'),$$

where a measurable field of automorphisms $x \to U_x'$ is chosen so that (for μ-a.e. $x \in X$

$$(4.17)\qquad U_x' \in \{R_s\}_{s=-\infty}^{\infty}, \quad \Phi(U_x) = \Phi(U_x').$$

The existence of such a field follows from the fact that, as stated by Remark 4.2, the function $\Phi(U_x)$ may be regarded as piecewise constant. It is clear that the group of automorphisms $\mathscr{G}'$ generated by automorphisms (4.14)—(4.16) will be weakly equivalent to the group $\mathscr{G}(Q, \varphi)$.

Let us define now, according to (4.9) and (4.13), the desired action of the group H satisfying (4.5):

$$(4.18)\qquad \rho_0(h)(x, x_1, \hat{g}, y') = (x, \rho'(h)(x_1, \hat{g}), \rho''(\tilde{h})\, y'), \quad h \in H,$$

where the element $\tilde{h}$ of the group $\tilde{H}$ is such that $h \in \tilde{h}$. It is easy to make sure that (4.18) defines the action of H. The action ρ_0 of the group H is not free, because its subgroup G_0, according to (4.12) and (4.13), acts identically. When $h_0 \in H_0$, (4.10) and (4.13) show that $\rho_0(h_0)$ preserves measure, i.e. the automorphism

$$V^0(h)(x_1, \hat{g}, y') = (x_1, \widehat{hg}, R_{\log \lambda(h)}y')$$

has the property: $\operatorname{mod} V^0(h_0) = 1$. From (4.13) and the properties of the flow $\{R_s\}_{s=-\infty}^{\infty}$ follows (4.7). If $h_0 \in G_0$, then (4.12) and (4.13) lead to the relation $\rho_0(h_0) = \mathbf{1}$, i.e. $\rho_0(G_0) \subset [\mathscr{G}]$. From (4.9), (4.11) and (4.17) follows the validity of (4.6). ▨

We shall show now that for any countable amenable group G one can realize the case (C). The case (A) is considered similarly.

Let H be an arbitrary normal subgroup of G. The role of the Lebesgue space $(X, \mathscr{B}, \mu)$ will be played by the group $\{0, 1\}^{G/H \times \mathbf{Z}}$ with the Haar measure, and we shall define the action of the group $G/H \times \mathbf{Z}$ as the Bernoulli shift on

$\{0, 1\}^{G/H\times \mathbf{Z}}$. Denote by a the action of the subgroup $G/H \times \{0\}$ and by $\{Q^n\}$ the action $e \times \mathbf{Z}$. The action a of $G/H \times \{0\}$ may be regarded as a non free action of G for which H acts identically. Standard arguments (see e.g. [16]) readily prove that the automorphism Q on $(X, \mathscr{B}, \mu)$ is ergodic. It follows from Lemma 4.3 that on the Lebesgue space $(Y, \mathscr{F}, \nu)$ there is an action V of the group G, such that $V(g) \in \mathrm{N}[S]$, $g \in G$, where S is a type II_∞ ergodic automorphism. Moreover, Lemma 4.3 shows that V may be chosen in such a way that $V(g)U = UV(g)$, where $U \in \mathrm{N}[S]$ and $\operatorname{mod} U = e^\varphi \neq 1$. Put

$$\rho(g)(x, y) = (a(g)x, V(g)y), \quad g \in G.$$

Then, $\rho(g) \in \mathrm{N}[\mathscr{G}(Q, \varphi)]$, $g \in G$ and the action ρ satisfies the case (C).

This example shows that there exist type III_0 groups $\mathscr{G}$ such that the cases (A) and (C) can be realized. In the meantime, for some groups of type III_0 automorphisms these cases cannot be realized. E.g., if a group $\mathscr{G}$ is such that the centralizer of the associated flow $\{W_\mathscr{G}(\cdot)\}$ is trivial (see, e.g. [14]), then the cases (A) and (C) are impossible. We shall discuss this point in more detail elsewhere on the basis of a study of the properties of cocycles of countable automorphisms groups. Here we shall formulate without proving the following statement:

Let $\{W_{Q,\varphi}(t)\}$ be a flow of automorphisms, associated with a type III_0 a.f. group $\mathscr{G}(Q, \varphi)$ such that there exists an action $\alpha : G \to C\{W_{Q,\varphi}(t)\}$ of G, and for a normal subgroup $H \subset G$ the automorphisms $\alpha(h) \in \{W_{Q,\varphi}(\cdot)\}$, $h \in H$. Then, there exists an action $\rho : G \to \mathrm{N}[\mathscr{G}(Q, \varphi)]$ of G, for which $\operatorname{mod}\rho(g) = \alpha(g)$, and the automorphisms $\rho(g)$, $g \in G$ satisfy the case (C), i.e. $a(h) = 1 \Leftrightarrow h \in H$.

4.2. Let the actions ρ_1 and ρ_2 of G be such that the p-actions $\tilde{\rho}_1$ and $\tilde{\rho}_2$ satisfy (4.1) and (4.2). From Lemma 4.1 it follows that the p-actions $\tilde{\rho}_1$ and $\tilde{\rho}_2$ of the group H may be thought to coincide. Moreover, if we consider the action $V^0(h)$, $h \in H$ of H constructed in Lemma 4.3 and such that $\operatorname{mod} V^0(h) = \operatorname{mod} \tilde{\rho}_i(h)$, $i = 1, 2$, then it follows from Lemma 4.1 that the actions ρ_1 and ρ_2 are isomorphic to the actions ρ_1' and ρ_2', such that for them $\tilde{\rho}_i'(h) = V^0(h)$, $h \in H$, $i = 1, 2$. Thus, we shall be able now to assume without loss of generality that the actions ρ_1 and ρ_2 satisfy the relations:

$$\tilde{\rho}_i(h)(x, y) = (x, V^0(h)y), \quad h \in H, \tag{4.19}$$

$$\tilde{\rho}_i(g)(x, y) = (a(g)x, V_x^i(g)y), \quad g \in G - H,\ i = 1, 2, \tag{4.20}$$

where H is a normal subgroup in G and $a(g) \neq \mathbf{1}$ for $g \notin H$.

The principal theorem in the case (C) is as follows:

THEOREM 4.4. *Let* $\mathscr{G}(Q, \varphi)$, G, H *be the same as above and the actions* ρ_1, ρ_2 *of the group* G *satisfy relations* (4.19), (4.20) *and* $\rho_i(g) \in \mathrm{N}[\mathscr{G}]$, $g \in G$, $i = 1, 2$.

Let also for $g \in G - H$

$$\Phi(V_x^1(g)) = \Phi(V_x^2(g)). \tag{4.21}$$

Then the actions ρ_1 *and* ρ_2 *are outer conjugate.*

Before proving this theorem, we will prove the following two lemmas.

LEMMA 4.5. *Let* S *be an ergodic type* II_∞ *automorphism of a Lebesgue space* $(Y, \mathscr{F}, \nu)$, $S \circ \nu = \nu$; H *a countable amenable group and* V *an action of* H *on* $(Y, \mathscr{F}, \nu)$ *such that* $V(h) \in \mathrm{N}[S]$, $h \in H$. *Suppose that there exists an automorphism* $R \in \mathrm{N}[S]$ *preserving the measure* ν *for which*

$$RV(h)R^{-1} = sV(h), \quad h \in H, \tag{4.22}$$

where $s = s(h) \in [S]$. *Let* $T \in \mathrm{Aut}(Y, \mathscr{F}, \nu)$ *be such that* $[T]$ *coincides with the full group generated by* S *and* $V(H)$. *Then* $R \in \mathrm{N}[T]$ *and there exists in* $[T]$ *a sequence of elements* $\{\sigma_m\}_{m=1}^\infty$ *such that* $\sigma_m \in [S]$ *and* $d(\sigma_m, R) \to 0$ *as* $m \to \infty$, *where the metric* d *was introduced in Subsection* 1.4 *and convergence of* σ_m *to* R *is considered in the group* $\mathrm{N}[T]$.

Proof. Consider in Y a subset $A(0)$ with measure $\nu(A(0)) = 1$ and let the automorphisms $\gamma_n \in [S]$ be such that the sets $A(0)$ and $\gamma_n A(0) = A(n)$, $n \in \mathbf{N}$ form a partition of Y. Denote $H_0 = \{h \in H : V(h) \in [S]\}$. Construct also, as was done in Section 3, a cocycle α on the full group $[T]$ with values in the group $\hat{H} = H/H_0$. For an automorphism $t \in [T]$ and $x \in Y$, we have

$$tx = sV(h)x, \quad s \in [S],\ h \in H. \tag{4.23}$$

Put for t defined in (4.23)

$$\alpha(x, t) = \hat{h} \in \hat{H}, \tag{4.24}$$

where $h \in \hat{h}$.

Use the result of [4], proving that on the set $A(0)$ there exists a sequence of T-arrays $\{\xi_n\}_{n=1}^\infty$ having the following properties:

(1) ξ_{n+1} is a refinement of ξ_n, $n \in \mathbf{N}$;

(2) the group $\bigcup_{n=1}^\infty \mathscr{G}(\xi_n)$ has the same trajectories as the automorphism $T_{A(0)}$;

(3) the σ-algebra generated by $\bigcup_{n=1}^\infty \mathscr{P}(\xi_n)$ coincides with $\mathscr{F}$;

(4) the cocycle α and the Radon-Nikodym cocycle r are constant on the elements of the array ξ_n, $n \in \mathbf{N}$.

Construct the required automorphisms σ_m, $m \in \mathbf{N}$. Let $\zeta_m = (Y, [0, N_m - 1], E_m(\cdot), \delta_m(\cdot, \cdot))$ be the array constructed by ξ_m and the automorphisms γ_n, $n \in \mathbf{N}$; consider the partition $R\xi_m = (Y, [0, N_m - 1], RE_m(\cdot))$. Since the measure of the sets $E_m(0)$ and $RE_m(0)$ is equal, there exists an automorphism $f_m \in [S]$ such that $f_m E_m(0) = RE_m(0)$. Put now for $y \in \gamma_n \delta_m(j) E_m(0)$, $n \in \mathbf{N}$, $m \in \mathbf{N}$, $j \in [0, N_m - 1]$

$$\sigma_m y = R\gamma_n \delta_m(j) R^{-1} f_m \delta_m(j)^{-1} \gamma_n^{-1} y, \tag{4.25}$$

where $\delta_m(j) = \delta_m(j, 0)$.

By virtue of (4.22) and the constancy of the cocycles α and r (see (4.24)) on elements of the array ξ_m, $m \in \mathbf{N}$ the automorphisms σ_m, $m \in \mathbf{N}$ are easily seen to belong to the group $[S]$.

It remains now to verify that, in the group $\mathrm{N}[T]$, the automorphisms σ_m converge (in the metric d) to R as $m \to \infty$. Let us prove first that $\{\sigma_m\}$ converges to R in the weak metric d_w (see Subsection 1.4). Indeed, let $\varepsilon > 0$ and let A be an arbitrary set from $\mathscr{F}$. Then there exists $M \in \mathbf{N}$ such that in $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{M} \mathscr{P}(\gamma_n \xi_m)$ there is for sure a set B for which $\nu(A \triangle B) < \varepsilon$. Since

$$\nu(RA \triangle \sigma_m A) \leqslant \nu(RA \triangle RB) + \nu(RB \triangle \sigma_m A) \tag{4.26}$$

and

$$\sigma_m(\gamma_n \delta_m(j) E_m(0)) = R(\gamma_n \delta_m(j) E_m(0))$$

for n, $m \in \mathbf{N}$, $j \in [0, N_m - 1]$, then $RB = \sigma_m B$ for all $m \geqslant M$. It follows from (4.26) that $\nu(RA \triangle \sigma_m A) < 2\varepsilon$ for $m \geqslant M$, i.e. $d_w(R, \sigma_m) \to 0$ as $m \to \infty$.

Now, by applying equality (4.25), we see that for $y \in R\gamma_n \delta_m(j) E_m(0)$, n, $n_1 \in \mathbf{N}$, $m \in \mathbf{N}$, $j \in [0, N_m - 1]$

$$\begin{aligned} &\sigma_m(\gamma_{n_1} \delta_m(j)^{-1} \gamma_n^{-1}) \sigma_m^{-1} y = \\ &= \sigma_m(\gamma_{n_1} \delta_m(j)^{-1} \gamma_n^{-1}) \gamma_n \delta_m(j) f_m^{-1} R \delta_m(j)^{-1} \gamma_n^{-1} R^{-1} y = \\ &= \sigma_m \gamma_{n_1} f_m^{-1} R \delta_m(j)^{-1} \gamma_n^{-1} R^{-1} y = \\ &= R\gamma_{n_1} R^{-1} f_m \gamma_{n_1}^{-1} \gamma_{n_1} f_m^{-1} R \delta_m(j)^{-1} \gamma_n^{-1} R^{-1} y = R(\gamma_{n_1} \delta_m(j)^{-1} \gamma_n^{-1}) R^{-1} y. \end{aligned} \tag{4.27}$$

Since the full group $[T]$ is generated by γ_n, $n \in \mathbf{N}$ and $\bigcup_{m=1}^{\infty} \mathscr{G}(\xi_m)$, the convergence of σ_m to R as $m \to \infty$ in the metric d is a corollary of the equality (4.27). Thus, it is proved that $d(\sigma_m, R) \to 0$ as $m \to \infty$. ▨

LEMMA 4.6. *Let the automorphism R be the limit in the metric d of a sequence of automorphisms $\{R_n\}_{n=1}^{\infty}$ from* N$[T]$ *such that $R_nV(h)R_n^{-1} = s_nV(h)$, $h \in H$, $n \in \mathrm{N}$, where H, T, S, α and V are the same as in Lemma 4.5. Then*

$$RV(h)R^{-1} = \sigma V(h), \quad h \in H,$$

where $\sigma = \sigma(h) \in [S]$.

Proof. From the convergence of R_n to R it follows that in the uniform metric d_u the sequence of automorphisms $\{R_nV(h)R_n^{-1}\}_{n=1}^{\infty}$ converges to $RV(h)R^{-1}$. As the cocycle α on the element $R_nV(h)R_n^{-1}$ is equal $\hat{h}$ $(h \in \hat{h})$ for all $n \in \mathrm{N}$, we have

$$\alpha(x, RV(h)R^{-1}) = \hat{h}.$$

From (4.23) and (4.24) it follows that $RV(h)R^{-1}x = \sigma_1 V(\bar{h})x = \sigma V(h)x$, where $\bar{h}$ and h belong to the same class $\hat{h}$. ▨

Proof of Theorem 4.4. (The proof will essentially use the proof of Theorem 1.4 presented in the Appendix.) Define on $(X, \mathscr{B}, \mu)$ a p-action of the group $\hat{G} = G/H$ assuming for μ-a.e. $x \in X$

$$a(\hat{g})x = a(\bar{g})x, \tag{4.28}$$

where $\bar{g}$ is an element from G such that $\bar{g} \in \hat{g}$. Since $a(H) = \mathbf{1}$, then (4.28) correctly defines a p-action of the group $\hat{G}$ independent of the choice of the elements $\bar{g}$. We have the obvious relations following from (2.4) and (2.6):

$$a(\hat{g}) \in \mathrm{N}[Q],$$

$$a(\hat{g}_1\hat{g}_2) = qa(\hat{g}_1)a(\hat{g}_2), \quad a(\hat{g})^{-1} = q'a(\hat{g}^{-1}),$$

where $q, q' \in [Q]$. Denote, as in Theorem 2.5, by $\mathscr{H}$ the a.f. groupoid generated by the p-action a of $\hat{G}$ and $[Q]$ on $(X, \mathscr{B}, \mu)$. Let $R \in \mathrm{Aut}(X, \mathscr{B}, \mu)$ be an automorphism such that its full group $[R]$ generates $\mathscr{H}$.

Construct two homomorphisms β_i', $i = 1, 2$ from $\mathscr{H}$ to N$[S]$. Because $Rx = Q^n a(\bar{g})x$, where $n = n(x)$, $\bar{g} = \bar{g}(x)$, then put

$$\beta_i(x, R) = W_x^i = Z(n, U, a(\bar{g})x)V_x^i(\bar{g}). \tag{4.29}$$

For $n \in \mathrm{N}$ put

$$\begin{gathered}\beta_i(x, R^n) = W^i_{R^{n-1}x} \ldots W^i_{Rx}W^i_x,\\ \beta_i(x, R^{-n}) = (W^i_{R^{-1}x})^{-1} \ldots (W^i_{R^{-n}x})^{-1}, \quad i = 1, 2, \ x \in X.\end{gathered} \tag{4.30}$$

Thus, relations (4.30) show that the maps β_i, defined by (4.29) on the element R, can be extended to homomorphisms of the whole groupoid $\mathscr{H}$.

From (4.21) it follows that $\beta_1 = \beta_2 \pmod{[\dot{S}]}$. Thus, all the conditions of Theorem 1.4 are fulfilled. We conclude from this theorem that there exist measurable maps $x \to s(x) \in [S]$ and $x \to P_0(x) \in N[S]$ for which

$$W_x^2 = P_0(Rx)^{-1} W_x^1 P_0(x)\, s(x). \tag{4.31}$$

The proof of Theorem 1.4 (see Appendix) implies that $P_0(x) = \lim_{k\to\infty} P_k(x)$, where $P_k(x) \in N[S]$. Let us show that for every $k \in \mathbf{N}$, $h \in H$

$$P_k(x) V^0(h) P_k(x)^{-1} = s^{(k)}(x) V^0(h), \quad s^{(k)}(x) \in [S] \tag{4.32}$$

for a.e. $x \in X$. The automorphism $P_k(x)$ is defined via the automorphism $P_{k-1}(x)$ by means of relations (A.2), (A.4) and (A.15). Therefore, it is enough to check (4.32) only for $P(x)$ and $Q(x)$ (see the Appendix) satisfying (A.4) and (A.15), respectively.

We have

$$\begin{aligned} P(x) &= s_1(x)\beta_2(x, R^n)\beta_1(x, R^n)^{-1} = \\ &= s_1(x) W^2_{R^{n-1}x} \ldots W^2_x (W^1_x)^{-1} \ldots (W^1_{R^{n-1}x})^{-1}, \quad s_1(x) \in [S], \end{aligned} \tag{4.33}$$

where we assume $n > 0$. Therefore,

$$\begin{aligned} P(x)^{-1} V^0(h) P(x) &= s_2(x) W^1_{R^{n-1}x} \ldots W^1_x (W^2_x)^{-1} \ldots (W^2_{R^{n-1}x})^{-1} \times \\ &\times V^0(h) W^2_{R^{n-1}x} \ldots W^2_x (W^1_x)^{-1} \ldots (W^1_{R^{n-1}x})^{-1}, \quad s_2(x) \in [S]. \end{aligned} \tag{4.34}$$

Consider for $i = 1, 2$ the expression

$$\begin{aligned} (W^i_y)^{-1} V^0(h) W^i_y &= (V^i_y(\bar{g}_0))^{-1} Z(m, U, a(\bar{g}_0)y)^{-1} \times \\ &\times V^0(h) Z(m, U, a(\bar{g}_0)y) V^i_y(\bar{g}_0), \quad y \in X, \end{aligned} \tag{4.35}$$

where the number m is defined from the equality $Ry = Q^m a(\bar{g}_0)y$. Since $\rho_i(h) \in N[\mathscr{G}]$, then, by (4.19) and Lemma 4.3

$$U_x^{-1} V^0(h) U_x = s'_x V^0(h)$$

and therefore (4.35), according to (2.9), becomes

$$(W_y^i)^{-1}V^0(h)W_y^i = \\ = s_y''(V_y^i(\bar{g}_0))^{-1}V^0(h)V_y^i(\bar{g}_0) = s_y'''V^0(h_1), \tag{4.36}$$

where $h_1 = \bar{g}_0^{-1}h\bar{g}_0$. Thus, the right-hand side of (4.34), according to the equality (4.36), becomes simpler and (4.34) leads to the following relation for μ-a.e. $x \in X$:

$$P(x)^{-1}V^0(h)P(x) = s_3(x)V^0(h), \quad h \in H. \tag{4.37}$$

Further, for

$$Q(x) = P(x)\beta_1(x, R^n)Q_1(x)\beta_1(x, R^n)^{-1}, \quad x \in X$$

(see (A.15)) the function $P(x)$ commutes with $V^0(h)$ up to $[S]$. Since $Q_1(x)$ is defined in quite the same fashion as $P(x)$, the function $Q_1(x)$ has the same property. Consider

$$Q(x)V^0(h)Q(x)^{-1} = P(x)\beta_1(x, R^n)Q_1(x)\beta_1(x, R^n)^{-1} \times \\ \times V^0(h)\beta_1(x, R^n)Q_1(x)^{-1}\beta_1(x, R^n)^{-1}P(x)^{-1}, \quad x \in X.$$

According to (4.36) and (4.37), we have for $x \in X$

$$Q(x)V^0(h)Q(x)^{-1} = s_4(x)V^0(h), \quad s_4(x) \in [S]. \tag{4.38}$$

Thus, equalities (4.37) and (4.38) prove (4.32).

Now let us apply Lemmas 4.5 and 4.6. Because $P_0(x)$ is, for almost everywhere fixed $x \in X$, the limit of $P_k(x)$ as $k \to \infty$ and because for $P_k(x)$ (4.32) is valid, then for $P_0(x)$ the following relation is true:

$$P_0(x)^{-1}V^0(h)P_0(x) = s_0(x)V^0(h), \quad h \in H \tag{4.39}$$

for a.e. $x \in X$.

Consider the automorphism $R_0^i(x, y) = (Rx, W_x^i y)$, $i = 1, 2$. Formula (4.31) shows that

$$P_0^{-1}R_0^1P_0s_0 = R_0^2, \tag{4.40}$$

where $P_0(x, y) = (x, P_0(x)y)$, $s_0 \in [S_0]$. Further, since $Qx = R^{n(x)}x$ and $a(\bar{g}) \notin [Q]$ for all $\bar{g}$, we have

$$(R_0^i)^{n(x)}(x, y) = (Qx, V^0(h)U_x s_5(x)y), \tag{4.41}$$

where $s_5(x) \in [S]$ and the element $h = h(x) \in H$ does not depend on i, because it is defined by group relations between representatives of co-sets. Therefore, it follows from (4.40) and (4.41) that for a.e. $x \in X$

$$P_0(Qx)^{-1}V^0(h)U_x P_0(x) = s_6(x)V^0(h)U_x$$

or according to (4.39),

$$P_0(Qx)^{-1}U_x P_0(x) = s_7(x)U_x, \tag{4.42}$$

where $s_6(x), s_7(x) \in [S]$. Equality (4.42) proves that $P_0 \in \mathrm{N}[\mathscr{G}]$.

For the same reasons we infer

$$P_0(a(\bar{g})x)^{-1}V_x^1(\bar{g})P_0(x) = V_x^2(\bar{g})s_8(x), \tag{4.43}$$

where $s_8(x) \in [S]$ and $\bar{g}$ is a representative of a co-set of the group G/H.

Let us conclude the proof of the theorem. Let g be an arbitrary element of the group G and $\bar{g}$ an element from G such that g and $\bar{g}$ lie in the same co-set of the group H, i.e. $g = \bar{g}h$. Make the following calculations by using (2.4), (2.5), (4.42) and (4.43):

$$\begin{aligned} P_0(a(g)x)^{-1}V_x^1(g)P_0(x) &= P_0(Q^n a(\bar{g})a(h)x)^{-1}V_x^1(\bar{g}h)P_0(x) = \\ &= P_0(Q^n a(\bar{g})x)^{-1}Z(n, U, a(\bar{g})x)V_x^1(\bar{g})V^0(h)P_0(x) = \\ &= P_0(Q^n a(\bar{g})x)^{-1}Z(n, U, a(\bar{g})x)P_0(a(\bar{g})x) \times \\ &\times P_0(a(\bar{g})x)^{-1}V_x^1(\bar{g})P_0(x)P_0(x)^{-1}V^0(h)P_0(x) = \\ &= Z(n, U, a(\bar{g})x)V_x^2(\bar{g})V^0(h)s_9(x) = V_x^2(g)s_{10}(x). \end{aligned} \tag{4.44}$$

Thus, (4.44) proves that $P_0^{-1}\tilde{\rho}_1(g)P_0 = \tilde{\rho}_2(g)s_0'$, $g \in G$, where $s_0' \in [S_0]$. Since $P_0 \in \mathrm{N}[\mathscr{G}]$, outer conjugacy of the actions ρ_1 and ρ_2 of G is proved. ▨

APPENDIX

Proof of Theorem 1.4 (sketch). The idea of the proof is as follows. Since $\mathscr{G}$ is an a.f. equivalence relation, then $\mathscr{G} = \bigcup_{k=1}^{\infty} \mathscr{G}_k$, $\mathscr{G}_k \subset \mathscr{G}_{k+1}$ and every equivalence relation $\mathscr{G}_k$ is finite type I. First, the statement of the theorem is proved for every $\mathscr{G}_k$ and the respective maps P_k and h_k are constructed so that they have limits P and h as $k \to \infty$. The required result will be obtained by passage to the limit.

LEMMA A.1. *Let $\mathcal{G}$ be a type I_n $(n < \infty)$ equivalence relation, $X, \beta_1, \beta_2, G, H$ be as above in Theorem* 1.4, *d be a complete metric on G compatible with the topology of the group G and $x \to \varepsilon(x) : X \to \mathbf{R}_+^*$ a Borel map. Then there exist Borel functions $h : \mathcal{G}_1 \to H$ and $P : X \to \bar{\bar{H}}$ satisfying* (1.13) *and*

$$d(P(x), e) < \varepsilon(x), \quad x \in X. \tag{A.1}$$

Proof. Let $A(0) \subset X$ be a Borel set intersecting each $\mathcal{G}$-orbit at exactly one point. Put

$$P(x) = e, \quad x \in A(0). \tag{A.2}$$

As in [13], let us construct the Borel map $h(\gamma)$ for $\gamma \in s^{-1}(A(0))$ such that

$$\begin{aligned} &d(h(\gamma)^{-1}\beta_2(\gamma)\beta_1(\gamma)^{-1}, e) < \varepsilon(s(\gamma)), \\ &h(\gamma) = e, \quad \gamma = (x, x) \in s^{-1}(A(0)). \end{aligned} \tag{A.3}$$

For $\gamma \in s^{-1}(A(0))$ and $y = r(\gamma)$, let us define

$$P(y) = h(\gamma)^{-1}\beta_2(\gamma)\beta_1(\gamma)^{-1}. \tag{A.4}$$

Then, P is defined almost everywhere on X and, according to (A.3), satisfies (A.1). For $\gamma \in s^{-1}(A(0))$, (1.13) is fulfilled, which follows from (A.2) and (A.4).

For $\gamma \in r^{-1}(A(0))$, put

$$h(\gamma) = P(r(\gamma))\beta_1(\gamma)P(s(\gamma))^{-1}h(\gamma^{-1})P(s(\gamma))\beta_1(\gamma)^{-1}P(r(\gamma)). \tag{A.5}$$

For any $\gamma \in \mathcal{G}$ there exist $\gamma_1 \in r^{-1}(A(0))$ and $\gamma_2 \in s^{-1}(A(0))$ such that $\gamma = \gamma_1\gamma_2$. Therefore, put

$$\begin{aligned} h(\gamma) = {} & h(\gamma_1)P(r(\gamma_1))\beta_1(\gamma_1)P(s(\gamma_1))^{-1}h(\gamma_2) \times \\ & \times P(s(\gamma_1))\beta_1(\gamma_1)^{-1}P(r(\gamma_1))^{-1}. \end{aligned} \tag{A.6}$$

Simple calculations using (A.5) and (A.6) show that (1.13) is fulfilled. From the equality $\beta_1 = \beta_2 \pmod{\bar{H}}$ it follows that $P \in \bar{H}$. ▨

LEMMA A.2. *Let $\mathcal{G}, X, \beta_1, \beta_2, G, \varepsilon(\cdot), d$ be the same as in Lemma* A.1. *Let $\mathcal{H}$ be a type I_m equivalence relation $(m < \infty)$ and $\mathcal{H} \supset \mathcal{G}$. If $\delta > 0$ is given, then there exist Borel maps $k : \mathcal{H} \to H$, $Q : X \to \bar{\bar{H}}$ such that, when $\gamma \in \mathcal{H}$.*

$$k(\gamma)Q(r(\gamma))\beta_1(\gamma)Q(s(\gamma))^{-1} = \beta_2(\gamma), \tag{A.7}$$

$$k(\gamma) = h(\gamma), \quad \gamma \in \mathcal{G}, \tag{A.8}$$

$$d(P(x), Q(x)) < \delta, \quad x \in X. \tag{A.9}$$

Proof. Let $A(0)$ and $P(x)$ be the same as in Lemma A.1. Put $\mathscr{H}_0 = \mathscr{H}|A(0)$,

$$(A.10)\qquad B(x_0) = \{g \in G : \sup_{(x,x_0)\in\mathscr{G}} d(P(x)\beta_1(x, x_0)g\beta_1(x, x_0)^{-1}, P(x)) \geqslant \delta\},$$

$$(A.11)\qquad \varepsilon_0(x) = \inf\{d(e, g) : g \in B(x_0)\} = \mathrm{dist}(e, B(x_0)).$$

By applying Lemma A.1 to the equivalence relation $\mathscr{H}_0$ and the function $\varepsilon_0(\cdot)$, we find the Borel functions $Q_0\colon A(0) \to \bar{H}$, $k_0\colon \mathscr{H}_0 \to H$ such that

$$(A.12)\qquad \beta_2(\gamma) = k_0(\gamma)Q_0(r(\gamma))\beta_1(\gamma)Q_0(s(\gamma))^{-1}, \quad \gamma \in \mathscr{H}_0,$$

$$(A.13)\qquad d(Q_0(y), e) < \varepsilon_0(y), \quad y \in A(0).$$

For $(y, x) \in \mathscr{H}$ there exist unique y_0, x_0 from $A(0)$ for which $(y, y_0) \in \mathscr{G}$, $(x, x_0) \in \mathscr{G}$, $(y_0, x_0) \in \mathscr{H}_0$. Put for $(y, x) \in \mathscr{H}$

$$k(y, x) = h(y, y_0)P(y)\beta_1(y, y_0)k_0(y_0, x_0)\beta_1(y, y_0)^{-1}P(y)^{-1} \times$$

$$(A.14)\qquad \times P(y)\beta_1(y, y_0)Q_0(y_0)\,\beta_1(y_0, x_0)Q_0(x_0)^{-1}h(x_0, x) \times$$

$$\times Q_0(x_0)\,\beta_1(y_0, x_0)^{-1}Q_0(y_0)^{-1}\beta_1(y, y_0)^{-1}P(y)^{-1},$$

$$(A.15)\qquad Q(x) = P(x)\beta_1(x, x_0)Q_0(x_0)\beta_1(x, x_0)^{-1}.$$

Calculation using (A.2), (A.6), (A.12), (A.14) and (A.15) proves that (A.7) and (A.8) are true. From (A.10), (A.11), (A.13) and (A.15), (A.9) follows. ▨

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