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SPECTRAL THEORY IN QUOTIENT FRÉCHET SPACES. II

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1. INTRODUCTION

In the first part of this work [17], we have defined a spectrum and constructed a holomorphic functional calculus for every morphism in the category of quotient Fréchet spaces.

The aim of the present paper, which is a continuation of [17], is to define a joint spectrum and to construct a holomorphic functional calculus for some finite families of mutually commuting morphisms in the category of quotient Fréchet spaces.

Throughout this work, by *Fréchet space* (briefly F-*space*) we mean a locally convex topological vector space, which is metrizable and complete. We mention that every linear space from this paper is automatically over the complex field **C**.

Let X be an F-space. A *Fréchet subspace* (briefly F-*subspace*) [21] of X is a linear subspace $Y \subset X$ that has an F-space structure which makes the inclusion $Y \subset X$ continuous. We shall designate in the following by $\mathrm{Lat}(X)$ the family of all F-subspaces of X (which is a lattice under the sum and intersection of subspaces; see, for instance, [17], Lemma 2.1). A *quotient Fréchet space* (briefly qF-*space*) is a linear space of the form X/Y, where X is an F-space and $Y \in \mathrm{Lat}(X)$ [21].

We shall also work with Banach spaces (B-spaces), Banach subspaces (B-subspaces) and quotient Banach spaces (qB-spaces), which are defined in a similar manner (see [18]). We shall occasionally use inductive limits of Fréchet spaces (LF-spaces) and quotients of such spaces (qLF-spaces), whose meaning can be easily deduced by the analogy with the above situations.

Let V be a differentiable (i.e. C^∞) manifold and let X be an F-space. The space of all indefinitely differentiable X-valued functions on V will be denoted by $\mathcal{E}(V, X)$. The space $\mathcal{E}(V, \mathbf{C})$ will be simply denoted by $\mathcal{E}(V)$. If V is an analytic manifold, then $\mathcal{O}(V, X)$ is the space of all X-valued functions, analytic on V. The space $\mathcal{O}(V, \mathbf{C})$ will be denoted by $\mathcal{O}(V)$.

Let L be a linear space and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of indeterminates. We denote by $\Lambda^p[\sigma, L]$ the linear space of all exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$, with coefficients in L. Obviously, $\Lambda^p[\sigma, L] = \{0\}$ if $p > n$ and $\Lambda^0[\sigma, L] = L$. The direct sum of the spaces $\Lambda^p[\sigma, L]$ will be designated by $\Lambda[\sigma, L]$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an arbitrary family of endomorphisms of L and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of indeterminates (which is often said to be *associated with* λ). Then we may define on $\Lambda[\sigma, L]$ the endomorphism

$$\delta(\lambda) = \lambda_1 \otimes \sigma_1 + \ldots + \lambda_n \otimes \sigma_n .$$

If λ is commutative, then $\delta(\lambda)\circ\delta(\lambda) = 0$. Therefore, if $\delta^p(\lambda) = \delta(\lambda)| \Lambda^p[\sigma, L]$, it is clear that the sequence

$$(1.1) \qquad K(L, \lambda) : 0 \to \Lambda^0[\sigma, L] \xrightarrow{\delta^0(\lambda)} \Lambda^1[\sigma, L] \xrightarrow{\delta^1(\lambda)} \ldots \xrightarrow{\delta^{n-1}(\lambda)} \Lambda^n[\sigma, L] \to 0$$

is a complex of linear spaces, which is called the *Koszul complex* associated to λ. The homology of the complex (1.1) will be designated by $\{H^p(L, \lambda)\}_{p\geqslant 0}$. The complex $K(L, \lambda)$ will be sometimes denoted by $K(L, \delta(\lambda))$ and its homology by $\{H^p(L, \delta(\lambda))\}_{p\geqslant 0}$ (when we neglect the source of $\delta(\lambda)$).

The family of endomorphisms $\lambda = (\lambda_1, \ldots, \lambda_n)$ is said to be *nonsingular* (resp. *singular*) if the complex $K(L, \lambda)$ is exact (resp. not exact). For further details see [13], [14], [16] etc.

Let us briefly describe the contents of the present work.

The next section is devoted to the study of some properties of linear operator (as we call here the morphisms) in qF-spaces. We take again some results from [17] in a more general setting (which is needed in the sequel), by introducing an abstract functor, whose properties systematize the subject.

The third section deals with the definition and general properties of the joint spectrum of a (commuting) multioperator. We prefer the term of multioperator, by analogy with multivector, multi-index etc. (and already used by J. L. Taylor [15]) to that of n-tuple of operators (which is often pleonastic) or that of commuting system of operators (since the word "system" has too many meanings).

The fourth section is dedicated to the construction of the Cauchy-Weil-Taylor integral in quotient Fréchet spaces. We have added the name of Taylor's to those of Cauchy's and Weil's (see [14] for the corresponding concept in Banach spaces) to emphasize his contribution to this remarkable construction. Although we basically use the method of J. L. Taylor (and our work relies heavily upon the papers [13] and [14]), we have taken into account the subsequent extensions and simplifications from [6], [11], [12], [16], [5] etc.

The last section contains the construction of the holomorphic functional calculus for some commuting multioperators, including the spectral mapping theorem (see also [20] for a different construction). A version of the Shilov idempotent theorem ends the present work.

Thanks are due to L. Waelbroeck, whose suggestions improved a preliminary version of this work.

2. LINEAR OPERATORS IN QUOTIENT FRÉCHET SPACES

Let X be an F-space. The family $\mathrm{Lat}(X)$ of all F-subspaces of X has some remarkable properties. We have already mentioned that $\mathrm{Lat}(X)$ is a lattice with respect to the sum and intersection of subspaces. It consists of all images of linear and continuous mappings, defined on arbitrary F-spaces, assuming values in X. The F-space topology of each member of $\mathrm{Lat}(X)$ is uniquely determined. Direct and inverse images of elements of $\mathrm{Lat}(X)$ via linear and continuous mappings are again F-subspaces. Finally, for each $Y \in \mathrm{Lat}(X)$ we have the heredity property $\mathrm{Lat}(Y) = \{Z \in \mathrm{Lat}(X) : Z \subset Y\}$ (see [17], Lemma 2.1). We also note that if X_j are F-spaces, $Y_j \in \mathrm{Lat}(X_j)$ $(j = 1, 2)$ and $u\colon X_1 \to X_2$ is a linear and continuous operator such that $u(Y_1) \subset Y_2$, then $u\colon Y_1 \to Y_2$ is also continuous, as a consequence of the closed graph theorem.

Now, let X_1/Y_1, X_2/Y_2 be qF-spaces. For every linear mapping $u : X_1/Y_1 \to X_2/Y_2$ we define the linear space

$$G_0(u) = \{(x, y) \in X_1 \times X_2 : y \in u(x + Y_1)\}, \tag{2.1}$$

which will be called the *lifted graph* of u. Note that the graph $G(u)$ of u is isomorphic to the quotient $G_0(u)/(Y_1 \times Y_2)$, which explains the name ascribed to $G_0(u)$. Note that the mapping u is completely determined by its lifted graph.

If $\mathrm{Ker}(u)$ (resp. $\mathrm{Im}(u)$) is the kernel (resp. the image) of the mapping u, we also set

$$\mathrm{Ker}_0(u) = \{x \in X_1 : x + Y_1 \in \mathrm{Ker}(u)\},$$

$$\mathrm{Im}_0(u) = \{y \in X_2 : y + Y_2 \in \mathrm{Im}(u)\}$$

(similar concepts have been defined in [2]).

2.1. Definition. A linear mapping $u\colon X_1/Y_1 \to X_2/Y_2$ such that $G_0(u) \in \mathrm{Lat}(X_1 \times X_2)$ will be called a *linear operator* (or simply *operator*) from X_1/Y_1 into X_2/Y_2.

There are several reasons to use the term of operator instead of that of *morphism* (in the category of quotient Fréchet spaces), as done in [17], Definition 2.3. One of them is the following result, which extends Lemma 2.5 from [17].

If L_1, L_2 are linear spaces, we denote by pr_1 (resp. pr_2) the projection of the Cartesian product $L_1 \times L_2$ onto L_1 (resp. L_2).

2.2. LEMMA. *Let X_1/Y_1, X_2/Y_2 be* qF-*spaces such that Y_j is closed in X_j ($j = 1, 2$). Then every linear operator u from X_1/Y_1 into X_2/Y_2 is continuous.*

Proof. The graph $G(u)$ of u is clearly isomorphic to X_1/Y_1. Therefore $G_0(u)/(Y_1 \times Y_2)$ is isomorphic to X_1/Y_1. Moreover, this isomorphism, say $\hat{\pi}_1: G_0(u)/(Y_1 \times Y_2) \to X_1/Y_1$, is induced by $\pi_1 = \mathrm{pr}_1|G_0(u)$. If $\pi_2 = \mathrm{pr}_2|G_0(u)$, then π_2 induces a mapping $\hat{\pi}_2: G_0(u)/(Y_1 \times Y_2) \to X_2/Y_2$ such that $u = \hat{\pi}_2 \circ \hat{\pi}_1^{-1}$.

We now use the hypothesis of the lemma. Since Y_1 is closed in X_1, then $Y_1 \times Y_2 = \pi_1^{-1}(Y_1)$ is closed in $G_0(u)$. Therefore $\hat{\pi}_1^{-1}$ is continuous, by the closed graph theorem. Then it is clear that $u = \hat{\pi} \circ \hat{\pi}_1^{-1}$ is also continuous.

Lemma 2.2 asserts, in fact, that the category of F-spaces (whose morphisms are the linear and continuous mappings) is a full subcategory of the category of qF-spaces (which will be discussed later; see Ramark 2.4.1°).

The set of all linear operators from the qF-space X_1/Y_1 into the qF-space X_2/Y_2 will be denoted by $\mathscr{L}(X_1/Y_1, X_2/Y_2)$. It is known that this set, endowed with the usual algebraic operations, is a linear space ([17], Proposition 2.6(1)). If $X_1/Y_1 = X/Y = X_2/Y_2$, the space $\mathscr{L}(X_1/Y_1, X_2/Y_2)$ is denoted by $\mathscr{L}(X/Y)$. The space $\mathscr{L}(X/Y)$ is a unital algebra with the property that the inverse of every bijective operator is still an operator ([17], Proposition 2.6).

2.3. LEMMA. *Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$. Then $\mathrm{Ker}_0(u) \in \mathrm{Lat}(X_1)$ and $\mathrm{Im}_0(u) \in \mathrm{Lat}(X_2)$.*

Proof. Since the projection pr_2 is continuous on $X_1 \times X_2$, $G_0(u) \in \mathrm{Lat}(X_1 \times X_2$ and $\mathrm{Im}_0(u) = \mathrm{pr}_2(G_0(u))$, we must have $\mathrm{Im}_0(u) \in \mathrm{Lat}(X_2)$, by Lemma 2.1 from [17]

To obtain the other assertion, note that

$$\mathrm{Ker}_0(u) \times Y_2 = G_0(u) \cap (X_1 \times Y_2). \tag{2.2}$$

Therefore $\mathrm{Ker}_0(u) \times Y_2 \in \mathrm{Lat}(X_1 \times X_2)$. Since $\mathrm{Ker}_0(u) = \mathrm{pr}_1(\mathrm{Ker}_0(u) \times Y_2)$, we infer that $\mathrm{Ker}_0(u) \in \mathrm{Lat}(X_1)$.

2.4. REMARKS. 1° As mentioned in [17], the family of all qF-spaces with the morphisms given by Definition 2.1 is a category that is designated by $\mathrm{q}\mathscr{F}$. It coincides with the category with the same objects and the same morphisms that was introduced in [21]. (It was L. Waelbroeck who observed, in a private communication, that the morphism $\hat{\pi}_1$, which occurs in the proof of Lemma 2.2, is what he calls a pseudo-isomorphism [21], so that Definition 2.3 from [17] and Definition 3.4 from [21] provide the same class of morphisms.)

Lemma 2.3 shows that if $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$, then $\mathrm{Im}(u) = \mathrm{Im}_0(u)/Y_2$ and $\mathrm{Ker}(u) = \mathrm{Ker}_0(u)/Y_1$ are objects in the category $\mathrm{q}\mathscr{F}$. The natural isomorphism between $G(u)$ and $G_0(u)/(Y_1 \times Y_2)$ indicates that $G(u)$ can also be regarded as an object in the category $\mathrm{q}\mathscr{F}$.

2° Let $u_0 \in \mathscr{L}(X_1, X_2)$ be such that $u_0(Y_1) \subset Y_2$. Then u_0 induces a linear mapping $u : X_1/Y_1 \to X_2/Y_2$ by the equality $u(x + Y_1) = u_0(x) + Y_2$ $(x \in X_1)$. It is known that $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ (see [17], Lemma 2.4). We shall say that u is a *strict operator* induced by u_0. Let us mention that this concept appears in [21] under the name of *strict morphism*.

A linear operator $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ is strict and induced by a certain operator $u_0 \in \mathscr{L}(X_1, X_2)$ if and only if

$$G_0(u) = G(u_0) + \{0\} \times Y_2, \tag{2.3}$$

where the sum from the right hand side is direct, as observed in [22]. Note that (2.3) provides, in particular, another argument to see that a strict operator is an operator.

3° Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ and let $Z_j \in \mathrm{Lat}(X_j)$ be such that $Z_j \supset Y_j$ $(j = 1, 2)$, and $u(Z_1/Y_1) \subset Z_2/Y_2$. Then the restriction $v = u|(Z_1/Y_1)$ is an element of $\mathscr{L}(Z_1/Y_1, Z_2/Y_2)$, since

$$G_0(v) = G_0(u) \cap (Z_1 \times Z_2) \in \mathrm{Lat}(Z_1 \times Z_2).$$

Note also that u induces a linear mapping $w : X_1/Z_1 \to X_2/Z_2$ by the formula $w(x_1 + Z_1) = x_2 + Z_2$, where $(x_1, x_2) \in G_0(u)$. Owing to the inclusion $u(Z_1/Y_1) \subset Z_2/Y_2$, the definition of w is correct. Moreover,

$$G_0(w) = G_0(u) + Z_1 \times Z_2 \in \mathrm{Lat}(X_1 \times X_2),$$

and so $w \in \mathscr{L}(X_1/Z_1, X_2/Z_2)$.

4° Let X/Y be a qF-space and let $Z \in \mathrm{Lat}(X)$ be such that $Z \supset Y$. Then $Y \in \mathrm{Lat}(Z)$ and the qF-space Z/Y is called a qF-*subspace* of X/Y [21].

Let $u \in \mathscr{L}(X/Y)$. A qF-subspace Z/Y of X/Y is said to be *invariant* under u if $u(Z/Y) \subset Z/Y$. In this case we can discuss the restriction $v \in \mathscr{L}(Z/Y)$ of u to Z/Y and the operator $w \in \mathscr{L}(X/Z)$ induced by u in X/Z, in virtue of the previous remark.

We shall meet in the following a slightly more general situation. Namely, let $X_0, Y_0 \in \mathrm{Lat}(X)$ be such that $X_0 \cap Y = Y_0$. Then the natural mapping $i_0 : X_0/Y_0 \to X/Y$, which is the strict operator induced by the inclusion $X_0 \subset X$, is injective. Thus the space X_0/Y_0 is isomorphic to the qF-subspace $(X_0 + Y)/Y$ of X/Y. The mapping i_0 will be called the *canonical embedding* of X_0/Y_0 into X/Y. The qF-space X_0/Y_0 will be said to be a qF-*subspace* of X/Y, by abuse of terminology. If $u \in \mathscr{L}(X/Y)$ and $(X_0 + Y)/Y$ is invariant under u, we shall say, again by abuse of terminology, that X_0/Y_0 is *invariant* under u. As a matter of fact, in this case u induces an operator $u_0 \in \mathscr{L}(X_0/Y_0)$ given by $u_0 = i_0^{-1}(u \mid ((X_0 + Y)/Y))i_0$.

More generally, if X_j/Y_j is a qF-space and X_{0j}/Y_{0j} is a qF-subspace of X_j/Y_j $(j = 1, 2)$, and $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ has the property $u((X_{01} + Y_1)/Y_1) \subset \subset (X_{02} + Y_2)/Y_2$, then u induces, as above, an operator $u_0 \in \mathscr{L}(X_{01}/Y_{01}, X_{02}/Y_{02})$. Since the construction of the operator u_0 is purely algebraic, it does exist as a linear mapping even if X_{0j}/Y_{0j} is no longer a qF-space but, say, a qLF-subspace of X_j/Y_j (that is, Y_{0j} is a LF-subspace of X_{0j} and $X_{0j} \cap Y_j = Y_{0j}$). In particular, we can discuss invariant qLF-subspaces under operators on qF-spaces and the linear mappings induced by them. We shall also say that u *extends* u_0, if u, u_0 are as above.

5° Let X_j/Y_j be qF-spaces, let $u_j \in \mathscr{L}(X_j/Y_j)$ $(j = 1, 2, 3)$ and let $\delta_1 \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$, $\delta_2 \in \mathscr{L}(X_2/Y_2, X_3/Y_3)$ satisfy the equations $\delta_2\delta_1 = 0$, $\delta_1 u_1 = u_2\delta_1$ and $\delta_2 u_2 = u_3\delta_2$. Then u_2 induces a linear operator w in the qF-space $\mathrm{Ker}_0(\delta_2)/\mathrm{Im}_0(\delta_1)$. Indeed, we obviously have $\mathrm{Im}(\delta_1) \subset \mathrm{Ker}(\delta_2)$, $u_2(\mathrm{Im}(\delta_1)) \subset \mathrm{Im}(\delta_1)$ and $u_2(\mathrm{Ker}(\delta_2)) \subset \subset \mathrm{Ker}(\delta_2)$. Therefore, by 3°, the restriction $u_2 \mid \mathrm{Ker}(\delta_2)$ induces an operator w in $\mathrm{Ker}_0(\delta_2)/\mathrm{Im}_0(\delta_1)$.

Note also that if u_1, u_2, u_3 are bijective, then w is bijective too. Indeed, in this case $\delta_1 u_1^{-1} = u_2^{-1}\delta_2$, $\delta_2 u_2^{-1} = u_3^{-1}\delta_2$, and the operator induced by u_2^{-1} in $\mathrm{Ker}_0(\delta_2)/\mathrm{Im}_0(\delta_1)$ provides the desired inverse of w.

The next result is a simple but useful description of the lifted graph of an operator.

2.5. LEMMA. *Let X_1/Y_1, X_2/Y_2 be qF-spaces and let $G_0 \in \mathrm{Lat}(X_1 \times X_2)$. The following conditions are equivalent:*

(1) *G_0 is the lifted graph of some operator $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$;*

(2) *$G_0 \cap (Y_1 \times X_2) = Y_1 \times Y_2$, $G_0 + Y_1 \times X_2 = X_1 \times X_2$;*

(3) *$G_0 \supset Y_1 \times Y_2$ and the complex of F-spaces*

$$0 \to Y_1 \times Y_2 \xrightarrow{\alpha} G_0 \times (Y_1 \times X_2) \xrightarrow{\beta} X_1 \times X_2 \to 0$$

is exact, where $\alpha(x, y) = ((x, y), -(x, y))$ $(x \in Y_1, y \in Y_2)$ and

$$\beta((x_1, x_2), (y_1, y_2)) = (x_1 + y_1, x_2 + y_2) \quad ((x_1, x_2) \in G_0,\ (y_1, y_2) \in Y_1 \times X_2).$$

Proof. Since the equivalence (2) ⇔ (3) is obvious, it is sufficient to prove the equivalence (1) ⇔ (2).

Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$, let $G_0 = G_0(u)$ and let $(x, y) \in X_1 \times X_2$. We can find a vector $y' \in X_2$ such that $(x, y') \in G_0(u)$. Thus

$$(x, y) = (x, y') + (0, y - y') \in G_0(u) + Y_1 \times X_2.$$

Moreover, if $(x, y) \in G_0 \cap (Y_1 \times X_2)$, then $y \in u(x + Y_1) = u(Y_1) = Y_2$, and so $(x, y) \in Y_1 \times Y_2$. Hence(1) ⇒ (2).

Conversely, let $G_0 \in \mathrm{Lat}(X_1 \times X_2)$ satisfy (2). Then for every $x \in X_1$ we can find pairs $(x_1, y) \in G_0$ and $(x_2, -y) \in Y_1 \times X_2$ such that

$$(x, 0) = (x_1, y) + (x_2, -y). \tag{2.4}$$

We define the mapping $u : X_1/Y_1 \to X_2/Y_2$ by the equality $u(x + Y_1) = y + Y_2$. When $x \in Y_1$, then in (2.4) we have $x_1 = x - x_2 \in Y_1$, and so $(x_1, y) \in G_0 \cap \cap (Y_1 \times X_2) = Y_1 \cap Y_2$. This shows that u is correctly defined. It is easily seen that u is also linear.

To end the proof, we have to infer the equality $G_0 = G_0(u)$. If $(x, y) \in G_0$, then the representation $(x, 0) = (x, y) + (0, -y)$ is as in (2.4). Hence $u(x + Y_1) = = y + Y_2$ by the definition of u, that is, $(x, y) \in G_0(u)$. Conversely, if $(x, y') \in G_0(u)$ and $(x, 0) = (x_1, y) + (x_2, -y)$ is as in (2.4), then $(x, y) = (x_1, y) + (x_2, 0) \in G_0$. Hence, as above, $u(x + Y_1) = y + Y_2$, so that $y' - y \in Y_2$. Finally, $(x, y') = (x, y) + + (0, y' - y) \in G_0$, which completes the proof of the lemma.

From now on we shall fix a nuclear Fréchet space N. For every F-space X we denote by $N \hat{\otimes} X$ the completion of the algebraic tensor product $N \otimes X$ with respect to the projective (or, equivalently, the injective) topology (see [8], [4] etc.).

2.6. REMARKS. 1° If X is an F-space and $Y \in \mathrm{Lat}(X)$, then the natural mapping from $N \hat{\otimes} Y$ into $N \hat{\otimes} X$ is (continuous and) injective. Therefore, we may identify $N \hat{\otimes} Y$ with its image in $N \hat{\otimes} X$, via this canonical monomorphism This allows us to assert that $N \hat{\otimes} Y \in \mathrm{Lat}(N \hat{\otimes} X)$.

We shall frequently use another identification. Namely, if X_1, X_2 are F-spaces then the space $N \hat{\otimes} (X_1 \times X_2)$ will be identified with $(N \hat{\otimes} X_1) \times (N \hat{\otimes} X_2)$.

2° Let X be an F-space and let $Y_1, Y_2 \in \mathrm{Lat}(X)$. Then, with the above identifications, we have the equalities

$$N \hat{\otimes} (Y_1 + Y_2) = N \hat{\otimes} Y_1 + N \hat{\otimes} Y_2, \quad N \hat{\otimes} (Y_1 \cap Y_2) = (N \hat{\otimes} Y_1) \cap (N \hat{\otimes} Y_2).$$

Indeed, we have the exact complex of F-spaces

$$0 \to Y_1 \cap Y_2 \xrightarrow{\alpha} Y_1 \times Y_2 \xrightarrow{\beta} Y_1 + Y_2 \to 0,$$

where $\alpha(x) = (x, -x)$ and $\beta(x, y) = x + y$. Then the diagram

$$\begin{array}{ccccccccc} 0 \to & N \hat{\otimes} (Y_1 \cap Y_2) & \xrightarrow{1 \hat{\otimes} \alpha} & N \hat{\otimes} (Y_1 \times Y_2) & \xrightarrow{1 \hat{\otimes} \beta} & N \hat{\otimes} (Y_1 + Y_2) & \to 0 \\ & \downarrow i_1 & & \downarrow \theta & & \uparrow i_2 & \\ 0 \to & (N \hat{\otimes} Y_1) \cap (N \hat{\otimes} Y_2) & \xrightarrow{\hat{\alpha}} & (N \hat{\otimes} Y_1) \times (N \hat{\otimes} Y_2) & \xrightarrow{\hat{\beta}} & N \hat{\otimes} Y_1 + N \hat{\otimes} Y_2 & \to 0 \end{array}$$

is commutative and has exact rows, where i_1, i_2 are the inclusions, θ is the canonical identification (see the previous remark) and $\hat{\alpha}$ (resp. $\hat{\beta}$) is defined as α (resp. β). The exactness of the lower row is obvious and the exactness of the upper row follows from the fact that the tensor multiplication of exact complexes of F-spaces by a nuclear F-space preserves the exactness [4] (this argument will be often used in this paper). Therefore i_1 and i_2 must be equalities, which is precisely our assertion.

3° Let X/Y be a qF-space. Then, by the first remark, we may define the qF-space

$$\varphi_N(X/Y) = (N \hat{\otimes} X)/(N \hat{\otimes} Y). \tag{2.5}$$

This notation is not ambiguous when Y is a closed subspace of X. Indeed, in such a situation we have the exactness of the complex of F-spaces

$$0 \to Y \to X \to X/Y \to 0,$$

where the mappings are the natural ones, which implies the exactness of the complex

$$0 \to N \hat{\otimes} Y \to N \hat{\otimes} X \to N \hat{\otimes} (X/Y) \to 0,$$

showing that the spaces $N \hat{\otimes} (X/Y)$ and $(N \hat{\otimes} X)/(N \hat{\otimes} Y)$ are isomorphic as F-spaces.

4° Let X_1, X_2 be F-spaces and let $v \in \mathscr{L}(X_1', X_2)$. Then $\mathrm{Im}(1_N \hat{\otimes} v) = N \hat{\otimes} \mathrm{Im}(v)$ and $\mathrm{Ker}(1_N \hat{\otimes} v) = N \hat{\otimes} \mathrm{Ker}(v)$ [8], where 1_N is the identity on N. Indeed, we have the exact complex of F-spaces

$$0 \to \mathrm{Ker}(v) \to X_1 \xrightarrow{v} \mathrm{Im}(v) \to 0,$$

where $\mathrm{Im}(v)$ is given the F-space structure induced by the algebraic isomorphism between $X_1/\mathrm{Ker}(v)$ and $\mathrm{Im}(v)$. Therefore we have the exact complex

$$0 \to N \hat{\otimes} \mathrm{Ker}(v) \to N \hat{\otimes} X_1 \xrightarrow{1_N \hat{\otimes} v} N \hat{\otimes} \mathrm{Im}(v) \to 0,$$

whence we obtain the desired conclusion.

2.7. LEMMA. *Let X be an F-space, let $Y \in \mathrm{Lat}(X)$ and assume that N is non-null. If $N \hat{\otimes} X = N \hat{\otimes} Y$, then $X = Y$.*

Proof. We fix an element $v_0 \in N$, $v_0 \neq 0$. Then the mapping $\theta : X \to N \hat{\otimes} X$ given by $\theta(x) = v_0 \otimes x$ is a topological embedding of X into $N \hat{\otimes} X$. Analogously, $\theta|Y : Y \to N \hat{\otimes} Y$ is a topological embedding of Y into $N \hat{\otimes} Y$. We first show that Y is closed in X. Indeed, if $\{x_k\}_k \subset Y$ is a sequence such that $x_k \to x$ $(k \to \infty)$

in the topology of X, then $v_0 \otimes x_k \to v_0 \otimes x$ in $N \hat{\otimes} X = N \hat{\otimes} Y$. Since $\theta|Y$ is a topological embedding, the sequence $\{x_k\}_k$ must be also convergent in the topology of Y, and hence $x \in Y$. But the space $N\hat{\otimes}(X/Y)$ is isomorphic to $(N\hat{\otimes}X)/(N\hat{\otimes}Y) = \{0\}$ (see Remark 2.6.3°). Therefore $X/Y = \{0\}$, that is $X = Y$.

2.8. LEMMA. *Let X_1/Y_1, X_2/Y_2 be* qF-*spaces. Then for every $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ there exists an operator $\varphi_N(u) \in \mathscr{L}(\varphi_N(X_1/Y_1), \varphi_N(X_2, Y_2))$ such that $G_0(\varphi_N(u)) = N \hat{\otimes} G_0(u)$. In particular, $\varphi_N(u)$ is uniquely determined by u.*

Proof. We consider the exact complex of F-spaces

$$0 \to Y_1 \times Y_2 \xrightarrow{\alpha} G_0(u) \times (Y_1 \times X_2) \xrightarrow{\beta} X_1 \times X_2 \to 0, \tag{2.6}$$

which appears in Lemma 2.5(3). If we multiply the complex (2.4) by N and use the canonical identifications from Remark 2.6.1°, we obtain the exact complex

$$0 \to (N \hat{\otimes} Y_1) \times (N \hat{\otimes} Y_2) \xrightarrow{\hat{\alpha}} (N \hat{\otimes} G_0(u)) \times ((N \hat{\otimes} Y_1) \times (N \hat{\otimes} X_2)) \to$$

$$\xrightarrow{\hat{\beta}} (N \hat{\otimes} X_1) \times (N \hat{\otimes} X_2) \to 0,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are defined as α and β, respectively. It follows by Lemma 2.5 that the space $N \hat{\otimes} G_0(u)$, which is a member of $\mathrm{Lat}((N \hat{\otimes} X_1) \times (N \hat{\otimes} X_2))$, is the lifted graph of an operator $\varphi_N(u) \in \mathscr{L}(\varphi_N(X_1/Y_1), \varphi_N(X_2/Y_2))$. Clearly, $\varphi_N(u)$ is uniquely determined by u.

2.9. LEMMA. *The mapping*

$$\mathscr{L}(X_1/Y_1, X_2/Y_2) \ni u \to \varphi_N(u) \in \mathscr{L}(\varphi_N(X_1/Y_1), \varphi_N(X_2/Y_2))$$

is linear.

Proof. Let u', u'' be in $\mathscr{L}(X_1/Y_1, X_2/Y_2)$. Consider the F-space

$$G_0 = \{((x, y_1), (x, y_2)) \in G_0(u') \times G_0(u'')\}.$$

The range of the linear and continuous operator

$$G_0 \ni ((x, y_1), (x, y_2)) \to (x, y_1 + y_2) \in X_1 \times X_2 \tag{2.7}$$

is precisely $G_0(u' + u'')$. Let α be the mapping (2.7) and let K_0 be the kernel of α. It is clear that

$$K_0 = \{((0, y), (0, -y)) \in G_0 : y \in Y_2\}.$$

We have an exact complex of F-spaces

$$0 \to K_0 \hookrightarrow G_0 \xrightarrow{\alpha} G_0(u' + u'') \to 0,$$

where the symbol $\hookrightarrow$ denotes inclusion, which leads to the exact complex

$$0 \to N \hat{\otimes} K_0 \hookrightarrow N \hat{\otimes} G_0 \xrightarrow{1 \hat{\otimes} \alpha} N \hat{\otimes} G_0(u' + u'') \to 0.$$

By canonical identifications, we have the equalities

$$N \hat{\otimes} G_0 = \{((f, g_1), (f, g_2)) \in (N \hat{\otimes} G_0(u')) \times (N \hat{\otimes} G_0(u''))\},$$

$$N \hat{\otimes} K_0 = \{((0, g), (0, -g)) \in N \hat{\otimes} G_0 : g \in N \hat{\otimes} Y_2\}.$$

Moreover, via these identifications, the mapping $1 \hat{\otimes} \alpha$ acts in the following way: $(1 \hat{\otimes} \alpha)((f, g_1), (f, g_2)) = (f, g_1 + g_2)$. Since $N \hat{\otimes} G_0(u') = G_0(\varphi_N(u'))$ and $N \hat{\otimes} G_0(u'') = G_0(\varphi_N(u''))$, it follows that $1 \hat{\otimes} \alpha$ coincides with the mapping (2.7) corresponding to the operators $\varphi_N(u')$ and $\varphi_N(u'')$. Therefore the image of $1 \hat{\otimes} \alpha$ is equal to $G_0(\varphi_N(u') + \varphi_N(u''))$. On the other hand, the image of $1 \hat{\otimes} \alpha$ is $N \hat{\otimes} G_0(u' + u'') = G_0(\varphi_N(u' + u''))$. Consequently, $\varphi_N(u' + u'') = \varphi_N(u') + \varphi_N(u'')$.

Next, let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ and $\lambda \in \mathbf{C} \setminus \{0\}$. Then

$$G_0(\lambda u) = \{(x, \lambda y) : (x, y) \in G_0(u)\}.$$

If γ is the linear mapping

$$G_0(u) \ni (x, y) \to (x, \lambda y) \in G_0(\lambda u),$$

then we have:

$$G_0(\varphi_N(\lambda u)) = N \hat{\otimes} \gamma(G_0(u)) = (1 \hat{\otimes} \gamma)(G_0(\varphi_N(u))) =$$

$$= \{(f, \lambda g) : (f, g) \in G_0(\varphi_N(u))\} = G_0(\lambda \varphi_N(u)).$$

Finally, if $u = 0$, then $G_0(u) = X_1 \hat{\otimes} Y_2$. Hence $G_0(\varphi_N(u)) = (N \hat{\otimes} X_1) \times (N \hat{\otimes} Y_2)$, showing that $\varphi_N(u) = 0$.

2.10. Lemma. *Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ and let $v \in \mathscr{L}(X_2/Y_2, X_3/Y_3)$. Then $\varphi_N(vu) = \varphi_N(v)\varphi_N(u)$.*

Proof. We proceed as in the previous proof. Consider the F-space

$$G_0 = \{((x, y), (y, z)) \in G_0(u) \times G_0(v)\}.$$

Then the range of the linear and continuous mapping

$$G_0 \ni ((x, y), (y, z)) \to (x, z) \in X_1 \times X_3 \tag{2.8}$$

is precisely $G_0(vu)$. Let α be the mapping (2.8). Then the kernel of α is the F-space

$$K_0 = \{((0, y), (y, 0)) : y \in Y_2\}.$$

Hence we have an exact complex of F-spaces

$$0 \to K_0 \hookrightarrow G_0 \xrightarrow{\alpha} G_0(vu) \to 0,$$

which leads to the exact complex

$$0 \to N \hat{\otimes} K_0 \hookrightarrow N \hat{\otimes} G_0 \xrightarrow{1 \hat{\otimes} \alpha} N \hat{\otimes} G_0(vu) \to 0.$$

Note that

$$N \hat{\otimes} G_0 = \{((f, g), (g, h)) \in (N \hat{\otimes} G_0(u)) \times (N \hat{\otimes} G_0(v))\},$$

$$N \hat{\otimes} K_0 = \{((0, g), (g, 0)) \in N \hat{\otimes} G_0 : g \in N \hat{\otimes} Y_2\},$$

and that $(1 \hat{\otimes} \alpha)((f, g), (g, h)) = (f, h)$ (via canonical identifications). Since $N \hat{\otimes} G_0(u) = G_0(\varphi_N(u))$ and $N \hat{\otimes} G_0(v) = G_0(\varphi_N(v))$, we infer that the mapping $1 \hat{\otimes} \alpha$ is precisely the mapping (2.8) corresponding to the operators $\varphi_N(u)$ and $\varphi_N(v)$. Thus the image of $1 \hat{\otimes} \alpha$ equals $G_0(\varphi_N(v)\, \varphi_N(u))$. On the other hand, the image of $1 \hat{\otimes} \alpha$ is $N \hat{\otimes} G_0(vu) = G_0(\varphi_N(vu))$, and the assertion is established.

2.11. LEMMA. *Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ be a strict operator induced by $u_0 \in \mathscr{L}(X_1, X_2)$. Then $\varphi_N(u)$ is the strict operator induced by $1_N \hat{\otimes} u_0$, where 1_N is the identity on N.*

Proof. From (2.3) we infer that

$$N \hat{\otimes} G_0(u) = N \hat{\otimes} G(u_0) + \{0\} \times (N \hat{\otimes} Y_2),$$

and the sum of the right hand side is direct. Note that the space $N \hat{\otimes} G(u_0)$ is precisely the graph of the operator $1_N \hat{\otimes} u_0$. Thus

$$G_0(\varphi_N(u)) = G(1_N \hat{\otimes} u_0) + \{0\} \times (N \hat{\otimes} Y_2),$$

whence it follows that $\varphi_N(u)$ is induced by $1_N \hat{\otimes} u_0$, in virtue of (2.3).

2.12. COROLLARY. *Let X/Y be a* qF-*space and let $1_{X/Y}$ be the identity of X/Y. Then $\varphi_N(1_{X/Y})$ is the identity on $\varphi_N(X/Y)$.*

Proof. Indeed, $1_{X/Y}$ is a strict operator induced by the identity of X. Therefore, by Lemma 2.11, $\varphi_N(1_{X/Y})$ is the strict operator induced by the identity of $N \hat{\otimes} X$.

2.13. REMARK. If for every object X/Y in the category q$\mathscr{F}$ we denote by $\varphi_N(X/Y)$ the object given by (2.5) and for every morphism u in the category q$\mathscr{F}$ we let $\varphi_N(u)$ be given by Lemma 2.8, then the results obtained so far show, in particular, that φ_N is a functor in the category q$\mathscr{F}$. An alternate construction of φ_N can be performed by using the general method indicated in [21].

We now summarize a part from the above results in the language of the theory of categories [3].

2.14. THEOREM. *Let N be a fixed nuclear* F*-space. Then φ_N is an additive and covariant functor of the category* q$\mathscr{F}$ *into itself, which leaves invariant the class of strict morphisms.*

Proof. The assertion follows from Lemmas 2.8–2.11 and Corollary 2.12.

The results obtained up to now extend and often simplify assertions contained in [17], Section 2. To obtain a complete extension of Theorem 2.9 from [17], we shall study the modifications of the functor φ_N when one acts upon the space N.

2.15. PROPOSITION. *Let N_1, N_2 be nuclear* F*-spaces and let $\theta \in \mathscr{L}(N_1, N_2)$. Let also $X_1/Y_1, X_2/Y_2$ be* qF*-spaces. If 1_j denotes the identity on X_j and $\theta_j \in \mathscr{L}(\varphi_{N_1}(X_j/Y_j), \varphi_{N_2}(X_j/Y_j))$ is the strict operator that is induced by $\theta \hat{\otimes} 1_j$ $(j = 1, 2)$, then for every $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ the diagram*

$$\begin{array}{ccc} \varphi_{N_1}(X_1/Y_1) & \xrightarrow{\varphi_{N_1}(u)} & \varphi_{N_1}(X_2/Y_2) \\ {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \theta_2} \\ \varphi_{N_2}(X_1/Y_1) & \xrightarrow{\varphi_{N_2}(u)} & \varphi_{N_2}(X_2/Y_2) \end{array}$$

is commutative.

Proof. Since $\theta \hat{\otimes} 1_j$ maps the space $N_1 \hat{\otimes} Y_j$ into $N_2 \hat{\otimes} Y_j$, it follows that $\theta \otimes 1_j$ induces a strict operator from $\varphi_{N_1}(X_j/Y_j)$ into $\varphi_{N_2}(X_j/Y_j)$. Let 1_0 be the identity of $G_0(u)$ and let $\theta_0 = \theta \hat{\otimes} 1_0$. Then the diagram

$$\begin{array}{ccc} N_1 \hat{\otimes} G_0(u) & \hookrightarrow & (N_1 \hat{\otimes} X_1) \times (N_1 \hat{\otimes} X_2) \\ {\scriptstyle \theta_0}\downarrow & & \downarrow{\scriptstyle \bar{\theta}} \\ N_2 \hat{\otimes} G_0(u) & \hookrightarrow & (N_2 \hat{\otimes} X_1) \times (N_2 \hat{\otimes} X_2) \end{array}$$

is commutative, where $\bar{\theta}(f,g) = ((\theta \hat{\otimes} 1_1)f, (\theta \hat{\otimes} 1_2)g)$. If $(f,g) \in G_0(\varphi_{N_1}(u))$, then $\theta_0^{\cdot}(f,g) \in G_0(\varphi_{N_2}(u))$. Therefore

$$(\theta \hat{\otimes} 1_2)g \in \varphi_{N_2}((\theta \hat{\otimes} 1_1)f + N_2 \hat{\otimes} Y_1) = \varphi_{N_2}(\theta_1(f + N_1 \hat{\otimes} Y_1)).$$

Since $(\theta \hat{\otimes} 1_2)g \in \theta_2\varphi_{N_1}(f + N_1 \hat{\otimes} Y_1))$, we infer easily the desired conclusion.

2.16. Remarks. 1° First of all we present an observation related to the previous proposition. Namely, if $X_1/Y_1, X_2/Y_2$ are qF-spaces, then for every $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ we have the equality

$$\varphi_{N_1\hat{\otimes}N_2}(u) = \varphi_{N_1}(\varphi_{N_2}(u)). \tag{2.9}$$

Indeed,

$$G_0(\varphi_{N_1\hat{\otimes}N_2}(u)) = N_1 \hat{\otimes} N_2 \hat{\otimes} G_0(u) = N_1 \hat{\otimes} G_0(\varphi_{N_2}(u)) = G_0(\varphi_{N_1}(\varphi_{N_2}(u))).$$

2° Let A be an arbitrary algebra. Assume that the nuclear F-space N is a left A-module such that the linear mapping L_a given by $L_a(v) = av$ $(v \in N)$ is continuous for every $a \in A$. Then for each F-space X the space $N \hat{\otimes} X$ is still an A-module (with the action induced by $L_a \hat{\otimes} 1_X$, $a \in A$). With these conditions, for every operator $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ the operator $\varphi_N(u)$ is an A-module homomorphism. This assertion is a simple consequence of the fact that $\varphi_N(X_1/Y_1)$, $\varphi_N(X_2/Y_2)$ and $G_0(\varphi_N(u))$ are A-modules.

3° Let N_2 be a nuclear F-space and let N_1 be a closed subspace of N_2. Then for every qF-space X/Y the natural mapping from $\varphi_{N_1}(X/Y)$ into $\varphi_{N_2}(X/Y)$ (see Proposition 2.15) is injective. This is clearly equivalent to the equality

$$(N_1 \hat{\otimes} X) \cap (N_2 \hat{\otimes} Y) = N_1 \hat{\otimes} Y,$$

and it follows from general arguments of short exact sequences of complexes. Let us sketch a direct proof. Let $N_3 = N_1/N_2$, which is also a nuclear F-space [8], let $i : N_1 \to N_2$ be the inclusion and let $k : N_2 \to N_3$ be the canonical mapping. If $f \in (N_1 \hat{\otimes} X) \cap (N_2 \hat{\otimes} Y)$, then

$$(k \hat{\otimes} 1_X)f = (k \hat{\otimes} 1_X)(i \hat{\otimes} 1_X)f = 0.$$

Therefore $(k \hat{\otimes} 1_Y)f = (k \hat{\otimes} 1_X)f = 0$, and so $f \in N_1 \hat{\otimes} Y$, by the exactness of the complex

$$0 \to N_1 \hat{\otimes} Y \xrightarrow{i\hat{\otimes}1_Y} N_2 \hat{\otimes} Y \xrightarrow{k\hat{\otimes}1_Y} N_3 \hat{\otimes} Y \to 0.$$

This shows that $\varphi_{N_1}(X/Y)$ is a qF-subspace of $\varphi_{N_2}(X/Y)$ (see Remark 2.4.4°). Hence, if $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$, then $\varphi_{N_1}(u)$ extends $\varphi_{N_2}(u)$, because of the relations

$$G_0(\varphi_{N_1}(u)) = N_1 \hat{\otimes} G_0(u) \subset N_2 \hat{\otimes} G_0(u) = G_0(\varphi_{N_2}(u)).$$

A particular case that interests us is the following. Let $N_2 = N$ be a nuclear F-space which is also a unital algebra whose multiplication is separately continuous, so that 2° applies. In addition, if $N_1 = \mathbf{C}\cdot 1$, then X/Y is canonically isomorphic to $\varphi_{N_1}(X/Y)$. In this case, the operator $\varphi_N(u)$ can be regarded as an extension of the operator $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$.

4° Let V be a differentiable manifold. Then $\mathscr{E}(V)$ is a nuclear F-space [8]. Therefore the results of this section can be applied to $N = \mathscr{E}(V)$. We shall denote in the following by $\varepsilon(V, \cdot)$ the functor $\varphi_{\mathscr{E}(V)}$ given by Theorem 2.14. Note that the morphisms obtained via $\varepsilon(V, \cdot)$ are also $\mathscr{E}(V)$-module homomorphisms, by 2°.

5° Let V be an analytic manifold. Then $\mathscr{O}(V)$ is a nuclear F-space [8]. Hence we may apply the results of this section to $N = \mathscr{O}(V)$. The functor $\varphi_{\mathscr{O}(V)}$ will be denoted in the following by $o(V, \cdot)$. It follows from 2° that this functor yields $\mathscr{O}(V)$-module homomorphisms.

Since $\mathscr{O}(V)$ is a closed subspace of $\mathscr{E}(V)$, it follows from 3° that $\varepsilon(V, u)$ extends $o(V, u)$ for every $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ (see also [9], Remarque 1). Note also that $\mathscr{O}(V)$ is an algebra as in the end of 3°. Hence the operator $o(V, u)$ extends in turn the operator u (if we regard X_j/Y_j as a qF-subspace of $o(V, X_j/Y_j)$, $j = 1, 2$). In particular, if $u \in \mathscr{L}(X/Y)$, then X/Y is invariant under $o(V, u)$ and $o(V, X/Y)$ is invariant under $\varepsilon(V, u)$ (see Remark 2.4.4°).

6° Let $\hat{\mathbf{C}}$ be the one-point compactification of the complex plane $\mathbf{C}$. For every open set $V \subset \hat{\mathbf{C}}$ we set ${}_0\mathscr{O}(V) = \mathscr{O}(V)$ if $V \not\ni \infty$ and ${}_0\mathscr{O}(V) = \{f \in \mathscr{O}(V) : :f(\infty) = 0\}$ if $V \ni \infty$. Since ${}_0\mathscr{O}(V)$ is nuclear, we may consider its functor ${}_0o(V, \cdot)$. It follows from 3° that $o(V, u)$ extends ${}_0o(V, u)$ for every $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$.

The functors ${}_0o(V, \cdot)$ and $o(V, \cdot)$ can be used to define a concept of spectrum for every $u \in \mathscr{L}(X/Y)$ [17]. Let us recall this definition. If $V \subset \hat{\mathbf{C}}$ is an open set, let ζ_V be the operator from ${}_0o(V, X/Y)$ into $o(V, X/Y)$ induced by the multiplication with the coordinate function ζ. Let also u_V be the operator ${}_0o(V, u)$ followed by the canonical embedding of ${}_0o(V, X/Y)$ into $o(V, X/Y)$.

A point $w \in \hat{\mathbf{C}}$ is said to be *regular* for u [17] if there exists an open set $W \ni w$ such that for each open subset $V \subset W$ the operator $\zeta_V - u_V \in \mathscr{L}({}_0o(V, X/Y), o(V, X/Y))$ is bijective. The complement in $\hat{\mathbf{C}}$ of the set of regular points for u is denoted by $\sigma(u, X/Y)$ and is called the *spectrum* of u. The set $\sigma(u, X/Y)$, which is obviously closed in $\hat{\mathbf{C}}$, is nonempty, provided $X \neq Y$ [17].

According to the standard terminology, we shall say that the operator $u \in \mathscr{L}(X/Y)$ is *regular* if the point ∞ is regular for u. This is equivalent to the compactness of the set $\sigma(u, X/Y)$ in $\mathbf{C}$.

In what follows N will be again a fixed nuclear F-space.

2.17. LEMMA. *Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$. Then we have the equalities* $\mathrm{Im}_0(\varphi_N(u)) = N \hat{\otimes} \mathrm{Im}_0(u)$ *and* $\mathrm{Ker}_0(\varphi_N(u)) = N \hat{\otimes} \mathrm{Ker}_0(u)$.

Proof. Since $\mathrm{Im}_0(u) = \mathrm{pr}_2(G_0(u))$ and N is nuclear, we have:

$$N \hat{\otimes} \mathrm{Im}_0(u) = N \hat{\otimes} \mathrm{pr}_2(G_0(u)) = (1 \hat{\otimes} \mathrm{pr}_2)(N \hat{\otimes} G_0(u)) =$$

$$= (1 \hat{\otimes} \mathrm{pr}_2)(G_0(\varphi_N(u))) = \mathrm{Im}_0(\varphi_N(u)),$$

because $1 \hat{\otimes} \mathrm{pr}_2$ is just the projection on the second coordinate in $(N \hat{\otimes} X_1) \times (N \hat{\otimes} X_2)$, and by virtue of Remark 2.6.4°.

To obtain the second equality, note that

$$(N \hat{\otimes} \mathrm{Ker}_0(u)) \times (N \hat{\otimes} Y_2) = G_0(\varphi_N(u)) \cap ((N \hat{\otimes} X_1) \times (N \hat{\otimes} Y_2)),$$

by (2.2) and Remark 2.6.2°. Thus $\mathrm{Ker}_0(\varphi_N(u)) = N \hat{\otimes} \mathrm{Ker}_0(u)$, again by (2.2).

2.18. COROLLARY. *Let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ and $v \in \mathscr{L}(X_2/Y_2, X_3/Y_3)$ be such that $vu = 0$. Then we have the equality*

$$\varphi_N(\mathrm{Ker}_0(v)/\mathrm{Im}_0(u)) = \mathrm{Ker}_0(\varphi_N(v))/\mathrm{Im}_0(\varphi_N(u)).$$

Proof. Since $vu = 0$, and hence $\mathrm{Im}_0(u) \subset \mathrm{Ker}_0(v)$, we also have $\varphi_N(v)\varphi_N(u) = 0$, and so $\mathrm{Im}_0(\varphi_N(u)) \subset \mathrm{Ker}_0(\varphi_N(v))$. The stated equality is now a simple consequence of Lemma 2.17.

The last two results can be applied to get some information about the complexes in the category $\mathrm{q}\mathscr{F}$. Let

$$(X/Y, u) : \ldots \xrightarrow{u^{p-1}} X^p/Y^p \xrightarrow{u^p} X^{p+1}/Y^{p+1} \xrightarrow{u^{p+1}} \ldots$$

be a complex in the category $\mathrm{q}\mathscr{F}$. Since $\mathrm{Ker}(u^p)/\mathrm{Im}(u^{p-1})$ is isomorphic to $\mathrm{Ker}_0(u^p)/\mathrm{Im}_0(u^{p-1})$, the homology of the above complex remains in $\mathrm{q}\mathscr{F}$, by Lemma 2.3. We shall designate this homology by $\{H^p(X/Y, u)\}_{p \in \mathbf{Z}}$.

2.19. PROPOSITION. *Let*

$$(X/Y, u) : \ldots \xrightarrow{u^{p-1}} X^p/Y^p \xrightarrow{u^p} X^p/Y^p \xrightarrow{u^{p+1}} \ldots$$

be a complex in the category $\mathrm{q}\mathscr{F}$ *and assume that* N *is non-null. Then*

$$(\varphi_N(X, Y), \varphi_N(u)) : \ldots \xrightarrow{\varphi_N(u^{p-1})} \varphi_N(X^p/Y^p) \xrightarrow{\varphi_N(u^p)} \ldots$$

is a complex in the category $\mathrm{q}\mathscr{F}$ *such that*

$$H^p(\varphi_N(X/Y), \varphi_N(u)) = \varphi_N(H^p(X/Y, u)), \quad p \in \mathbf{Z}.$$

In particular, $(X/Y, u)$ *is exact if and only if* $(\varphi_N(X/Y), \varphi_N(u))$ *is exact.*

Proof. That $(\varphi_N(X/Y), \varphi_N(u))$ is a complex whose homology is described as in the statement follows from Corollary 2.18. In particular, the exactness of $(X/Y, u)$ implies the exactness of $(\varphi_N(X/Y), \varphi_N(u))$.

Conversely, if $(\varphi_N(X/Y), \varphi_N(u))$ is exact, then we have the equality $N\hat{\otimes}\mathrm{Im}_0(u^{p-1}) = N\hat{\otimes}\mathrm{Ker}_0(u^p)$ for each p. From Lemma 2.7 we deduce that $\mathrm{Im}_0(u^{p-1}) = \mathrm{Ker}_0(u^p)$ for all p, that is the exactness of the complex $(X/Y, u)$.

The last result of this section is an F-space version of Proposition 5 from [19].

2.20. PROPOSITION. *Let*

$$(X/Y, u) : \ldots \xrightarrow{u^{p-1}} X^p/Y^p \xrightarrow{u^p} X^{p+1}/Y^{p+1} \xrightarrow{u^{p+1}} \ldots$$

be a complex in the category $\mathrm{q}\mathscr{F}$ *such that* u^p *is a strict operator induced by* $u_0^p \in \mathscr{L}(X^p, X^{p+1})$ *for all* p. *The complex* $(X/Y, u)$ *is exact if and only if the complex of* F-*spaces*

$$\ldots \xrightarrow{v^{p-1}} X^p \times Y^{p+1} \xrightarrow{v^p} X^{p+1} \times Y^{p+2} \xrightarrow{v^{p+1}} \ldots \tag{2.10}$$

is exact, where

$$v^p = \begin{pmatrix} u_0^p & -1_{p+1} \\ u_0^{p+1}u_0^p & -u_0^{p+1} \mid Y^{p+1} \end{pmatrix}$$

and 1_p *is the identity on* Y^p *for all* $p \in \mathbf{Z}$.

Proof. Since $(X/Y, u)$ is a complex, we must have $\mathrm{Im}_0(u^{p-1}) \subset \mathrm{Ker}_0(u^p)$ for every p. Note that $\mathrm{Im}_0(u^{p-1}) = \mathrm{Im}(u_0^{p-1}) + Y^p$ and $\mathrm{Ker}_0(u^p) = (u_0^p)^{-1}(Y^{p+1})$, from which we derive, in particular, that (2.10) is a complex. Therefore the complex $(X/Y, u)$ is exact if and only if

$$\mathrm{Im}(u_0^{p-1}) + Y^p = (u_0^p)^{-1}(Y^{p+1}), \quad p \in \mathbf{Z}.$$

This is equivalent to the fact that the system of equations

$$u_0^{p-1}x_{p-1} - y_p = x_p$$

$$u_0^p(u_0^{p-1}x_{p-1} - y_p) = y_{p-1}$$

has a solution $(x_{p-1}, y_p) \in X^{p-1} \times Y^p$ for each given pair $(x_p, y_{p+1}) \in X^p \times Y^{p+1}$ with the property that $u_0^p x_p - y_{p+1} = 0$, where p is arbitrary, which means just the exactness of the complex (2.10).

3. THE JOINT SPECTRUM OF A MULTIOPERATOR

Let X/Y be a fixed qF-space. In this section we intend to define a joint spectrum for some finite families of commuting operators from $\mathscr{L}(X/Y)$.

3.1. DEFINITION. A finite family $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ will be designated as a *multioperator*. If $u_j u_k = u_k u_j$ $(j, k = 1, \ldots, n)$, then u is said to be a *commuting* multioperator. If every u_j $(j = 1, \ldots, n)$ is regular (see Remark 2.16.6°), then the multioperator u will be called *regular*. If each u_j $(j = 1, \ldots, n)$ is a strict operator (see Remark 2.4.2°), then the multioperator u is said to be *strict*.

In this section we shall mainly work with commuting regular multioperators, which will be briefly designated by c.r.m. We denote by $\zeta = (\zeta_1, \ldots, \zeta_n)$ the family of coordinate functions in $\mathbf{C}^n$, that is $\zeta_j(z) = z_j$ for all $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $j = 1, \ldots, n$. When no confusion is possible, we shall also denote by ζ_j the multiplication operator by the function ζ_j in any of the spaces $\mathcal{O}(V, X/Y)$, $\varepsilon(V, X/Y)$ (see Remarks 2.16.4° and 2.16.5°), as well as in direct products of such spaces etc., where $V \subset \mathbf{C}^n$ is an arbitrary open set. For every multioperator $u = (u_1, \ldots, u_n)$ we denote by $\mathcal{O}(V, u)$ (resp. $\varepsilon(V, u)$) the multioperator $(\mathcal{O}(V, u_1), \ldots, \mathcal{O}(V, u_n))$ (resp. $(\varepsilon(V, u_1), \ldots, \varepsilon(V, u_n))$).

Let us observe that if $X_1/Y_1, \ldots, X_n/Y_n$ are qF-spaces, then the natural isomorphism

$$(X_1/Y_1) \times \ldots \times (X_n/Y_n) \to (X_1 \times \ldots \times X_n)/(Y_1 \times \ldots \times Y_n)$$

allows us to regard direct products (or direct sums) as qF-spaces. We shall make no distinction between such spaces in the following. In particular, the Koszul complex obtained from qF-spaces is a complex of qF-spaces.

3.2. DEFINITION. Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a c.r.m. The *resolvent set* $\rho(u, X/Y)$ of u is the set of those points $w \in \mathbf{C}^n$ for which there exists an open set $W \ni w$ such that

$$H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)) = \{0\}, \quad p \geqslant 0,$$

for each open polydisc $D \subset W$. The set $\sigma(u, X/Y) = \mathbf{C}^n \setminus \rho(u, X/Y)$ will be called the *joint spectrum* of the multioperator u (compare with [13], [11], [19], [27], etc.).

The family of differentials $d\bar{\zeta} = (d\bar{\zeta}_1, \ldots, d\bar{\zeta}_n)$ will be regarded in what follows as a system of indeterminates associated to the multioperator $\bar{\nabla} = (\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_n)$. Therefore the operator $\delta(\bar{\nabla})$ (see the Introduction) is precisely $\bar{\partial} = (\partial/\partial\bar{z}_1)d\bar{\zeta}_1 + \ldots + (\partial/\partial\bar{z}_n)d\bar{\zeta}_n$.

If $v \in \mathscr{L}(X/Y)$, since $G_0(\varepsilon(V, v)) = \mathscr{E}(V, G_0(v))$, it is clear that $\varepsilon(V, v)$ commutes with the operator induced by $\partial/\partial\bar{z}_j$ in $\varepsilon(V, X/Y)$ (also denoted by $\partial/\partial\bar{z}_j$) for all open $V \subset \mathbf{C}^n$ and $j = 1, \ldots, n$. In particular, $(\varepsilon(V, u), \bar{\nabla})$ is a commuting multioperator for every commuting multioperator $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$.

3.3. LEMMA. *Let $V \subset \mathbf{C}^n$ be a Stein open set. Then the complex*

$$0 \to o(V, X/Y) \xrightarrow{i_V} \varepsilon(V, X/Y) \xrightarrow{\bar{\partial}} \Lambda^1[d\bar{\zeta}, \varepsilon(V, X/Y)] \xrightarrow{\bar{\partial}} \ldots$$

$$\ldots \xrightarrow{\bar{\partial}} \Lambda^n[d\bar{\zeta}, \varepsilon(V, X/Y)] \to 0,$$

is exact, where i_V is the canonical embedding.

Proof. The exactness of the complex

$$0 \to \mathscr{E}(V, Y) \hookrightarrow \mathscr{E}(V, X) \to \varepsilon(V, X/Y) \to 0,$$

where the mappings are the natural ones, implies the exactness of the sequence of complexes

$$0 \to K(\mathscr{E}(V, Y), \bar{\partial}) \to K(\mathscr{E}(V, X), \bar{\partial}) \to K(\varepsilon(V, X/Y), \bar{\partial}) \to 0,$$

from which we derive the exactness of the long homology sequence

$$\ldots \to H^p(\mathscr{E}(V, Y), \bar{\partial}) \to H^p(\mathscr{E}(V, X), \bar{\partial}) \to H^p(\varepsilon(V, X/Y), \bar{\partial}) \to$$

$$\to H^{p+1}(\mathscr{E}(V, Y), \bar{\partial}) \to \ldots$$

(see, for instance, [7], I. 2.2). Since V is a Stein manifold, then we must have $H^p(\mathscr{E}(V, Y), \bar{\partial}) = \{0\}$ and $H^p(\mathscr{E}(V, X), \bar{\partial}) = \{0\}$ for all $p \geq 1$. (These facts are well known for scalar-valued functions [10]; the case of vector-valued functions can be easily derived by tensor multiplication.) The same argument implies the exactness of the complex

$$0 \to \mathscr{O}(V, Y) \hookrightarrow \mathscr{O}(V, X) \to H^0(\varepsilon(V, X/Y), \bar{\partial}) \to 0,$$

which shows that $\mathrm{Ker}(\bar{\partial}|\varepsilon(V, X/Y))$ is isomorphic to $\mathcal{O}(V, X/Y)$, via the canonical embedding i_V.

3.4. REMARK. The argument from the proof of the previous lemma also works for finite direct products of complexes of the given type. For instance, if $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a system of indeterminates, then the complex

$$0 \to \Lambda[\sigma, \mathcal{O}(V, X/Y)] \xrightarrow{i_V} \Lambda[\sigma, \varepsilon(V, X/Y)] \xrightarrow{\bar{\partial}}$$

$$\to \Lambda^1[\mathrm{d}\bar{\zeta}, \Lambda[\sigma, \varepsilon(V, X/Y)]] \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Lambda^n[\mathrm{d}\bar{\zeta}, \Lambda[\sigma, \varepsilon(V, X/Y)]] \to 0$$

is exact for every open Stein set $V \subset \mathbf{C}^n$.

3.5. LEMMA. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* V *be a Stein open set in* $\mathbf{C}^n$. *We have*

$$H^p(\mathcal{O}(V, X/Y), \zeta - \mathcal{O}(V, u)) = \{0\}, \quad p \geqslant 0$$

if and only if

$$H^p(\varepsilon(V, X/Y), (\zeta - \varepsilon(V, u), \bar{\nabla})) = \{0\}, \quad p \geqslant 0.$$

Proof. The complex $K = K(\varepsilon(V, X/Y), (\zeta - \varepsilon(V, u), \bar{\nabla}))$ can be regarded as a double complex $K = (K^{p,q})_{p\geqslant 0, q\geqslant 0}$, with the differentials $\delta(\zeta - \varepsilon(V, u))$ and $\bar{\partial}$, where $K^{p,q}$ are the exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$ and of degree q in $\mathrm{d}\bar{\zeta}_1, \ldots, \mathrm{d}\bar{\zeta}_n$, with coefficients in $\varepsilon(V, X/Y)$. If $''H^q(K)$ is the q^{th} homology space of K with respect to $\bar{\partial}$, then, acocording to Remark 3.4, we must have $''H^q(K) = \{0\}$ if $q \geqslant 1$. This shows that the conditions of Theorem I.4.8.1 from [7] are fulfilled. In virtue of this theorem, if we denote by L the subcomplex of K consisting of those exterior forms which can be written as $\xi = \xi_{00} + \xi_{10} + \ldots$, where ξ_{p0} is a form of degree p in $\sigma_1, \ldots, \sigma_n$, of degree zero in $\mathrm{d}\bar{\zeta}_1, \ldots \mathrm{d}\bar{\zeta}_n$, and $\bar{\partial}\xi_{p0} = 0$, the embedding $L \to K$ provides an isomorphism between the homology of L and that of K. It follows from Remark 3.4 that the complex L is isomorphic to the complex of exterior forms in $\sigma_1 \ldots, \sigma_n$, with coefficients in $\mathcal{O}(V, X/Y)$.

3.6. REMARK. The proof of Lemma 3.5 actually shows that the space

$$H^p(\mathcal{O}(V, X/Y), \zeta - \mathcal{O}(V, u))$$

is isomorphic to the space

$$H^p(\varepsilon(V, X/Y), (\zeta - \varepsilon(V, u), \bar{\nabla}))$$

for every $p \geqslant 0$. This fact might be used as the starting point of an approach to a local spectral theory in this context (see also [6], [16], [5] etc.).

3.7. LEMMA. *Let $W \subset \mathbf{C}^n$ be an arbitrary open set and let $\varepsilon_W(X/Y)$ be the sheaf associated to the presheaf $U \to \varepsilon(U, X/Y)$ ($U \subset W$ open). Then $\varepsilon_W(X/Y)$ is a fine sheaf whose space of global sections on W coincides with $\varepsilon(W, X/Y)$.*

Proof. The sheaves $\varepsilon_W(X) = \varepsilon_W(X/\{0\})$, $\varepsilon_W(Y) = \varepsilon_W(Y/\{0\})$ and $\varepsilon_W(X/Y)$ are fine since they are invariant under multiplication by indefinitely differentiable functions with compact support. Moreover, if $\Gamma(\varepsilon_W(X/Y))$ is the space of global sections of $\varepsilon_W(X/Y)$, then, from the exactness of the complex of sheaves

$$0 \to \varepsilon_W(Y) \hookrightarrow \varepsilon_W(X) \to \varepsilon_W(X/Y) \to 0,$$

where the mappings are the natural sheaf morphisms, we deduce the exactness of the complex of global sections

$$0 \to \Gamma(\varepsilon_W(Y)) \hookrightarrow \Gamma(\varepsilon_W(X)) \to \Gamma(\varepsilon_W(X/Y)) \to 0$$

(see [10], VI.A for details). But it is visible that $\Gamma(\varepsilon_W(Y)) = \mathscr{E}(W, Y)$ and $\Gamma(\varepsilon_W(X)) = \mathscr{E}(W, X)$. Therefore $\Gamma(\varepsilon_W(X/Y)) = \varepsilon(W, X/Y)$, as claimed.

3.8. THEOREM. *Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a* c.r.m. *Then:*

(1) *for every open Stein set $V \subset \rho(u, X/Y)$ we have*

$$H^p(\mathscr{O}(V, X/Y), \zeta - \mathscr{O}(V, u)) = \{0\}, \quad p \geqslant 0;$$

(2) *for every open set $W \subset \rho(u, X/Y)$ we have*

$$H^p(\varepsilon(W, X/Y), (\zeta - \varepsilon(W, u), \bar{\nabla})) = \{0\}, \quad p \geqslant 0.$$

Proof. (1) Let $\mathscr{O}_V(X/Y)$ be the sheaf associated to the presheaf $U \to \mathscr{O}(U, X/Y)$ ($U \subset V$ open) and let $\mathscr{O}_{V,p}(X/Y) = \Lambda^p[\sigma, \mathscr{O}_V(X/Y)]$ ($p \geqslant 0$). Definition 3.2 implies the exactness of the complex of sheaves

$$(3.1) \qquad 0 \to \mathscr{O}_{V,0}(X/Y) \xrightarrow{\delta^0} \mathscr{O}_{V,1}(X/Y) \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{n-1}} \mathscr{O}_{V,n}(X/Y) \to 0,$$

where the sheaf morphisms are induced by the family $\{\delta(\zeta - \mathscr{O}(U, u))\}_{U \subset V}$. Note that every sheaf $\mathscr{O}_{V,p}(X/Y)$ is acyclic. Indeed, if we set

$$\varepsilon_{V,p,q}(X/Y) = \Lambda^q[d\bar{\zeta}, \Lambda^p[\sigma, \varepsilon_V(X/Y)]], \quad q \geqslant 0,$$

with $\varepsilon_V(X/Y)$ given by Lemma 3.7, then the complex of sheaves

$$(3.2) \qquad 0 \to \mathscr{O}_{V,p}(X/Y) \xrightarrow{i_V} \varepsilon_{V,p,0}(X/Y) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \varepsilon_{V,p,n}(X/Y) \to 0$$

is exact, by Remark 3.4. Let us observe that (3.2) is a fine resolution of the sheaf

$\mathcal{O}_{V,p}(X/Y)$, as a consequence of Lemma 3.7 (since every $\varepsilon_{V,p,q}(X/Y)$ is a finite direct product of copies of $\varepsilon_V(X/Y)$). Hence the complex of the spaces of global sections associated with (3.1) is exact (see [10], VI.B). But this is precisely our assertion (1).

(2) Set

$$\varepsilon_{W,m}(X/Y) = \Lambda^m[(\sigma, d\bar{\zeta}), \varepsilon_W(X/Y)], \quad m \geqslant 0.$$

By Lemmas 3.5 and 3.7, the complex of fine sheaves

$$0 \to \varepsilon_{W,0}(X/Y) \xrightarrow{d^0} \varepsilon_{W,1}(X/Y) \xrightarrow{d^1} \dots \tag{3.3}$$

is exact, where the sheaf morphisms d^m are induced by the family $\{\delta(\zeta - \varepsilon(U, u)) + \bar{\partial}\}_{U \subset W}$. Then the complex of the spaces of global sections associated to (3.3) is exact (by Corollary VI.A.4 of [10]), which is just our assertion (2).

We shall prove in the following some elementary properties of the joint spectrum (see also [13] or [16] for Banach or Fréchet space operators).

3.9. LEMMA. *Let* $u = (u_1, \dots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *Then we have the inclusion*

$$\sigma(u, X/Y) \subset \sigma(u_1, X/Y) \times \dots \times \sigma(u_n, X/Y). \tag{3.4}$$

Proof. Let $w \in \mathbf{C}^n$ be a point that is not in the right hand side of (3.4) together with an open neighbourhood W of it. Let $D \subset W$ be an open polydisc, $D = D_1 \times \dots \times D_n$, where D_j is a disc ($j = 1, \dots, n$). Then there exists an index j such that $D_j \cap \sigma(u_j, X/Y) = \emptyset$. From the definition of the spectrum of an operator (see Remark 2.16.6°), it follows that the operator $\zeta_j - \mathcal{O}(D_j, u_j)$ is invertible on $\mathcal{O}(D_j, X/Y)$. Then the operator $\mathcal{O}(G_j, \zeta_j - \mathcal{O}(D_j, u_j))$ is also invertible on $\mathcal{O}(G_j, \mathcal{O}(D_j, X/Y))$ (as follows from Theorem 2.14), where

$$G_j = D_1 \times \dots \times D_{j-1} \times D_{j+1} \times \dots \times D_n.$$

But the space $\mathcal{O}(G_j, \mathcal{O}(D_j, X/Y))$ is isomorphic to $\mathcal{O}(D, X/Y)$ and the operator $\mathcal{O}(G_j, \zeta_j - \mathcal{O}(D_j, u_j))$ can be identified with the operator $\zeta_j - \mathcal{O}(D, u_j)$, via this isomorphism, by Proposition 2.15 and Remark 2.16.1°. Therefore the operator $\zeta_j - \mathcal{O}(D, u_j)$ is invertible on $\mathcal{O}(D, X/Y)$. This implies that the multioperator $\zeta - \mathcal{O}(D, u)$ is nonsingular, by Lemma 1.1 from [13]. In other words

$$H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)) = \{0\}$$

for all $p \geqslant 0$. Since $D \subset W$ is arbitrary, it follows that $w \in \rho(u, X/Y)$.

3.10. THEOREM. *Let* $u' = (u_1, \ldots, u_n, u_{n+1}) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* $u = (u_1, \ldots, u_n)$. *If* $\pi_n: \mathbf{C}^{n+1} \to \mathbf{C}^n$ *is the projection on the first n coordinates, then we have the eqaulity* $\pi_n(\sigma(u', X/Y)) = \sigma(u, X/Y)$.

Proof. Let $w \in \rho(u, X/Y)$ and let $W \subset \rho(u, X/Y)$ be a neighbourhood of w as in Definition 3.2. Let also $D \subset W$ be an open polydisc. Then the complex $K(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u))$ is exact. Let D_{n+1} be an arbitrary disc and let $D' = D \times D_{n+1}$. From Proposition 2.19 and Remark 2.16.1° we deduce readily the exactness of the complex $K(\mathscr{O}(D', X/Y), \zeta - \mathscr{O}(D', u))$. Then the multioperator

$$(\zeta - \mathscr{O}(D', u), \zeta_{n+1} - \mathscr{O}(D', u_{n+1})) = (\zeta' - \mathscr{O}(D', u'))$$

is nonsingular, by Lemma 1.3 from [13], where $\zeta' = (\zeta, \zeta_{n+1})$. Hence no point $w' = (w, w_{n+1}) \in \mathbf{C}^{n+1}$ can be in $\sigma(u', X/Y)$, that is $w \notin \pi_n(\sigma(u', X/Y))$.

Conversely, let $D \subset \mathbf{C}^n$ be an arbitrary polydisc. Let us first observe that the operator $\mathscr{O}(D, u_{n+1})$ acts naturally on the space $\Lambda[\sigma, \mathscr{O}(D, X/Y)]$ (which is a direct product of copies of $\mathscr{O}(D, X/Y)$). Moreover, we have

$$\delta(\zeta - \mathscr{O}(D, u))\mathscr{O}(D, u_{n+1}) = \mathscr{O}(D, u_{n+1})\, \delta(\zeta - \mathscr{O}(D, u)),$$

which shows that $\mathscr{O}(D, u_{n+1})$ induces an operator, say $\mathscr{O}^p(D, u_{n+1})$, in the space $H^p(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u))$ (see Remark 2.4.5°).

Let us assume now that $\sigma(u, X/Y) \not\subset \pi_n(\sigma(u', X/Y))$. We shall prove that this hypothesis leads to a contradiction. For, let $D_{n+1} \subset \mathbf{C}$ be an arbitrary disc. We first show that the operator $\zeta_{n+1} - \mathscr{O}(D_{n+1}, \mathscr{O}^p(D, u_{n+1}))$ is bijective on the space $\mathscr{O}(D_{n+1}, H^p(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u)))$. Indeed, let $w \in \sigma(u, X/Y) \setminus \pi_n(\sigma(u', X/Y))$ and let $w_{n+1} \in \mathbf{C}$ be fixed. Then $w' = (w, w_{n+1}) \in \rho(u', X/Y)$. Let W' be a neighbourhood of w' provided by Definition 3.2. With no loss of generality, we may assume that $W' = W_1 \times \ldots \times W_n \times W_{n+1}$, with $W_j \subset \mathbf{C}$ $(j = 1, \ldots, n, n+1)$. If $D_j \subset W_j$ are arbitrary discs, we let D denote $D_1 \times \ldots \times D_n$ and $D' = D \times D_{n+1}$. Then the complex $K(\mathscr{O}(D', X/Y), \zeta' - \mathscr{O}(D', u'))$ is exact. If we proceed as in the proof of Theorem 3.1 from [13] (see also [19], Proposition 1), we infer that the operator $\zeta_{n+1} - \mathscr{O}(D', u_{n+1})$ induces an isomorphism of the space $H^p(\mathscr{O}(D', X/Y), \zeta - \mathscr{O}(D', u))$ onto itself. Cut this space is isomorphic to $\mathscr{O}(D_{n+1}, H^p(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u)))$ and the isomorphism induced by $\zeta_{n+1} - \mathscr{O}(D', u_{n+1})$ becomes $\mathscr{O}(D_{n+1}, \zeta_{n+1} - \mathscr{O}^p(D, u_{n+1}))$. This establishes the bijectivity of the operator $\mathscr{O}(D_{n+1}, \zeta_{n+1} - \mathscr{O}^p(D, u_{n+1}))$. Since D_{n+1} is arbitrary, we have obtained the inclusion

$$\mathbf{C} \subset \rho(\mathscr{O}^p(D, u_{n+1}), H^p(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u))).$$

We shall prove that the operator $\mathscr{O}^p(D, u_{n+1})$ is regular. For, let $G \subset \hat{\mathbf{C}}$ be an open neighbourhood of the infinity such that the operator

$$\zeta_{n+1} - (u_{n+1})_G : {}_0\mathscr{O}(G, X/Y) \to \mathscr{O}(G, X/Y)$$

is bijective (the operator $(u_{n+1})_G$ is defined as in Remark 2.16.6°). Then the operator

$$\Lambda^p[\sigma, {}_0\mathcal{O}(G, \mathcal{O}(D, X/Y))] \to \Lambda^p[\sigma, \mathcal{O}(G, \mathcal{O}(D, X/Y))]$$

induced by $\zeta_{n+1} - (u_{n+1})_G$ is also bijective, where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a system of indeterminates associated with the multioperator u. Therefore the operator

$${}_0\mathcal{O}(G, H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u))) \to \mathcal{O}(G, H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)))$$

induced by $\zeta_{n+1} - \mathcal{O}^p(D, u_{n+1})$ is bijective too (see Remark 2.4.5°), so that $\mathcal{O}^p(D, u_{n+1})$ is regular. In other words, the spectrum of $\mathcal{O}^p(D, u_{n+1})$ is empty for each $p \geqslant 0$, which is possible only if $H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u))$ is null for all p (and D). On the other hand, since $w \in \sigma(u, X/Y)$, we can find a polydisc D and an index p such that $H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)) \neq \{0\}$. This contradiction concludes the proof of the theorem.

The next result is a version of the so-called *projection property* of the joint spectrum.

3.11. THEOREM. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* $J = \{j_1, \ldots, j_p\} \subset \{1, \ldots, n\}$ *be a family of distinct integers, with* $1 \leqslant p \leqslant n$. *Set* $u_J = \{u_{j_1}, \ldots, u_{j_p}\}$. *If* $\pi_J: \mathbf{C}^n \to \mathbf{C}^p$ *is the projection* $z = (z_1, \ldots, z_n) \to z_J = (z_{j_1}, \ldots, z_{j_p})$, *then we have the equality* $\sigma(u_J, X/Y) = \pi_J(\sigma(u, X/Y))$.

Proof. The assertion can be easily obtained, by combining Theorem 3.10 with the method from [16], Lemma III.9.6.

3.12. COROLLARY. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$. *Then the joint spectrum* $\sigma(u, X/Y)$ *is compact and nonempty, provided* $X \neq Y$.

Proof. The compactness of $\sigma(u, X/Y)$ follows from Lemma 3.9. Since the projection of $\sigma(u, X/Y)$ on the j^{th} component equals the spectrum of u_j, and the latter is nonempty for all $j = 1, \ldots, n$, then $\sigma(u, X/Y)$ is nonempty.

Let X_0/Y_0 be a qF-subspace of X/Y that is invariant under the c.r.m. $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ (i.e. X_0/Y_0 is invariant under each $u_j, j = 1, \ldots, n$). Assume that the multioperators induced by u in both X_0/Y_0 and in $X/(X_0 + Y)$ (see Remarks 2.4.3° and 2.4.4°) are regular. Then we can discuss the joint spectra of the multioperators that are induced by u in X_0/Y_0 and in $X/(X_0 + Y)$, which are denoted by $\sigma(u, X_0/Y_0)$ and $\sigma(u, X/(X_0 + Y))$, respectively.

3.13. PROPOSITION. *The union of any two of the sets* $\sigma(u, X/Y)$, $\sigma(u, X_0/Y_0)$ *and* $\sigma(u, X/(X_0 + Y))$ *contains the third.*

Proof. This result is a version of Lemma 1.2 from [13] (see also [16], Remark IV.1.7). Let $i_0 : X_0/Y_0 \to X/Y$ be the canonical embedding and let $k_0 : X/Y \to X/(X_0 + Y)$ be the canonical mapping. Let also $D \subset \mathbf{C}^n$ be an arbitrary polydisc. Since the complex

$$0 \to X_0/Y_0 \xrightarrow{i_0} X/Y \xrightarrow{k_0} X/(X_0 + Y) \to 0$$

is exact, then, in virtue of Proposition 2.19, the complex

$$0 \to \mathcal{O}(D, X_0/Y_0) \xrightarrow{\mathcal{O}(D, i_0)} \mathcal{O}(D, X/Y) \xrightarrow{\mathcal{O}(D, k_0)} \mathcal{O}(D, X/(X_0 + Y)) \to 0$$

is also exact. Hence the sequence of complexes

$$0 \to K(\mathcal{O}(D, X_0/Y_0), \zeta - \mathcal{O}(D, u)) \to K(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)) \to$$

$$\to K(\mathcal{O}(D, X/(X_0 + Y)), \zeta - \mathcal{O}(D, u)) \to 0$$

is exact. Therefore there exists a long exact homology complex

$$\ldots \to H^p(\mathcal{O}(D, X_0/Y_0), \zeta - \mathcal{O}(D, u)) \to H^p(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u)) \to$$

$$\to H^p(\mathcal{O}(D, X/(X_0 + Y)), \zeta - \mathcal{O}(D, u)) \to \ldots$$

from which we easily derive our assertion.

As in the case of Banach space operators [13], for commuting multioperators in qB-spaces one can give a "pointwise" definition of the spectrum [19]. We shall show in what follows that Definition 3.2 and the corresponding definition from [19] provide the same concept of joint spectrum in qB-spaces.

3.14. REMARK. A morphism in the category of qB-spaces (briefly designated by $q\mathscr{B}$) is defined in [18] as a finite superposition of strict morphisms and inverses of bijective strict morphisms (called in [18] pseudo-isomorphisms). On the other hand, if X_1/Y_1, X_2/Y_2 are qB-spaces and $v: X_1/Y_1 \to X_2/Y_2$ is a linear mapping, we may say that v is an *operator* (in $q\mathscr{B}$) if $G_0(v)$ is a B-subspace of $X_1 \times X_2$, by adapting Definition 2.1 to this case. One can see that a linear mapping is a morphism in $q\mathscr{B}$ if and only if it is an operator in $q\mathscr{B}$. Indeed, strict morphisms and inverses of bijective strict morphisms are operators (as in [17], Lemma 2.4), so that their superpositions are still operators. Conversely, every operator can be written as a composite of a strict morphism and the inverse of a bijective strict morphism (see the proof of Lemma 2.2).

Nevertheless, a natural question arises in this context. If X_1/Y_1, X_2/Y_2 are qB-spaces and $v : X_1/Y_1 \to X_2/Y_2$ is a morphism in the category $q\mathscr{F}$, is v also a morphism in the category $q\mathscr{B}$? (In other words, is $q\mathscr{B}$ a full subcategory of $q\mathscr{F}$?) Before proving the announced equivalence of joint spectra, we shall settle this question, whose answer happens to be affirmative.

3.15. PROPOSITION. *Let* X_1/Y_1, X_2/Y_2 *be* qB-*spaces. Then for every* $v \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ *the lifted graph* $G_0(v)$ *is a* B-*subspace of the* B-*space* $X_1 \times X_2$.

The proof of this assertion is based upon an auxiliary result:

3.16. LEMMA. *Let* $0 \to X_0 \to X \to X_1 \to 0$ *be an exact complex of* F-*spaces and continuous mappings. If any two of the spaces* X_0, X, X_1 *are* B-*spaces, then the third is also a* B-*space.*

Proof. If either (X_0, X) or (X, X_1) is a pair of B-spaces, then the assertion is easily obtained. Therefore it suffices to deal with the case when (X_0, X_1) is pair of B-spaces.

With no loss of generality we may assume that $X_0 \subset X$ and $X_1 = X/X_0$. Let $\{s_n\}_{n \geqslant 1}$ be an increasing family of seminorms which defines the topology of X. Since X_0 is closed in X and the restriction of the topology of X to X_0 induces a B-space structure, there exists an index n_0 such that each seminorm $s_n^0 = s_n | X_0$ is a norm for $n \geqslant n_0$, and any two norms from the family $\{s_n^0\}_{n \geqslant n_0}$ are equivalent.

Similarly, since X_1 has a B-space structure, there exists an index n_1 such that each seminorm s_n^1 that is defined by s_n on X_1 (i.e. $s_n^1(x + X_0) = \inf\{s_n(x + y) : : y \in X_0\}$, $x \in X$) is a norm for $n \geqslant n_1$, and any two norms from the family $\{s_n\}_{n \geqslant n_1}$ are equivalent.

Let $n = \max\{n_0, n_1\}$ and let $s' = s_{n'}$, $s'' = s_{n''}$, where $n'' \geqslant n' \geqslant n$. Hence $s' \leqslant s''$. We shall show that there is a constant $C > 0$ such that $s'' \leqslant Cs'$. If this were not true, we could find a sequence $\{x_k\}_k \subset X$ such that $s'(x_k) \leqslant 1$ for all k and $s''(x_k) \to \infty$ $(k \to \infty)$. Let $C_1 > 0$ be a constant such that $s_{n''}^1 \leqslant C_1 s_{n'}^1$. Therefore

$$s_{n''}^1(x_k + X_0) \leqslant C_1 s_{n'}^1(x_k + X_0) \leqslant C_1 s'(x_k) \leqslant C_1.$$

The definition of $s_{n''}^1$ then shows that we can choose a sequence $\{y_k\}_k \subset X_0$ such that $s''(x_k + y_k) \leqslant C_1 + 1$.

Next, let $C_0 > 0$ be a constant such that $s_{n''}^0 \leqslant C_0 s_{n'}^0$. Then

$$s'(y_k) \leqslant s'(x_k + y_k) + s'(x_k) \leqslant s''(x_k + y_k) + 1 \leqslant C_1 + 2.$$

Therefore $s''(y_k) \leqslant C_0(C_1 + 2)$, so that

$$s''(x_k) \leqslant s''(x_k + y_k) + s''(y_k) \leqslant C_1 + 1 + C_0(C_1 + 2) < \infty,$$

which contradicts the choice of the sequence $\{x_k\}_k$. This shows that any two seminorms of the family $\{s_m\}_{m \geqslant n}$ are equivalent. Therefore they must be equivalent norms, inducing a B-space structure on X that is equivalent to the original structure.

Proof of Proposition 3.15. We have the exact complex of F-spaces

$$0 \to \{0\} \times X_2 \to G_0(v) \xrightarrow{\mathrm{pr}_1} X_1 \to 0.$$

Since both $\{0\} \times X_2$ and X_1 are B-spaces, it follows from Lemma 3.16 that $G_0(v)$ must be a B-space too.

We shall deal in the following with commuting multioperators acting in the qB-space X/Y. Let us remark that a multioperator on a qB-space is automatically regular, since every operator on a qB-space in regular, as follows from [18].

Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a commuting multioperator (briefly, a.c.m.). Then the *joint spectrum* of u can be defined as the set of those points $w \in \mathbf{C}^n$ such that the complex $K(X/Y, w - u)$ is not exact [19] (which adapts verbatim the corresponding definition from [13]). Let us denote, for the moment, by $\sigma_0(u, X/Y)$ the joint spectrum of u obtained in this way. We intend to prove the equality $\sigma_0(u, X/Y) = \sigma(u, X/Y)$ (which holds for Banach space operators [6], [16]).

3.17. PROPOSITION. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.m. *which is strict. Then we have the equality* $\sigma_0(u, X/Y) = \sigma(u, X/Y)$.

Proof. Let $u^0 = (u_1^0, \ldots, u_n^0) \subset \mathscr{L}(X)$ be a (not necessarily commuting) multioperator such that u_j^0 induces u_j $(j = 1, \ldots, n)$. Let $w \notin \sigma_0(u, X/Y)$. Then there exists an open neighbourhood W of w such that the complex $K(X/Y, z - u)$ is exact for each $z \in W$ (see [19], Proposition 6). By Proposition 2.20, we infer that the complex

$$\text{(3.5)} \qquad \begin{aligned} \ldots \to \Lambda^p[\sigma, X] \times \Lambda^{p+1}[\sigma, Y] &\xrightarrow{d^p(z-u^0)} \\ \xrightarrow{d^p(z-u^0)} \Lambda^{p+1}[\sigma, X] &\times \Lambda^{p+2}[\sigma, Y] \to \ldots \end{aligned}$$

is exact for every $z \in W$, where

$$d^p(z - u^0) = \begin{pmatrix} \delta^p(z - u^0) & -1_{p+1} \\ \delta^{p+1}(z - u^0)\delta^p(z - u^0) & -\delta^{p+1}(z - u^0)|\Lambda^{p+1}[\sigma, Y] \end{pmatrix},$$

$\delta^p(z - u^0)$ is defined as in the Introduction, 1_p is the identity on $\Lambda^p[\sigma, Y]$ $(p \geqslant 0)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a system of indeterminates.

With the terminology of [13], (3.5) is an analitically parametrized complex (of finite length) on W. By Theorem 2.2 from [13], it follows that the complex

$$\text{(3.6)} \qquad \begin{aligned} \ldots \to \Lambda^p[\sigma, \mathscr{O}(D, X)] \times \Lambda^{p+1}[\sigma, \mathscr{O}(D, Y)] &\xrightarrow{d^p} \\ \xrightarrow{d^p} \Lambda^{p+1}[\sigma, \mathscr{O}(D, X)] \times \Lambda^{p+2}[\sigma, \mathscr{O}(D, Y)] &\to \ldots \end{aligned}$$

is exact for each polydisc $D \subset W$, where d^p is the multiplication by the function $z \to d^p(z - u^0)$. The exactness of (3. 6) implies, again by Proposition 2.20, that the complex $K(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D,u))$ is exact (the multioperator $\mathcal{O}(D, u)$ is induced as a strict multioperator by u^0 in a natural way; see Lemma 2.11 from [17]). Since $D \subset W$ is arbitrary, we must have $w \notin \sigma(u, X/Y)$.

Conversely, let $w \notin \sigma(u, X/Y)$ and let $D \ni w$ be a polydisc such that the complex $K(\mathcal{O}(D, X/Y), \zeta - \mathcal{O}(D, u))$ is exact. Then the complex (3.6) is exact, by Proposition 2.20. To obtain the exactness of (3.5), we can use an inductive argument from the proof of Proposition III.9.1 in [16]. We leave the details to the reader.

3.18. REMARK. Definition 3.2 is invariant under similarities in the category q$\mathscr{F}$. In other words, with the notation of Definition 3.2, if X_1/Y_1 is another qF-space and if $\theta \in \mathscr{L}(X/Y, X_1/Y_1)$ is bijective, then $\sigma(v, X_1/Y_1) = \sigma(u, X/Y)$, where $v = (v_1, \dots, v_n)$ and $v_j = \theta u_j \theta^{-1}$ $(j = 1, \dots, n)$. Indeed, for every polydisc $D \subset \mathbf{C}^n$ we have

$$\mathcal{O}(D, v) = (\mathcal{O}(D, \theta)\mathcal{O}(D, u_1)\mathcal{O}(D, \theta)^{-1}, \dots, \mathcal{O}(D, \theta)\mathcal{O}(D, u_n)\mathcal{O}(D, \theta)^{-1}),$$

that is $\mathcal{O}(D, v)$ and $\mathcal{O}(D, u)$ are similar, whence we derive our assertion.

Analogously, the definition of the joint spectrum denoted here by $\sigma_0(u, X/Y)$ is invariant under similarities in the category q$\mathscr{B}$.

3.19. THEOREM. *Let X/Y be a quotient Banach space and let $u = (u_1, \dots, u_n) \subset \mathscr{L}(X/Y)$ be a* c.m. *Then we have the equality $\sigma_0(u, X/Y) = \sigma(u, X/Y)$.*

Proof. It is known that every qB-space is isomorphic in the category q$\mathscr{B}$ to a so-called standard qB-space. Moreover, every operator defined on a standard qB-space is strict (see [18] for details). Therefore every commuting multioperator is similar to a commuting multioperator which is strict. Our assertion then follows from Proposition 3.17 and Remark 3.18.

For $n = 1$ see also [23].

4. THE CAUCHY-WEIL-TAYLOR INTEGRAL

In this section we shall construct a certain linear mapping, which will be called the Cauchy-Weil-Taylor integral in quotient Fréchet spaces. First of all we shall present the algebraic framework, mainly due to J. L. Taylor [14], in a slightly modified form (see also [6], [11], [16] etc.).

Let A be a complex algebra with unit, let L_0, L be left A-modules and let $i_0\colon L_0 \to L$ be an injective A-module homomorphism. Then the elements of A can (and will) be regarded as endomorphisms on each of the spaces L, L_0 and $L/i_0(L_0)$ (as a matter of fact, L_0 and $i_0(L_0)$ will be sometimes identified).

4.1. DEFINITION. Let A, L_0, L and i_0 be as above. Let also $\lambda = (\lambda_1, \ldots \ldots, \lambda_q) \subset A$ be a commuting family. The triple (L, L_0, λ) will be called a *Cauchy--Weil-Taylor system* (briefly a CWT-*system)* if the family of endomorphisms induced by λ in $L/i_0(L_0)$ is nonsingular.

We shall fix in the following a commutative family of elements of A of the form (a, d), where $a = (a_1, \ldots, a_n)$ and $d = (d_1, \ldots, d_m)$. The families a and d will be associated with the families of indeterminates $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots \ldots, \tau_m)$, respectively.

Assume that $(L, L_0, (a, d))$ is a CWT-system. We shall define a linear mapping $R(a)^p : H^p(L, d) \to H^{n+p}(L_0, d)$ for each $p \geqslant 0$, which will play a central rôle in our constructions. This mapping will be obtained in three steps:

1) Let $s : \Lambda[\tau, L] \to \Lambda[(\sigma, \tau), L]$ be given by the equality $s\xi = \xi \wedge \sigma_1 \wedge \ldots \ldots \wedge \sigma_n$ ($\xi \in \Lambda[\tau, L]$). It is clear that s defines a chain map of degree n of the complex $K(L, d)$ into the complex $K(L, (a, d))$. In particular, s determines a linear mapping

$$s^p : H^p(L, d) \to H^{n+p}(L, (a, d)) \tag{4.1}$$

for each $p \geqslant 0$.

2) The exact complex of A-modules

$$0 \to L_0 \xrightarrow{i_0} L \xrightarrow{k_0} L/i_0(L_0) \to 0,$$

where k_0 is the canonical mapping, induces the exact homology complex

$$\ldots \to H^p(L_0, (a, d)) \xrightarrow{i_0^p} H^p(L, (a, d)) \xrightarrow{k_0^p} H^p(L/i_0(L_0), (a, d)) \to \ldots .$$

Since (a, d) is a CWT-system, we have $H^p(L/i_0(L_0), (a, d)) = \{0\}$ for all $p \geqslant 0$. Therefore there exists an isomorphism

$$i_0^p : H^p(L_0, (a, d)) \to H^p(L, (a, d)) \tag{4.2}$$

that is induced by i_0 for each $p \geqslant 0$.

3) Let π_σ be the projection of the space $\Lambda[(\sigma, \tau), L_0]$ onto the space $\Lambda[\tau, L_0]$. It is easily seen that π_σ induces a chain map of degree zero from the complex $K(L_0, (a, d))$ into the complex $K(L_0, d)$. In particular, there exists a linear mapping

$$\pi_\sigma^p : H^p(L_0, (a, d)) \to H(L_0, d) \tag{4.3}$$

induced by π_σ for every $p \geqslant 0$.

We can now define the desired mapping by the formula

$$R(a)^p = (-1)^n \pi_\sigma^{n+p} (i_0^{n+p})^{-1} s^p, \tag{4.4}$$

for each integer $p \geqslant 0$.

4.2. REMARKS. 1° Let us mention that $(L, L_0, (a, d))$ (with $L_0 \subset L$) is called in [14] a *Cauchy-Weil system* if a is nonsingular on L/L_0. From this property of a one can derive the nonsingularity of the family (a, d) on L/L_0 (via Lemma 1.3 from [13]), and so the construction of (4.4) can be performed. It is easily seen that the general properties of the mapping (4.4) (see [14], Section 1) are the same if we use either the concept of Cauchy-Weil system from [14] or that slightly more general of CWT-system (more adequate to our methods) given by Definition 4.1. For this reason, we shall refer to [14] when discussing the properties of the mapping (4.4); their proof is practically the same (see also [6] and [11]).

2° Let V be a differentiable manifold and let X be an F-space. We shall denote by $\mathscr{E}_0(V, X)$ the set of those functions from $\mathscr{E}(V, X)$ that have compact support. Clearly, $\mathscr{E}_0(V, X)$ is an LF-space. If $Y \in \mathrm{Lat}(X)$, then we have $\mathscr{E}_0(V, Y) \subset \subset \mathscr{E}_0(V, X)$ and the inclusion is continuous. Therefore we may define the qLF-space

$$\varepsilon_0(V, X/Y) = \varepsilon_0(V, X)/\varepsilon_0(V, Y). \tag{4.5}$$

Since $\mathscr{E}_0(V, X) \cap \mathscr{E}(V, Y) = \mathscr{E}_0(V, Y)$, it follows that $\varepsilon_0(V, X/Y)$ is a qLF-subspace of the qF-space $\varepsilon(V, X/Y)$ (see Remark 2.4.4°).

Now, let X_1/Y_1, X_2/Y_2 be qF-spaces and let $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$. Since $\varepsilon(V, u)$ is an $\mathscr{E}(V)$-module homomorphism (Remark 2.16.4°), it is easily seen that

$$\varepsilon(V, u)((\mathscr{E}_0(V, X_1) + \mathscr{E}(V, Y_1))/\mathscr{E}(V, Y_1)) \subset (\mathscr{E}_0(V, X_2) + \mathscr{E}(V, Y_2))/\mathscr{E}(V, Y_2).$$

Therefore there exists a linear mapping

$$\varepsilon_0(V, u) : \varepsilon_0(V, X_1/Y_1) \to \varepsilon_0(V, X_2/Y_2) \tag{4.6}$$

that is induced by $\varepsilon(V, u)$ (Remark 2.4.4°). In particular, if $X_1/Y_1 = X/Y = X_2/Y_2$, then $\varepsilon_0(V, X/Y)$ is invariant under $\varepsilon(V, u)$.

3° Let V as above and let $L \subset V$ be a closed set. We denote by $\mathscr{E}(L)$ the space of those functions from $\mathscr{E}(V)$ whose support is contained in L. The space $\mathscr{E}(L)$ is closed in $\mathscr{E}(V)$, and hence it is nuclear [8]. Let $\varepsilon(L, \cdot)$ be the functor φ_N given by Theorem 2.14 for $N = \mathscr{E}(L)$. If X/Y is an arbitrary qF-space, since $\mathscr{E}(L, X) \cap \cap \mathscr{E}(V, Y) = \mathscr{E}(L, Y)$, it follows that $\varepsilon(L, X/Y)$ is a qF-subspace of $\varepsilon(V, X/Y)$ (th meaning of $\mathscr{E}(L, X)$ is obvious). If L is compact, then there also exists a canonical embedding of $\varepsilon(L, X/Y)$ into $\varepsilon_0(V, X/Y)$.

4° Let $U \subset \mathbf{C}^{n+m}$ be an open set. A closed subset $L \subset U$ is said to be $\mathbf{C}^n$-*compact* in U if the subset $L \cap (\mathbf{C}^n \times K)$ is compact for every compact subset $K \subset V$, where V is the projection of U on the last m coordinates ([14], Definition 3.2).

If X is an F-space, we denote by $\mathscr{E}_1(U, X)$ the family of all functions from $\mathscr{E}(U, X)$ whose support is $\mathbf{C}^n$-compact in U. Clearly, $\mathscr{E}_1(U, X)$ has a natural structure of LF-space. Moreover, if $Y \in \mathrm{Lat}(X)$, then $\mathscr{E}_1(U, Y) \subset \mathscr{E}_1(U, X)$ and the inclusion is continuous. Therefore we may define the qLF-space $\varepsilon_1(U, X/Y)$ as the quotient $\mathscr{E}_1(U, X)/\mathscr{E}_1(U, Y)$, by analogy with (4.5). Furthermore, if X_1/Y_1, X_2/Y_2 are qF-spaces, there exists a linear mapping $\varepsilon_1(U, u)$ from $\varepsilon_1(U, X_1/Y_1)$ into $\varepsilon_1(U, X_2/Y_2)$, which is defined by analogy with (4.6), for each $u \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$. The operator $\varepsilon_1(U, u)$ is the operator induced by $\varepsilon(U, u)$ (Remark 2.4.4°).

4.3. DEFINITON. Let X/Y be a qF-space, let $U \subset \mathbf{C}^{n+m}$ be an open set and let $\varepsilon_U(X/Y)$ be the sheaf defined in Lemma 3.7. Let also $\alpha = (\alpha^1, \ldots, \alpha^n)$ be a family of sheaf morphisms of $\varepsilon_U(X/Y)$ into itself. We say that α is an *admissible system of sheaf morphisms of* $\varepsilon_U(X/Y)$ if the following conditions are fulfilled:

(1) If α^p_W is the mapping induced by α^p on $\varepsilon(W, X/Y)$, then α^p_W is an $\mathscr{E}(W)$--module homomorphism and $(\alpha^1_W, \ldots, \alpha^n_W, \partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_{n+m})$ is a commuting multioperator on $\varepsilon(W, X/Y)$ for all open $W \subset U$ and $p = 1, \ldots, n$, where $(z_1, \ldots, z_{n+m})$ is the variable in $\mathbf{C}^{n+m}$.

(2) For every $\mathbf{C}^n$-compact subset $L \subset U$ the qF-space $\varepsilon(L, X/Y)$ is invariant under $\alpha^1_U, \ldots, \alpha^n_U$ (Remark 2.4.4°).

For an admissible system of sheaf morphisms $\alpha = (\alpha^1, \ldots, \alpha^n)$ of $\varepsilon_U(X/Y)$ we shall denote by $\sigma_U(\alpha)$ the complement in U of the union of those open sets $W \subset U$ such that if $\alpha_G = (\alpha^1_G, \ldots, \alpha^n_G)$, then

$$H^p(\varepsilon(G, X/Y), (\alpha_G, \bar{\nabla})) = \{0\}, \quad p \geqslant 0, \tag{4.7}$$

for every open subset $G \subset W$, where $\bar{\nabla} = (\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_{n+m})$.

4.4. REMARK. Since $\varepsilon_U(X/Y)$ is a fine sheaf (Lemma 3.7), equation (4.7) holds for each open set $G \subset U \setminus \sigma_U(\alpha)$, in virtue of [10], Corollary VI.A.4 (see also the proof of Proposition 3.8(2) above).

4.5. LEMMA. *Let $\zeta = (\zeta_1, \ldots, \zeta_{n+m})$ be the system of coordinate functions in $\mathbf{C}^{n+m}$, let $\alpha = (\alpha^1, \ldots, \alpha^n)$ be as in Definition* 4.3 *and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of indeterminates associated with α. Let also L, M be closed neighbourhoods of $\sigma_U(\alpha)$ in U such that L is in the interior of M and let $W = U \setminus L$. Then for every $\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(U, X/Y)]$ such that $(\delta(\alpha_W) + \bar{\partial})(\eta|W) = 0$ there exists $\eta_0 \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(M, X/Y)]$ with the property that $\eta - i_{M,U}(\eta_0) \in \mathrm{Im}(\delta(\alpha_U) + \bar{\partial})$, where $i_{M,U}$ is induced by the canonical embedding $\varepsilon(M, X/Y) \to \varepsilon(U, X/Y)$.*

Proof. We choose a function $\theta \in \mathscr{E}(U)$ such that $\theta = 0$ in a neighbourhood of L and $\theta = 1$ in a neighbourhood of $\overline{U \setminus M}$. Since $(\delta(\alpha_W) + \bar{\partial})(\eta|W) = 0$, the equation $\eta|W = (\delta(\alpha_W) + \bar{\partial})\xi$ has a solution $\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(W, X/Y)]$, by Remark 4.4. Then the form $\theta\xi$, extended with zero on L, has the property that $\eta - (\delta(\alpha_U) + \bar{\partial})\theta\xi$ is null in a neighbourhood of $\overline{U \setminus M}$, by the choice of θ and ξ. It is then clear that there exists a form $\eta_0 \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(M, X/Y)]$ such that $i_{M,U}(\eta_0) = \eta - (\delta(\alpha_U) + \bar{\partial})\theta\xi$.

4.6. Lemma. *Let α be as in Definition* 4.3. *If the set $\sigma_U(\alpha)$ is $\mathbf{C}^n$-compact in U, then $(\varepsilon(U, X/Y), \varepsilon_1(U, X/Y), (\alpha_U, \bar{\nabla}))$ is a* CWT-*system.*

Proof. We have to show that the multioperator induced by $(\alpha_U, \bar{\nabla})$ in $\varepsilon(U, X/Y)/i_U^1(\varepsilon_1(U, X/Y))$ is nonsingular, where i_U^1 is the canonical embedding $\varepsilon_1(U, X/Y) \to \varepsilon(U, X/Y)$. This is equivalent to saying that if $\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(U, X/Y)]$ is such that

$$(\delta(\alpha_U) + \bar{\partial})\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_U^1(\varepsilon_1(U, X/Y))],$$

then there exists $\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(U, X/Y)]$ with the property

$$\eta - (\delta(\alpha_U) + \bar{\partial})\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_U^1(\varepsilon_1(U, X/Y))].$$

But the choice of η shows that $(\delta(\alpha_W) + \bar{\partial})(\eta|W) = 0$, where $W = U \setminus L_1$ and $L_1 \subset U$ is $\mathbf{C}^n$-compact. If L_2 is a $\mathbf{C}^n$-compact neighbourhood of $\sigma_U(\alpha)$ and $L = L_1 \cup L_2$, then, by Lemma 4.5, there exists $\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon(U, X/Y)]$ such that

$$\eta - (\delta(\alpha_U) + \bar{\partial})\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_{M,U}(\varepsilon(M, X/Y))],$$

where M is a $\mathbf{C}^n$-compact neighbourhood of L, from which we derive easily our assertion.

4.7. Remarks. 1° Lemma 4.6 shows, in particular, that if $U \subset \mathbf{C}^n$ and $\sigma_U(\alpha)$ is compact, then $(\varepsilon(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \bar{\nabla}))$ is a CWT-system.

2° We need later a different version of Lemma 4.6. Namely, if $\sigma_U(\alpha) \subset \mathbf{C}^n \times K$, where $K \subset \mathbf{C}^m$ is compact, then $(\varepsilon_1(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \bar{\nabla}))$ is a CWT-system. The proof of this assertion follows along the lines of the proof of Lemma 4.6. Nevertheless, one needs a finer version of Lemma 4.5. Specifically, if η from the statement of this lemma satisfies

$$\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_{S,U}(\varepsilon(S, X/Y))],$$

then we can choose a form

$$\xi_0 \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_{T,U}(\varepsilon(T, X/Y))]$$

such that $\eta - i_{M,U}(\eta_0) = (\delta(\alpha_U) + \bar{\partial})\xi_0$, where S, T are arbitrary closed subsets of U, with S contained in the interior of T, and $i_{M,U}$, $i_{S,U}$ are the canonical embeddings in $\varepsilon(U, X/Y)$.

Indeed, since

$$\eta | W \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_{S,W}(\varepsilon(W, X/Y))],$$

it follows that we can choose a form

$$\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_{T,W}(\varepsilon(W, X/Y))]$$

such that $\eta | W = (\delta(\alpha_W) + \bar{\partial})\xi$, by applying two times the argument from the proof of Lemma 4.5 (as in the proof of Corollary III.8.2 from [16]). If θ is chosen as in the proof of Lemma 4.5, then the form $\xi_0 = \theta\xi$ satisfies our requirement.

Now, if $\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon_1(U, X/Y)]$ is such that

$$(\delta(\alpha_U) + \bar{\partial})\eta \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_U^0(\varepsilon_0(U, X/Y))],$$

by the above remark there exists a form $\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), \varepsilon_1(U, X/Y)]$ with the property

$$\eta - (\delta(\alpha_U) + \bar{\partial})\xi \in \Lambda[(\sigma, \mathrm{d}\bar{\zeta}), i_U^0(\varepsilon_0(U, X/Y))],$$

showing that $(\varepsilon_1(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \bar{\nabla}))$ is a CWT-system.

We shall give in the following some details concerning the "integration" in certain quotient topological vector spaces introduced above. First of all we shall make our notation more specific.

Let $U \subset \mathbf{C}^{n+m}$ be an open set and let $V = \pi_m(U)$, where $\pi_m \colon \mathbf{C}^{n+m} \to \mathbf{C}^m$ is the projection on the last m coordinates. We denote by (z, w) the variable in $\mathbf{C}^{n+m}$, where $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $w = (w_1, \ldots, w_m) \in \mathbf{C}^m$. Let also $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\omega = (\omega_1, \ldots, \omega_m)$ be the coordinate functions in $\mathbf{C}^n$ and $\mathbf{C}^m$, respectively. We set $\bar{\partial}' = (\partial/\partial\bar{z}_1)\mathrm{d}\bar{\zeta}_1 + \ldots + (\partial/\partial\bar{z}_n)\mathrm{d}\bar{\zeta}_n$, $\bar{\partial}'' = (\partial/\partial\bar{w}_1)\mathrm{d}\bar{\omega}_1 + \ldots + (\partial/\partial\bar{w}_m)\mathrm{d}\bar{\omega}_m$ and $\bar{\partial} = \bar{\partial}' + \bar{\partial}''$. We also use the operators $\partial' = (\partial/\partial z_1)\mathrm{d}\zeta_1 + \ldots + (\partial/\partial z_n)\mathrm{d}\zeta_n$, $\partial'' = (\partial/\partial w_1)\mathrm{d}\omega_1 + \ldots + (\partial/\partial w_m)\mathrm{d}\omega_m$, $\partial = \partial' + \partial''$, $\mathrm{d}' = \partial' + \bar{\partial}'$, $\mathrm{d}'' = \partial'' + \bar{\partial}''$, $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$, as well as the systems of differentials $\mathrm{d}\zeta = (\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_n)$, $\mathrm{d}\bar{\zeta} = (\mathrm{d}\bar{\zeta}_1, \ldots, \mathrm{d}\bar{\zeta}_n)$ and $\mathrm{d}\bar{\omega} = (\mathrm{d}\bar{\omega}_1, \ldots, \mathrm{d}\bar{\omega}_m)$, which are regarded as systems of indeterminates.

We now define some linear mappings. The first of them is the mapping $\eta \to \eta \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n$ of the space $\Lambda[(\mathrm{d}\bar{\zeta}, \mathrm{d}\bar{\omega}), \varepsilon_1(U, X/Y)]$ into $\Lambda[(\mathrm{d}\zeta, \mathrm{d}\bar{\zeta}, \mathrm{d}\bar{\omega}), \varepsilon_1(U, X/Y)]$, which obviously induces a chain map of degree n of $K(\varepsilon_1(X/Y), \bar{\partial})$ into $K(\varepsilon_1(X/Y), \partial' + \bar{\partial})$.

The next mapping is defined from the space $\Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon_1(X/Y)]$ into $\Lambda[d\bar{\omega}, \varepsilon(V, X/Y)]$ in the following way. We write the form $\eta \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon_1(U, X/Y)]$ as $\eta = \psi d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \ldots \wedge d\bar{\zeta}_n \wedge d\zeta_n + \chi$, where χ contains only terms of degree less than $2n$ in $d\zeta_1, \ldots, d\zeta_n$, $d\bar{\zeta}_1, \ldots, d\bar{\zeta}_n$. Then $\psi = g + \Lambda[d\bar{\omega}, \mathscr{E}_1(U, Y)]$, with $g \in \Lambda[d\bar{\omega}, \mathscr{E}_1(U, X)]$, and we may define

$$(4.8) \qquad \int \eta = \int g\, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \ldots \wedge d\bar{\zeta}_n \wedge d\zeta_n + \Lambda[d\bar{\omega}, \mathscr{E}(V, Y)],$$

where $d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \ldots \wedge d\bar{\zeta}_n \wedge d\zeta_n$ is regarded as $(2i)^n$ times the Lebesgue measure in $\mathbf{C}^n$. Since $g \in \Lambda[d\bar{\omega}, \mathscr{E}_1(U, X/Y)]$, the integral from the right hand side of (4.8) makes sense for each $w \in V$ and yields an element of $\Lambda[d\bar{\omega}, \mathscr{E}(V, Y)]$. Moreover, if $g \in \Lambda[d\bar{\omega}, \mathscr{E}_1(U, Y)]$, then the integral of g is an element of $\Lambda[d\bar{\omega}, \mathscr{E}(V, Y)]$, so that the right hand side of (4.8) depends only on the coset ψ, and therefore only on η. Let us observe that (4.8) induces a chain map of degree $-2n$ from $K(\varepsilon_1(U, X/Y), \partial' + \bar{\partial})$ into $K(\varepsilon(V, X/Y), \bar{\partial}'')$. Indeed, we have

$$\int (d' + \bar{\partial}'')\eta = \int \bar{\partial}''\psi d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \ldots \wedge d\bar{\zeta}_n \wedge d\zeta_n +$$

$$+ \int d'\chi + \int \bar{\partial}''\chi = \bar{\partial}'' \int \eta,$$

since $\int d'\chi = 0$ by Stokes's formula (note that the coefficients of χ are classes of functions with support $\mathbf{C}^n$-compact).

The composite of the above chain maps provides a mapping

$$(4.9) \qquad H^{n+p}(\varepsilon_1(U, X/Y), \bar{\partial}) \ni \xi \to \int \xi\, d\zeta_1 \wedge \ldots \wedge d\zeta_n \in H^p(\varepsilon(V, X/Y), \bar{\partial}'')$$

for every $p \geqslant 0$.

4.8. Remarks. 1° If we replace the space $\varepsilon_1(U, X/Y)$ by the space $\varepsilon_0(U, X/Y$ and perform the previous construction, we obtain a similar mapping

$$(4.10) \quad H^{n+p}(\varepsilon_0(U, X/Y), \bar{\partial}) \ni \xi \to \int \xi d\zeta_1 \wedge \ldots \wedge d\zeta_n \in H^p(\varepsilon_0(V, X/Y), \bar{\partial}'').$$

Indeed, the corresponding modification of (4.8) shows that $\int \eta$ is an element of $\Lambda[d\bar{\omega}, \varepsilon_0(V, X/Y)]$ if $\eta \in \Lambda[d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon_0(U, X/Y)]$.

In addition, it is easily seen that the diagram

$$
(4.11)\qquad \begin{array}{ccc}
H^{n+p}(\varepsilon_0(U, X/Y), \bar{\partial}) & \to & H^{n+p}(\varepsilon_1(U, X/Y), \bar{\partial}) \\
\downarrow{\scriptstyle \rho^{n+p}} & & \downarrow{\scriptstyle \rho_0^{n+p}} \\
H^p(\varepsilon_0(V, X/Y), \bar{\partial}'') & \to & H^p(\varepsilon(V, X/Y), \bar{\partial}'')
\end{array}
$$

is commutative, where the horizontal mappings are induced by the canonical embeddings, ρ^{n+p} is the mapping (4.9) and ρ_0^{n+p} is (4.10).

2° Let $L \subset U$ be a $\mathbf{C}^n$-compact subset. Then we have a mapping

$$
(4.12)\qquad H^{n+p}(\varepsilon(L, X/Y), \bar{\partial}) \ni \xi \to \int \xi \, \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n \in H^p(\varepsilon(V, X/Y), \bar{\partial}'')
$$

such that the following diagram

$$
(4.13)\qquad \begin{array}{ccc}
H^{n+p}(\varepsilon(L, X/Y), \bar{\partial}) & \to & H^{n+p}(\varepsilon_1(U, X/Y), \bar{\partial}) \\
 & \searrow{\scriptstyle \rho_L^{n+p}} & \downarrow{\scriptstyle \rho^{n+p}} \\
 & & H^p(\varepsilon(V, X/Y), \bar{\partial}'')
\end{array}
$$

is commutative, where the horizontal mapping is induced by the canonical embedding $\varepsilon(L, X/Y) \to \varepsilon_1(U, X/Y)$ and ρ_L^{n+p} is the mapping (4.12). Indeed, the construction of (4.12) can be obtained in the same way as that of (4.9).

Unlike (4.9) and (4.10), the mapping (4.12) is a morphism in the category $\mathrm{q}\mathscr{F}$ (it is easily seen that (4.12) is the strict operator induced by the usual integration).

4.9. DEFINITION. Let $U \subset \mathbf{C}^{n+m}$ be open, let $V = \pi_m(U)$, let X/Y be a qF-space and let $\alpha = (\alpha^1, \ldots, \alpha^n)$ be an admissible system of sheaf morphisms of $\varepsilon_U(X/Y)$ such that the set $\sigma_U(\alpha)$ is $\mathbf{C}^n$-compact in U. Then there exists a mapping

$$
R(\alpha_U)^p : H^p(\varepsilon(U, X/Y), \bar{\partial}) \to H^{n+p}(\varepsilon_1(U, X/Y), \bar{\partial})
$$

for each $p \geqslant 0$, given by (4.4) and Lemma 4.6. Then the element

$$
\int R(\alpha_U)^p \psi \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n \in H^p(\varepsilon(V, X/Y), \bar{\partial}''),
$$

which is obtained via (4.9), will be called the *Cauchy-Weil-Taylor integral* (briefly, the CWT-*integral)* of $\psi \in H^p(\varepsilon(U, X/Y), \bar{\partial})$.

In particular, the CWT-integral maps the space $\mathscr{O}(U, X/Y)$, identified with $H^0(\varepsilon(U, X/Y), \bar\partial)$ (see Lemma 3.3), into the space $\mathscr{O}(V, X/Y)$, identified with $H_0(\varepsilon(V, X/Y), \bar\partial'')$.

All the previous considerations remain valid for $m = 0$, when we take $\mathbf{C}^m = \{0\}$, $V = \{0\}$ and $\varepsilon(V, X/Y) = X/Y$. In this case the CWT-integral takes values in X/Y for $p = 0$ and is null for $p > 0$.

4.10. Proposition. *The* CWT-*integral is a linear operator.*

Proof. We shall show that the CWT-integral can be obtained as a superposition of morphisms in the category $q\mathscr{F}$, so that it is itself a morphism in this category. Since the space $\varepsilon_1(U, X/Y)$ is a qLF-space but not a qF-space, the corresponding mapping (4.2)

$$i_U^{p,1} : H^p(\varepsilon_1(U, X/Y), \delta(\alpha_U) + \bar\partial) \to H^p(\varepsilon(U, X/Y), \delta(\alpha_U) + \bar\partial)$$

is not a morphism in the category $q\mathscr{F}$. It is therefore necessary to express the CWT integral in a different way.

Let L, M be $\mathbf{C}^n$-compact neighbourhood of $\sigma_U(\alpha)$ in U such that L is contained in the interior of M. Let δ_U^p be the restriction of $\delta(\alpha_U) + \bar\partial$ to the space $\Lambda^p[(\sigma, d\zeta, d\bar\omega), \varepsilon(U, X/Y)]$ for a fixed $p \geqslant 0$. Analogously, let δ_L^p (resp. $\delta_U^{p,1}$) be the corresponding operator that is obtained when $\varepsilon(U, \cdot)$ is replaced by $\varepsilon(L, \cdot)$ (resp. by $\varepsilon_1(U, \cdot)$). If $Z_U^p = \mathrm{Ker}(\delta_U^p)$, $Z_L^p = \mathrm{Ker}(\delta_L^p)$ and $Z_U^{p,1} = \mathrm{Ker}(\delta_U^{p,1})$, then we have the commutative diagram

$$\begin{array}{ccc} Z_U^{p,1} & \xrightarrow{\ i\ } & Z_U^p \\ {\scriptstyle i_1}\nwarrow & & \nearrow{\scriptstyle i_0} \\ & Z_L^p & \end{array}$$

in which the mappings are induced by the canonical embeddings. Let also $B_U^p = \mathrm{Im}(\delta_U^{p-1})$. Then the composite of i_0 and the canonical mapping

$$Z_U^p \to Z_U^p/B_U^p = H^p(\varepsilon(U, X/Y), \delta(\alpha_U) + \bar\partial)$$

yields a surjective mapping j_0. Indeed, by Lemma 4.5, for every $\eta \in Z_U^p$ there exists $\eta_0 \in Z_L^p$ such that $\eta - i_0(\eta_0) \in B_U^p$. Let $N_L^p = \mathrm{Ker}(j_0)$. Then j_0 induces an isomorphism, say $j_{L,U}^p$, from Z_L^p/N_L^p onto Z_U^p/B_U^p.

Similarly, if $B_U^{p,1} = \mathrm{Im}(\delta_U^{p-1,1})$, then the composite of i_1 and the canonical mapping $Z_U^{p,1} \to Z_U^{p,1}/B_U^{p,1}$ yields, as above, a surjective mapping j_1. Since $Z_U^{p,1}/B_U^{p,1}$ and Z_U^p/B_U^p are isomorphic via the corresponding mapping (4.2) (designated above by $i_U^{p,1}$), we must have the equality $\mathrm{Ker}(j_1) = N_L^p$. Therefore, we have an isomorphism

$j^{p,1}_{L,U}$ from Z^p_L/N^p_L onto $Z^{p,1}_U/B^{p,1}_U$, which is induced by j_1, such that the diagram

$$\begin{array}{ccc} H^p(\varepsilon_1(U,X/Y),\delta(\alpha_U)+\bar\partial) & \xrightarrow{i^{p,1}_U} & H^p(\varepsilon(U,X/Y),\delta(\alpha_U)+\bar\partial) \\ & {}_{j^{p,1}_{L,U}}\nwarrow \quad \nearrow_{j^p_{L,U}} & \\ & Z^p_L/N^p_L & \end{array} \tag{4.14}$$

is commutative. Moreover, Z^p_L/N^p_L is an object in the category $\mathrm{q}\mathscr{F}$, since j_0 is a morphism in this category (see Remark 2.4.3°).

We now show that there exists a natural mapping from Z^p_L/N^p_L into $H^p(\varepsilon(M,X/Y),\bar\partial)$. Indeed, let π_σ be the projection of the space $\Lambda[(\sigma,\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(L,X/Y)]$ onto $\Lambda[(\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(L,X/Y)]$ (see also equation (4.3)) and let

$$\pi_{\sigma,M}\colon Z^p_L\to\Lambda[(\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(M,X/Y)]$$

be the composite of the restriction of π_σ to Z^p_L and the map $i_{L,M}$ induced by the canonical embedding $\varepsilon(L,X/Y)\to\varepsilon(M,X/Y)$. Since π_σ intertwines $\delta(\alpha_U)+\bar\partial$ and $\bar\partial$, we must have

$$\pi_{\sigma,M}(Z^p_L)\subset\mathrm{Ker}(\bar\partial|\Lambda^p[(\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(M,X/Y)]).$$

We also have

$$\pi_{\sigma,M}(N^p_L)\subset\mathrm{Im}(\bar\partial|\Lambda^{p-1}[(\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(M,X/Y)]). \tag{4.15}$$

Indeed, if $\eta\in N^p_L$, then $i_0(\eta)=\delta^{p-1}_U(\xi_1)$ for some $\xi_1\in\Lambda^{p-1}[(\sigma,\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(U,X/Y)]$. Moreover, $\delta^{p-1}_U\xi_1|(U\setminus L)=0$. Hence, by Lemma 4.5, we can find $\xi\in\Lambda^{p-1}[(\sigma,\mathrm{d}\bar\zeta,\mathrm{d}\bar\omega),\varepsilon(M,X/Y)]$ such that $\xi_1-i_{M,U}(\xi)\in\mathrm{Im}(\delta^{p-1}_U)$, where $i_{M,U}$ is induced by the canonical embedding $\varepsilon(M,X/Y)\to\varepsilon(U,X/Y)$. Therefore $i_0(\eta)=$ $=\delta^{p-1}_U(\xi_1)=\delta^{p-1}_U i_{M,U}(\xi)$, so that $i_{L,M}(\eta)=\delta^{p-1}_M\xi$. From this equality we obtain the inclusion (4.15). Consequently there is a mapping

$$\pi^p_{\sigma,M}\colon Z^p_L/N^p_L\to H^p(\varepsilon(M,X/Y),\bar\partial) \tag{4.16}$$

induced by $\pi_{\sigma,M}$, which is a morphism in the category $\mathrm{q}\mathscr{F}$. Moreover, the diagram

$$\begin{array}{ccc} H^p(\varepsilon_1(U,X/Y),\delta(\alpha_U)+\bar\partial) & \xrightarrow{\pi^p_\sigma} & H^p(\varepsilon_1(U,X/Y),\bar\partial) \\ {\scriptstyle j^{p,1}_{L,U}}\uparrow & & \uparrow{\scriptstyle i^{p,1}_{M,U}} \\ Z^p_L/N^p_L & \xrightarrow{\pi^p_{\sigma,M}} & H^p(\varepsilon(M,X/Y),\bar\partial) \end{array} \tag{4.17}$$

is commutative, where π_σ^p is given by (4.3), $j_{L,U}^{p,1}$ occurs in (4.14) and $i_{M,U}^{p,1}$ is induced by the embedding $\varepsilon(M, X/Y) \to \varepsilon_1(U, X/Y)$.

The commutativity of the diagrams (4.13), (4.14) and (4.17) shows that

$$\rho^{n+p}\pi_\sigma^{n+p}(i_U^{n+p,1})^{-1}s^p = \rho_M^{n+p}\pi_{\sigma,M}^{n+p}(j_{L,U}^{n+p})^{-1}s^p, \tag{4.18}$$

and the right hand side of (4.18) is a morphism in the category $q\mathscr{F}$, that is the CWT--integral is a morphism in the category $q\mathscr{F}$.

4.11. Remark. Let $U \subset \mathbf{C}^{n+m}$ be open and let X/Y be a qF-space. Let also $v \in \mathscr{L}(X/Y)$. As we have mentioned in the previous section, the operator $\varepsilon(U, v)$ commutes with every differential operator $\partial/\partial\bar{z}_j$, $j = 1, \ldots, n$ or $\partial/\partial\bar{w}_k$, $k = 1, \ldots, m$. This leads to the conclusion that $\varepsilon(U, v)$ induces an operator $\varepsilon^p(U, v)$, acting in $H^p(\varepsilon(U, X/Y), \bar{\partial})$ for each $p \geqslant 0$ (see Remark 2.4.5°). Similarly, if $V = \pi_m(U)$, then $\varepsilon(V, v)$ induces an operator $\varepsilon^p(V, v)$ in $H^p(\varepsilon(V, X/Y), \bar{\partial}'')$ for all $p \geqslant 0$.

Let also note that the family $\{\varepsilon(W, v) : W \subset U \text{ open}\}$ defines a sheaf morphism of the sheaf $\varepsilon_U(X/Y)$. Let $\varepsilon(v)$ denote this sheaf morphism. Then we have the following.

4.12. Proposition. *Assume that $(\alpha^1, \ldots, \alpha^n, \varepsilon(v))$ is an admissible system of sheaf morphism of $\varepsilon_U(X/Y)$. Then we have the equality*

$$\int R(\alpha_U)^p \varepsilon^p(U, v)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \varepsilon^p(V, v)\int R(\alpha_U)^p \psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for all $\psi \in H^p(\varepsilon(U, X/Y), \bar{\partial})$ and $p \geqslant 0$, provided $\sigma_U(\alpha)$ is $\mathbf{C}^n$-compact in U, with $\alpha = (\alpha^1, \ldots, \alpha^n)$.

Proof. If we denote by $\tilde{\varepsilon}(U, v)$ the action induced by $\varepsilon(U, v)$ in $\Lambda[(\sigma, d\bar{\zeta}, d\bar{\omega}), \varepsilon(U, X/Y)]$, it follows from the hypothesis that

$$\tilde{\varepsilon}(U, v)(\delta(\alpha_U) + \bar{\partial}) = (\delta(\alpha_U) + \bar{\partial})\tilde{\varepsilon}(U, v).$$

Therefore $\tilde{\varepsilon}(U, v)$ induces an operator $\tilde{\varepsilon}^p(U, v)$ acting in $H^p(\varepsilon(U, X/Y), \delta(\alpha_U) + \bar{\partial})$ (by Remark 2.4.5°). It is clear that $s^p\varepsilon^p(U, v) = \tilde{\varepsilon}^p(U, v)s^p$, where s^p is the corresponding mapping (4.1). One can easily see that $\varepsilon(L, v)$ (resp. $\varepsilon(M, v)$) induces in Z_L^p/N_N^p (resp. in $H^p(\varepsilon(M, X/Y), \bar{\partial})$) a mapping $\tilde{\varepsilon}^p(L, v)$ (resp. $\varepsilon^p(M, v)$) such that $\tilde{\varepsilon}^p(U, v)j_{L,U}^p = j_{L,U}^p\tilde{\varepsilon}^p(L, v)$ and $\pi_{\sigma,M}^p\tilde{\varepsilon}^p(L, v) = \varepsilon^p(M, v)\,\pi_{\sigma,M}^p$ (for notation see the proof of Proposition 4.10). The only fact which remains to be proved is the equality

$$\rho_M^{n+p}\varepsilon^{n+p}(M, v) = \varepsilon^p(V, v)\,\rho_M^{n+p}. \tag{4.19}$$

Equation (4.19) will be derived from the formula

$$(4.20)\qquad \int \varepsilon(M, v)\eta = \varepsilon(V, v)\int \eta$$

where $\eta \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon(M, X/Y)]$ and $\int \eta$ is defined by (4.8), where one replaces $\varepsilon_1(U, \cdot)$ by $\varepsilon(M, \cdot)$, and its value is also in $\Lambda[d\bar{\omega}, \varepsilon(V, X/Y)]$. Notice that if $M_0 \supset M$ is $\mathbf{C}^n$-compact in U, then

$$(4.21)\qquad \int i_{M,M_0}(\eta) = \int \eta$$

for all $\eta \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon(M, X/Y)]$, where i_{M,M_0} is induced by the embedding $\varepsilon(M, X/Y) \to \varepsilon(M_0, X/Y)$.

First we obtain a version of (4.20). Namely, if $M' \times M'' \subset U$ is compact, where $M' \subset \mathbf{C}^n$ and $M'' \subset \mathbf{C}^m$, then

$$(4.22)\qquad \int^0 \varepsilon(M' \times M'', v)\eta = \varepsilon(M'', v)\int^0 \eta$$

for every $\eta \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon(M' \times M'', X/Y)]$, where $\int^0 \eta$ is defined to be an element of $\Lambda[d\bar{\omega}, \varepsilon(M'', X/Y)]$, which is possible in this case by an obvious modification of (4.8) (see also Remark 4.8.1°). We have in fact

$$(4.23)\qquad \int \eta = i_{M'',V}\int^0 \eta,$$

with $i_{M'',V}$ induced by the embedding $\varepsilon(M'', X/Y) \to \varepsilon(V, X/Y)$. Let us also observe that in (4.22) (as well as in (4.20)), the operators constructed from v via ε are, as a matter of fact, direct products of copies of the corresponding operators, for which we keep the same notation.

Let us prove (4.22). Since $G_0(\varepsilon(M' \times M'', v)) = \mathscr{E}(M') \hat{\otimes} G_0(\varepsilon(M'', V))$, if $(\psi_1, \psi_2) \in G_0(\varepsilon(M' \times M'', v))$, then $\left(\int \psi_1, \int \psi_2\right) \in G_0(\varepsilon(M'', v))$, from which we derive easily that (4.22) holds (as in Lemma 4.3 from [17]).

We can now obtain (4.20). Let $\{f_j\}_{j\in J} \subset \mathscr{E}_0(V)$ and $\{g_k\}_{k\in K} \subset \mathscr{E}_0(U)$ be partitions of unitary such that $\operatorname{supp}(f_j g_k) \subset M'_{jk} \times M''_{jk}$ which are compact subsets of U so that $M'_{jk} \subset \mathbf{C}^n$ and $M''_{jk} \subset \mathbf{C}^m$. Let also $M_{jk} = M \cup (M'_{jk} \times M''_{jk})$. We fix a form $\eta \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon(M, X/Y)]$. Since M is $\mathbf{C}^n$-compact, the family $\{f_j g_k\}_{k \in K}$ contains only a finite number of non-null forms for each index j. Moreover, for every pair (j, k) there exists a form

$$\eta_{jk} \in \Lambda[(d\zeta, d\bar{\zeta}, d\bar{\omega}), \varepsilon(M'_{jk} \times M''_{jk}, X/Y)]$$

such that $i_{M,M_{jk}}(f_j g_k \eta) = i_{M'_{jk} \times M''_{jk}, M_{jk}}(\eta_{jk})$. Then, by (4.21) and (4.23) we have:

$$\int \varepsilon(M, v)\eta = \sum_{j \in J} f_j \int \varepsilon(M, v)\eta = \sum_{j \in J} \sum_{k \in K} \int \varepsilon(M, v) f_j g_k \eta =$$

$$= \sum_{j \in J} \sum_{k \in K} \int i_{M'_{jk} \times M''_{jk}, M_{jk}} (\varepsilon(M'_{jk} \times M''_{jk}, v)\eta_{jk}) =$$

$$= \sum_{j \in J} \sum_{k \in K} i_{M''_{jk}, V} \int^0 \varepsilon(M'_{jk} \times M''_{jk}, v)\eta_{jk} =$$

$$= \sum_{j \in J} \sum_{k \in K} i_{M''_{jk}, V} \varepsilon(M''_{jk}, v) \int^0 \eta_{jk} =$$

$$= \sum_{j \in J} \sum_{k \in K} \varepsilon(V, v) \int i_{M, M_{jk}}(f_j g_k \eta) = \sum_{j \in J} \varepsilon(V, v) \int f_j \eta = \varepsilon(V, v) \int \eta,$$

where we have used the equalities

$$i_{M, M_{jk}} \varepsilon(M, v) = \varepsilon(M_{jk}, v) i_{M, M_{jk}},$$

$$\varepsilon(M_{jk}, v) i_{M'_{jk} \times M''_{jk}, M_{jk}} = i_{M'_{jk} \times M''_{jk}, M_{jk}} \varepsilon(M'_{jk} \times M''_{jk}, v)$$

and

$$i_{M''_{jk}, V} \varepsilon(M''_{jk}, v) = \varepsilon(V, v) i_{M''_{jk}, V},$$

which follow from Proposition 2.15. Since $\varepsilon(M, v)$, $\varepsilon(V, v)$ and the integral are chain maps, from (4.20) we infer (4.19), which completes the proof of the proposition.

4.13. REMARKS. 1° Let $U \subset \mathbf{C}^{n+m}$ be open, let X_1/Y_1, X_2/Y_2 be qF-spaces and let $\alpha = (\alpha^1, \ldots, \alpha^n)$ and $\beta = (\beta^1, \ldots, \beta^n)$ be admissible systems of sheaf morphisms of $\varepsilon_U(X_1/Y_1)$ and $\varepsilon_U(X_2/Y_2)$, respectively. Let $v \in \mathscr{L}(X_1/Y_1, X_2/Y_2)$ be such that $\varepsilon(W, v)\alpha^j_W = \beta^j_W \varepsilon(W, v)$ for all open $W \subset U$ and $j = 1, \ldots, n$. Then

$$\varepsilon(W, v)(\delta(\alpha_W) + \bar{c}) = (\delta(\beta_W) + \bar{c})\varepsilon(W, v)$$

and $\varepsilon(U, v)$ induces an operator

$$\varepsilon^p(U, v) : H^p(\varepsilon(U, X_1/Y_1), \bar{\partial}) \to H^p(\varepsilon(U, X_2/Y_2), \bar{\partial})$$

for each $p \geqslant 0$. If, in addition, both sets $\sigma_U(\alpha)$ and $\sigma_U(\beta)$ are $\mathbf{C}^n$-compact in U, then we have the equality

$$\int R(\beta_U)^p \varepsilon^p(U, v)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \varepsilon^p(V, v) \int R(\alpha_U)^p \psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for all $\psi \in H^p(\varepsilon(U, X_1/Y_1), \bar{\partial})$ and $p \geqslant 0$. The proof of this assertion is practically the same as the proof of Proposition 4.12. We leave the details to the reader.

2° The above result holds for $m = 0$ too. In this case it can be stated as follows:

$$\int R(\beta_U)^0 \mathcal{O}(U, v)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = v \int R(\alpha_U)^0 \psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for every $\psi \in \mathcal{O}(U, X_1/Y_1)$ (see the comment after Definition 4.9).

4.14. PROPOSITION. *Let U', U'' be open sets in $\mathbf{C}^{n+m}$, let $U = U' \cup U''$ and let $\alpha = (\alpha^1, \ldots, \alpha^n)$ be an admissible system of sheaf morphisms of $\varepsilon_U(X/Y)$. If the set $\sigma_U(\alpha)$ is $\mathbf{C}^n$-compact in U and $\sigma_U(\alpha) \subset U' \cap U''$, then*

$$\int R(\alpha_{U'})^p(\psi|U') \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \int R(\alpha_{U''})^p(\psi|U'') \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for every $\psi \in H^p(\varepsilon(U, X/Y), \bar{\partial})$ and $p \geqslant 0$.

Proof. With no loss of generality we may assume that $U' \subset U'' = U$, and therefore $\sigma_U(\alpha) \subset U'$. Then $\sigma_{U'}(\alpha) \subset \sigma_U(\alpha)$ and $\sigma_{U'}(\alpha)$ is $\mathbf{C}^n$-compact in U'. By Lemma 4.6, $(\varepsilon(U, X/Y), \varepsilon_1(U, X/Y), (\alpha_U, \bar{\nabla}))$ and $(\varepsilon(U', X/Y), \varepsilon_1(U', X/Y), (\alpha_{U'}, \bar{\nabla}))$ are CWT-systems. If we use the embedding $\varepsilon_1(U', X/Y) \to \varepsilon(U, X/Y)$, the proof of Lemma 4.6 (with minor modifications) shows that $(\varepsilon(U, X/Y), \varepsilon_1(U', X/Y), (\alpha_U, \bar{\nabla})$ is a CWT-system. If $r : \varepsilon(U, X/Y) \to \varepsilon(U', X/Y)$ is the restriction and i designates various canonical embeddings, then we have the commutative diagram

$$\begin{array}{ccc} \varepsilon(U, X/Y) & \xleftarrow{\ i\ } & \varepsilon_1(U, X/Y) \\ {\scriptstyle r}\downarrow & \nwarrow^{i} & \uparrow{\scriptstyle i} \\ \varepsilon(U', X/Y) & \xleftarrow{\ i\ } & \varepsilon_1(U', X/Y). \end{array}$$

By Proposition 1.10 from [14] we infer that the diagram

$$(4.24)\qquad \begin{array}{ccc} H^p(\varepsilon(U, X/Y), \bar{\partial}) & \xrightarrow{R(\alpha_U)^p} & H^{n+p}(\varepsilon_1(U, X/Y), \bar{\partial}) \\ {\scriptstyle r^p}\big\downarrow & \searrow^{R(\alpha_U)^p} & \big\uparrow{\scriptstyle i^{n+p}} \\ H^p(\varepsilon(U', X/Y), \bar{\partial}) & \xrightarrow{R(\alpha_{U'})^p} & H^{n+p}(\varepsilon_1(U', X/Y), \bar{\partial}) \end{array}$$

is commutative, where r^p (resp. i^{n+p}) is induced by r (resp. by i). Since the composite of i^{n+p} and the mapping (4.9) equals the mapping (4.9) when defined from $H^{n+p}(\varepsilon_1(U', X/Y), \bar{\partial})$ (as in Remark 4.2.8°), our assertion is a straightforward conse quence of (4.24).

4.15. Remark. Proposition 4.14 shows a certain independence of the CWT-integral with respect to the set U containing the "joint spectrum" of α (see also [14], Proposition 3.11). From now on the CWT-integral will be often denoted by

$$(4.25)\qquad \int R(\alpha)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for arbitrary $\psi \in H^p(\varepsilon(U, X/Y), \bar{\partial})$ and $p \geq 0$.

4.16. Definition. Let $U \subset \mathbf{C}^{n+m}$ be open, let $V = \pi_m(U)$ and let γ be a morphism of the sheaf $\varepsilon_V(X/Y)$. We say that γ has a *proper extension* to the sheaf $\varepsilon_U(X/Y)$ if there exists a morphism $\hat{\gamma}$ of $\varepsilon_U(X/Y)$ such that $\hat{\gamma}_{W' \times W''} = \varepsilon(W', \gamma_{W''})$ for every open $W' \times W'' \subset U$ with $W' \subset \mathbf{C}^n$ and $W'' \subset \mathbf{C}^m$ (here we identify the spaces $\varepsilon(W' \times W'', X/Y)$ and $\varepsilon(W', \varepsilon(W'', X/Y))$). More generally, if $\beta = (\beta^1, \ldots, \beta^m)$ is a family of morphisms of $\varepsilon_V(X/Y)$, we say that β has a *proper extension* to the sheaf $\varepsilon_U(X/Y)$ if there exists a family $\hat{\beta} = (\hat{\beta}^1, \ldots, \hat{\beta}^m)$ of morphisms of $\varepsilon_U(X/Y)$ such that $\hat{\beta}^q$ is a proper extension of β^q for all $q = 1, \ldots, m$.

The next result is a Fubini type theorem for the CWT-integral (see also [14], Theorem 3.6).

4.17. Theorem. *Let $U \subset \mathbf{C}^{n+m}$ be an open set, let V the projection of U on the last m coordinates, let X/Y be a* qF-*space and let $\alpha = (\alpha^1, \ldots, \alpha^n)$ (*resp. *$\beta = (\beta^1, \ldots, \beta^m)$) be an admissible system of morphisms of $\varepsilon_U(X/Y)$ (*resp. *$\varepsilon_V(X/Y)$) such that the set $\sigma_U(\alpha)$ (*resp. *$\sigma_V(\beta)$) is $\mathbf{C}^n$-compact in U (*resp. *compact in V). Assume that β has a proper extension $\hat{\beta}$ to the sheaf $\varepsilon_U(X/Y)$ such that $(\alpha, \hat{\beta})$ is an admissible system of morphisms of $\varepsilon_U(X/Y)$. Then the set $\sigma_U(\alpha, \hat{\beta})$ is compact* *and we have*

$$\int R(\beta)\left(\int R(\alpha)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n\right) \wedge d\omega_1 \wedge \ldots \wedge d\omega_m =$$

$$= \int R(\alpha, \hat{\beta})\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\omega_1 \wedge \ldots \wedge d\omega_m$$

for every $\psi \in \mathcal{E}(U, X/Y)$.

Proof. We first note that $\sigma_U(\hat{\beta}) \subset \mathbf{C}^n \times \sigma_V(\beta)$. Indeed, if $W'' \subset V$ is an open set such that

$$H^p(\varepsilon(W'', X/Y), \delta(\beta_{W''}) + \bar{\partial}'') = \{0\}, \quad p \geqslant 0,$$

and $W' \times W'' \subset U$ for some $W' \subset \mathbf{C}^n$, W' open, then, by Proposition 2.19 above and Lemma 1.3 from [13] we have:

$$H^p(\varepsilon(W', \varepsilon(W'', X/Y)), \varepsilon(W', \delta(\beta_{W''}) + \bar{\partial}'')) =$$

$$= H^p(\varepsilon(W' \times W'', X/Y), \delta(\hat{\beta}_{W' \times W''}) + \bar{\partial}'') =$$

$$= H^p(\varepsilon(W' \times W'', X/Y), \delta(\hat{\beta}_{W' \times W''}) + \bar{\partial}) = \{0\}$$

for all $p \geqslant 0$. From this observations we obtain readily the inclusion $\sigma_U(\hat{\beta}) \subset \mathbf{C}^n \times \sigma_V(\beta)$. Using again Lemma 1.3 from [13] (or Lemma I.2.3 from [16]) we obtain the inclusion

$$\sigma_U(\alpha, \hat{\beta}) \subset \sigma_U(\alpha) \cap (\mathbf{C}^n \times \sigma_V(\beta)). \tag{4.26}$$

As the right hand side of (4.26) is compact, it follows that the set $\sigma_U(\alpha, \hat{\beta})$ is compact. In particular, $(\varepsilon(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \hat{\beta}_U, \bar{\nabla}))$ is a CWT-system, by Remark 4.7.1°. Note also that $(\varepsilon_1(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \hat{\beta}_U, \bar{\nabla}))$ is a CWT-system, by Remark 4.7.2°. Therefore $(\varepsilon(U, X/Y), \varepsilon_0(U, X/Y), (\alpha_U, \hat{\beta}_U, \bar{\nabla}))$ is a CWT-system and $R(\alpha_U, \hat{\beta}_U)^0 = R(\hat{\beta}_U)^n R(\alpha_U)^0$, by virtue of Theorem 1.13 from [14]. For this reason we have the equality

$$\begin{aligned} &\int R(\alpha, \hat{\beta})\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\omega_1 \wedge \ldots \wedge d\omega_m = \\ &= \int\left(\int R(\hat{\beta})R(\alpha)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n\right) \wedge d\omega_1 \wedge \ldots \wedge d\omega_m \end{aligned} \tag{4.27}$$

for all $\psi \in \mathcal{E}(U, X/Y)$ (see Remark 4.15).

Next we obtain the equality

$$\int \delta(\hat{\beta}_U)\xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \tag{4.28}$$

$$= \delta(\beta_V)\int \xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for all $\xi \in \Lambda[(\sigma, \tau, d\bar{\zeta}, d\bar{\omega}), \varepsilon_1(U, X/Y)]$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots, \tau_m)$ are systems of indeterminates associated with α and β (or $\hat{\beta}$), respectively.

Equality (4.28) follows in the same way as (4.20). We only note that if $W' \times W'' \subset U$ is an open set with $W' \subset \mathbf{C}^n$ and $W'' \subset \mathbf{C}^m$, then $\delta(\hat{\beta}_{W' \times W''}) = \varepsilon(W', \delta(\beta_{W''}))$, since $\hat{\beta}$ is a proper extension of β (Definition 4.16). This makes available the argument from the proof of (4.20). We leave the routine details to the reader.

We can now proceed as in the proof of Theorem 3.6 from [14]. Namely, from (4.28) we infer that the diagram of complexes

$$\begin{array}{ccc}
K(\varepsilon_1(U, X/Y), \delta(\hat{\beta}_U) + \bar{\partial}) & \xleftarrow{\ i\ } & K(\varepsilon_0(U, X/Y), \delta(\hat{\beta}_U) + \bar{\partial}) \\
\downarrow \rho & & \downarrow \rho \\
K(\varepsilon(V, X/Y), \delta(\beta_V) + \bar{\partial}'') & \xleftarrow{\ i\ } & K(\varepsilon_0(V, X/Y), \delta(\beta_V) + \bar{\partial}'')
\end{array}$$

is commutative, where the chain maps i are induced by the canonical embeddings and the chain maps ρ are defined as in (4.8) (see also Remark 4.8.1°).

Since the diagrams

$$\begin{array}{ccc}
K(\varepsilon_1(U, X/Y), \bar{\partial}) & \xrightarrow{\ s\ } & K(\varepsilon_1(U, X/Y), \delta(\hat{\beta}_U) + \bar{\partial}) \\
\downarrow \rho & & \downarrow \rho \\
K(\varepsilon(V, X/Y), \bar{\partial}'') & \xrightarrow{\ s\ } & K(\varepsilon(V, X/Y), \delta(\beta_V) + \bar{\partial}'')
\end{array}$$

and

$$\begin{array}{ccc}
K(\varepsilon_0(U, X/Y), \delta(\hat{\beta}_U) + \bar{\partial}) & \xrightarrow{\ \pi_\sigma\ } & K(\varepsilon_0(U, X/Y), \bar{\partial}) \\
\downarrow \rho & & \downarrow \rho \\
K(\varepsilon_0(V, X/Y), \delta(\beta_V) + \bar{\partial}'') & \xrightarrow{\ \pi_\sigma\ } & K(\varepsilon_0(V, X/Y), \bar{\partial}'')
\end{array}$$

are obviusly commutative, where s and π_σ are defined as in (4.1) and, respectively, (4.3), we obtain the commutative diagram

$$
\begin{array}{ccc}
H^p(\varepsilon_1(U, X/Y), \bar{\partial}) & \xrightarrow{R(\hat{\beta}_U)^n} & H^{n+m}(\varepsilon_0(U, X/Y), \bar{\partial}) \\
\rho^n \downarrow & & \downarrow \rho_0^{n+m} \\
H^0(\varepsilon(V, X/Y), \bar{\partial}'') & \xrightarrow{R(\beta_V)^0} & H^m(\varepsilon_0(V, X/Y), \bar{\partial}'')
\end{array}
\tag{4.29}
$$

with ρ^n and ρ_0^{n+m} given by (4.9) and, respectively, (4.10). Finally, from (4.27) and (4.29) we infer easily the desired conclusion.

4.18. Proposition. *Let $U \subset \mathbf{C}^{n+m}$ be an open set and let $(\alpha^1, \ldots, \alpha^n, \gamma_{11}, \ldots, \ldots, \gamma_{1n}, \ldots, \gamma_{n1}, \ldots, \gamma_{nn})$ be an admissible system of morphisms of the sheaf $\varepsilon_U(X/Y)$. Let also $\beta^j = \gamma_{j1}\alpha^1 + \ldots + \gamma_{jn}\alpha^n$ $(i = 1, \ldots, n)$, $\alpha = (\alpha^1, \ldots, \alpha^n)$ and $\beta = (\beta^1, \ldots, \beta^n)$. If the sets $\sigma_U(\alpha)$ and $\sigma_U(\beta)$ are $\mathbf{C}^n$-compact in U, then*

$$\int R(\beta)(\det(\gamma_{jk})_{j,k=1}^n)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n =$$

$$= \int R(\alpha)\psi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

for all $\psi \in \mathcal{O}(U, X/Y)$ (where "det" *stands for determinant).*

Proof. This assertion follows easily from Definition 4.9 above and Proposition 1.11 from [14].

4.19. Lemma. *Let $U \subset \mathbf{C}$ be an open set and let α be a morphism of the sheaf $\varepsilon_U(X/Y)$ such that the singleton (α) is an admissible system. Assume that there exists a compact set $K \subset U$ such that $\alpha_{U \setminus K}$ is bijective. Then*

$$\int R(\alpha)\psi \wedge d\zeta = \int_\Gamma (\alpha_{U \setminus K})^{-1}(\psi | U \setminus K) d\zeta$$

for every $\psi \in \mathcal{O}(U, X/Y)$, where Γ is a finite system of Jordan curves that surrounds K.

Proof. We proceed as for the proof of Lemma 3.14 from [14]. Let $\psi \in \mathcal{O}(U, X/Y) = H^0(\varepsilon(U, X/Y), \bar{\partial})$ and let $\theta \in \mathcal{E}(U)$ be equal to zero in a compact neighbourhood of K and equal to one outside another compact neighbourhood of K. Then we define the element $\varphi \in \varepsilon(U, X/Y)$ by the formula $\varphi = (\alpha_{U \setminus K})^{-1}(\psi | U \setminus K)$ in $U \setminus K$ and $\varphi = 0$ on K (via a canonical embedding). Since $\partial/\partial\bar{z}$ and $(\alpha_{U \setminus K})^{-1}$ commute, it is easily seen that $\bar{\partial}\varphi \in \Lambda^1[d\bar{\zeta}, \varepsilon_0(U, X/Y)]$ and that $R(\alpha)\psi$ equals the homology class of $\bar{\partial}\varphi$ in $H^1(\varepsilon_0(U, X/Y), \bar{\partial})$. Therefore

$$\int R(\alpha)\psi \wedge d\zeta = \int \bar{\partial}\varphi \wedge d\zeta = \int d(\varphi d\zeta) = \int_\Gamma \varphi d\zeta =$$

$$= \int_\Gamma (\alpha_{U\setminus K})^{-1}(\psi | U\setminus K) d\zeta,$$

by Stokes's formula.

4.20. COROLLARY. *Let $U \subset \mathbf{C}$ be an open set and let $V \subset \bar{V} \subset U$ be an open set with $\bar{V}$ compact. If ζ and ω are the coordinate functions on U and V, respectively, then we have*

$$\psi | V = \frac{1}{2\pi i}\int R(\zeta - \omega)\psi \wedge d\zeta$$

for every $\psi \in \mathcal{O}(U, X/Y)$.

Proof. We consider the qF-space $\mathcal{O}(V, X/Y)$ and the operator (induced by) ω, acting on this space. Note that $\zeta - \omega$ induces an admissible system on the sheaf $\varepsilon_U(\mathcal{O}(V, X/Y))$ and that $\zeta - \omega$ is bijective on $\varepsilon(W, \mathcal{O}(V, X/Y))$, with $W = U\setminus\bar{V}$. By the previous lemma we have

$$\frac{1}{2\pi i}\int R(\zeta - \omega)\psi \wedge d\zeta = \frac{1}{2\pi i}\int_\Gamma (\zeta - \omega)^{-1}\psi d\zeta$$

for every $\psi \in \mathcal{O}(U, \mathcal{O}(V, X/Y))$, where Γ is a contour surrounding the set $\bar{V}$ in U. In particular, if $\psi \in \mathcal{O}(U, X/Y) \subset \mathcal{O}(U, \mathcal{O}(V, X/Y))$ and if $\psi = g + \mathcal{O}(U, Y)$, where $g \in \mathcal{O}(U, X)$, then

$$\int_\Gamma (\zeta - \omega)^{-1}\psi d\zeta = \int_\Gamma (\zeta - \omega)^{-1} g d\zeta + \mathcal{O}(V, Y) =$$

$$= 2\pi i g | V + \mathcal{O}(V, Y) = 2\pi i \psi | V.$$

4.21. LEMMA. *Let $U = U_1 \times \ldots \times U_n$ be an open set in $\mathbf{C}^n$ and let β^j be a morphism of the sheaf $\varepsilon_{U_j}(X/Y)$ that has a proper extension $\hat{\beta}^j$ $(j = 1, \ldots, n)$ to the sheaf $\varepsilon_U(X/Y)$ such that $\hat{\beta} = (\hat{\beta}^1, \ldots, \hat{\beta}^n)$ is an admissible system of morphisms of $\varepsilon(U, X/Y)$. Assume that for each index j there exists a compact set $K_j \subset U_j$ with the* *property that* $\beta^j_{W_j}$ *is bijective, with* $W_j = U_j \setminus K_j$. *Then we have the equality*

$$\int R(\hat{\beta})\psi \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n =$$

$$= \int_{\Gamma_1} \ldots \int_{\Gamma_n} \alpha^1 \ldots \alpha^n(\psi(W_1 \times \ldots \times W_n))\, \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n$$

for every $\psi \in \mathcal{E}(U, X/Y)$, *where* Γ_j *surrounds* K_j *in* U_j *and*

$$\alpha^j = \mathcal{E}(W_1 \times \ldots \times W_{j-1} \times W_{j+1} \times \ldots \times W_n, (\beta^j_{W_j})^{-1}).$$

Proof. The assertion follows from Theorem 4.17 and Lemma 4.19, by induction.

4.22. COROLLARY. *Let* $U = U_1 \times \ldots \times U_n \subset \mathbf{C}^n$ *be an open set, let* $V_j \subset \bar{V}_j \subset U_j$ *be an open set with* $\bar{V}_j$ *compact, let* ζ_j *(*resp. ω_j*) be the coordinate function on* U_j *(*resp. V_j*)* $(j = 1, \ldots, n)$ *and let* $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\omega = (\omega_1, \ldots, \omega_n)$. *Then we have*

$$\psi | V = \frac{1}{(2\pi \mathrm{i})^n} \int R(\zeta - \omega)\psi \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n$$

for all $\psi \in \mathcal{E}(U, X/Y)$, *where* $V = V \times \ldots \times V_n$.

Corollary 4.22 does not provide a strong enough assertion for our purpose. Nevertheless, by using it we shall prove a stronger version. We start with an auxiliary result.

4.23. LEMMA. *Let* $U \subset \mathbf{C}^n$ *be an open set and let* $\beta = (\beta^1, \ldots, \beta^n)$ *be an admissible system of morphisms of the sheaf* $\varepsilon_{U \times U}(X/Y)$. *Assume that*

$$(\varepsilon(U \times U, X/Y), \varepsilon_1(U \times U, X/Y), (\beta_{U \times U}, \bar{\nabla})),$$

$$(\varepsilon(U, \varepsilon(V, X/Y)), \varepsilon_0(U, \varepsilon(V, X/Y)), (\beta_{U \times V}, \bar{\nabla}))$$

are CWT-*systems for some open* $V \subset U$. *Then we have the equality*

$$\int R(\beta_{U \times V})(\psi U \times V) \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n = \left(\int R(\beta_{U \times U})\psi \wedge \mathrm{d}\zeta_1 \wedge \ldots \wedge \mathrm{d}\zeta_n\right) | V$$

for all $\psi \in \mathcal{E}(U \times U, X/Y)$.

Proof. The following diagrams of complexes are commutative:

$$
\begin{array}{ccccc}
K(\varepsilon(U\times U, X/Y), \bar{\partial}) & \xrightarrow{s} & K(\varepsilon(U\times U, X/Y), \delta(\beta_{U\times U})+\bar{\partial}) & \xleftarrow{i} & K(\varepsilon_1(U\times U, X/Y), \delta(\beta_{U\times U})+\bar{\partial}) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \uparrow{\scriptstyle i} \\
K(\varepsilon(U, \varepsilon(V, X/Y)), \bar{\partial}) & \xrightarrow{s} & K(\varepsilon(U, \varepsilon(V, X/Y)), \delta(\beta_{U\times V})+\bar{\partial}) & \xleftarrow{i} & K(\varepsilon_0(U, \varepsilon(V, X/Y)), \delta(\beta_{U\times V})+\bar{\partial}),
\end{array}
$$

$$
\begin{array}{ccccc}
K(\varepsilon_1(U\times U, X/Y), \delta(\beta_{U\times U}) + \bar{\partial}) & \xrightarrow{\pi_\sigma} & K(\varepsilon_1(U\times U, X/Y), \bar{\partial}) & \xrightarrow{\rho} & K(\varepsilon(U, X/Y), \bar{\partial}'') \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle r} \\
K(\varepsilon_0(U, \varepsilon(V, X/Y)), \delta(\beta_{U\times V}) + \bar{\partial}) & \xrightarrow{\pi_\sigma} & K(\varepsilon_0(U, \varepsilon(V, X/Y)), \bar{\partial}) & \xrightarrow{\rho} & K(\varepsilon(V, X/Y), \bar{\partial}'')
\end{array}
$$

where the mappings s and π_σ are given by (4.1) and, respectively, (4.3), ρ is the mapping (4.8) and i (resp. r) designates various canonical embeddings (resp. restrictions). When passing to homology spaces, the commutativity of the above diagrams implies readily the desired equality.

4.24. PROPOSITION. *Let $U \subset \mathbf{C}^n$ be open and let $(\zeta, \omega) = (\zeta_1, \ldots, \zeta_n, \omega_1, \ldots, \omega_n)$ be the coordinate functions in $U\times U$. Then we have the equality*

$$\psi = \frac{1}{(2\pi i)^n}\int R(\zeta - \omega)\psi \wedge d\zeta_1 \wedge \wedge \ldots \wedge d\zeta_n \tag{4.30}$$

for every $\psi \in \mathcal{E}(U, X/Y)$.

Proof. We check first that the hypothesis of Lemma 4.23 is fulfilled, when $V = D \subset \bar{D} \subset U$ is a polydisc and $\beta = \zeta - \omega$. Indeed, it is easily seen that

$$\sigma_{U\times U}(\beta) = \{(z, w) \in U\times U: z = w\} \tag{4.31}$$

and the right hand side of (4.31) is obviously $\mathbf{C}^n$-compat in $U \times U$. We also note that if $D \subset \bar{D} \subset U$ is a polydisc, then $\zeta - \omega \mid D$, when regarded as a sheaf morphism of the sheaf $\varepsilon_U(\varepsilon(D, X/Y))$, satisfies

$$\sigma_U(\zeta - \omega D) \subset \bar{D}, \tag{4.32}$$

and the right hand side of (4.32) is compact. The inclusions (4.31) and (4.32) show that

$$(\varepsilon(U\times U, X/Y), \varepsilon_1(U\times U, X/Y), (\zeta - \omega), \bar{\nabla})),$$

$$(\varepsilon(U, \varepsilon(D, X/Y)), \varepsilon_0(U, \varepsilon(D, X/Y)), (\zeta - \omega D, \bar{\nabla}))$$

are CWT-systems, by virtue of Lemma 4.6 (see also Remark 4.7.1°). Therefore, if φ denotes the element defined by the integral (4.30), then, by Lemma 4.23, we have

$$\varphi \mid D = \frac{1}{(2\pi i)^n} \int R(\zeta - \omega|D)(\psi|D) \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \psi \mid D,$$

where the last equality follows from Corollary 4.22. Since $\varphi \mid D = \psi \mid D$ for every polydisc $D \subset U$, we must have $\varphi = \psi$, and this completes the proof of the proposition.

5. ANALYTIC FUNCTIONAL CALCULUS

Let X/Y be a fixed (non-null) qF-space. In this section we shall construct an analytic functional calculus associated with every commuting regular multioperator from $\mathscr{L}(X/Y)$.

5.1. DEFINITION. Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a c.r.m. Then the set $\sigma(u, X/Y)$ is compact and nonempty in $\mathbf{C}^n$. Let $U \supset \sigma(u, X/Y)$ be an open set. If $\varepsilon(u) = (\varepsilon(u_1), \ldots, \varepsilon(u_n))$ is the family of sheaf morphisms induced by u on $\varepsilon_U(X/Y)$ (more precisely, induced by the family $\{(\varepsilon(W, u_1), \ldots, \varepsilon(W, u_n)) : W \subset U \text{ open}\}$), it follows from the previous results that $\zeta - \varepsilon(u) = (\zeta_1 - \varepsilon(u_1), \ldots, \zeta_n - \varepsilon(u_n))$ is an admissible system of sheaf morphisms of $\varepsilon_U(X/Y)$. Moreover, $\sigma_U(\zeta - \varepsilon(u)) = \sigma(u, X/Y)$, as a direct consequence of Theorem 3.8. Let $f \in \mathscr{O}(U)$ and let $\xi \in X/Y$. Since X/Y is a qF-subspace of $\mathscr{O}(U, X/Y)$, then $f\xi \in \mathscr{O}(U, X/Y)$ and we may define the integral

$$f(u)\xi = \frac{1}{(2\pi i)^n} \int R(\zeta - \varepsilon(U, u)) f\xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n. \tag{5.1}$$

Formula (5.1) provides an *analytic functional calculus* for the multioperator u, whose precise meaning is given by the following result (which is a version of Theorem 4.3 from [14]).

5.2. THEOREM. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* $U \supset \sigma(u, X/Y)$ *be an open set. Then* (5.1) *provides a unital algebra homomorphism of* $\mathscr{O}(U)$ *into the subalgebra of* $\mathscr{L}(X/Y)$ *consisting of all operators on* X/Y *which commute with every operator commuting with* $u_1, \ldots, u_n$. *Moreover,* $\zeta_j(u) = u_j$ *for all* $j = 1, \ldots, n$

Proof. Let $i_U: X/Y \to \mathscr{O}(U, X/Y)$ be the canonical embedding. Since (5.1) is $2\pi i)^{-n}$ times the composite of i_U and the CWT-integral, it follows from Proposition 4.10 that $f(u) \in \mathscr{L}(X/Y)$ for every $f \in \mathscr{O}(U)$.

As a matter of fact, $f(u)$ commutes with every operator $v \in \mathscr{L}(X/Y)$ that commutes with $u_1, \ldots, u_n$. We note first that

$$i_U v = \mathscr{O}(U, v) i_U \tag{5.2}$$

for every $v \in \mathscr{L}(X/Y)$, which follows from Proposition 2.15 (with $N_1 = \mathbf{C}$, $N_2 = \mathscr{O}(U)$ and θ the inclusion $\mathbf{C} \subset \mathscr{O}(U)$). If, in addition, v commutes with $u_1, \ldots, u_n$, then, by (5.2),

$$f(u)v\xi = \frac{1}{(2\pi i)^n}\int R(\zeta - \varepsilon(U, u))\varrho(U, v)(f\xi) \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n =$$

$$= \frac{1}{(2\pi i)^n} v \int R(\zeta - \varepsilon(U, u))f\xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = vf(u)\xi$$

for each $\xi \in X/Y$, as a consequence of Proposition 4.12 (see also Remark 4.13.2°)

Let P be a complex polynomial. If we want to compute $P(u)$ as given by (5.1). we may assume, with no loss of generality, that $U = \mathbf{C}^n$, in virtue of Proposition 4.14. Therefore

$$P(u) = \frac{1}{(2\pi i)^n}\int R(\zeta - \varepsilon(\mathbf{C}^n, u))P\xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n =$$

(5.3)

$$= \frac{1}{(2\pi i)^n}\int_{\Gamma_1} \ldots \int_{\Gamma_n} \alpha^1 \ldots \alpha^n P\xi \, d\zeta_1 \ldots d\zeta_n,$$

by Lemma 4.21, where

$$\alpha^j = \varrho(W_1 \times \ldots \times W_{j-1} \times W_{j+1} \times \ldots \times W_n, (\zeta_j - \varrho(W_j, u))^{-1}),$$

$W_j = \mathbf{C} \setminus \sigma(u_j, X/Y)$ $(j = 1, \ldots, n)$ and $\Gamma_1, \ldots, \Gamma_n$ are sufficiently large circles. Note that $\alpha^j = (\zeta_j - \varrho(W, u))^{-1}$, with $W = W_1 \times \ldots \times W_n$ (via a natural isomorphism). Therefore, if we take the monomial $P_n(z) = z_n^{k_n}$ $(k_n \geqslant 0)$, then

$$\frac{1}{(2\pi i)^n}\int_{\Gamma_1} \ldots \int_{\Gamma_n} \alpha^1 \ldots \alpha^n P_n \xi d\zeta_1 \ldots d\zeta_n =$$

$$= \frac{1}{(2\pi i)^n}\int_{\Gamma_1} \ldots \int_{\Gamma_{n-1}} \alpha^{(1,\ldots,n-1)} \int_{\Gamma_n} (\zeta_n - \varrho(W_n, u))^{-1} P_n \xi d\zeta_1 \ldots d\zeta_n =$$

$$= \frac{1}{(2\pi i)^n}\int_{\Gamma_1} \ldots \int_{\Gamma_{n-1}} \alpha^{(1,\ldots,n-1)} u_n^{k_n} d\zeta_1 \ldots d\zeta_{n-1},$$

where

$$\alpha^{(1,\ldots,n-1)} = (\zeta_1 - \varrho(W', u_1))^{-1} \ldots (\zeta_{n-1} - \varrho(W', u_{n-1}))^{-1}$$

and $W' = W_1 \times \ldots \times W_{n-1}$ (we have used here Lemma 4.3 and Theorem 4.11 from [17]).

This shows, by induction, that the right hand side of (5.3) is precisely the operator obtained from P when the variable z_j is replaced by u_j ($j = 1, \ldots, n$). In particular, the identity of $\mathcal{O}(U)$ is mapped on the identity of X/Y and $\zeta_j(u) = u_j$ for all j.

The mapping $f \to f(u)$ is obviously linear. Let us prove that it is multiplicative. If $f, g \in \mathcal{O}(U)$, we have:

$$f(u)g(u)\xi = \frac{1}{(2\pi i)^{2n}}\int R(\zeta - \varepsilon(U, u))f(\zeta)\left(\int R(\omega - \varepsilon(U, u))g(\omega)\xi \wedge d\bar{\omega}\right) \wedge d\bar{\xi} =$$

$$= \frac{1}{(2\pi i)^{2n}}\int R(\omega - \varepsilon(U \times U, u),\ \zeta - \varepsilon(U \times U, u))f(\zeta)g(\omega)\xi \wedge d\bar{\omega} \wedge d\bar{\zeta},$$

by Proposition 1.10 from [14] and Theorem 4.17 above (which applies, since the sheaf morphisms we work with have evidently proper extensions), where we have put, for simplicity, $d\bar{\omega} = d\omega_1 \wedge \ldots \wedge d\omega_n$ and $d\bar{\zeta} = d\zeta_1 \wedge \ldots \wedge d\zeta_n$.

As in the proof of Theorem 4.17 from [14], we tranform the multioperator $(\omega_1 - u_1, \ldots, \omega_n - u_n,\ \zeta_1 - u_1, \ldots, \zeta_n - u_n)$ by the matrix $(\gamma_{jk})_{j,k=1}^{2n}$, where $\gamma_{jk} = 1$ if $k = j$, $\gamma_{jk} = -1$ if $k = n + j$ and $\gamma_{jk} = 0$ otherwise. Since $\det(\gamma_{jk})_{j,k} = 1$, from Proposition 4.18 we infer that

$$\frac{1}{(2\pi i)^{2n}}\int R(\omega - \varepsilon(U \times U, u),\ \zeta - \varepsilon(U \times U, u))f(\zeta)g(\omega)\xi \wedge d\bar{\omega} \wedge d\bar{\zeta} =$$

$$= \frac{1}{(2\pi i)^{2n}}\int R(\omega - \zeta,\ \zeta - \varepsilon(U \times U, u))f(\zeta)g(\omega)\xi \wedge d\bar{\omega} \wedge d\bar{\zeta} =$$

$$= \frac{1}{(2\pi i)^{2n}}\int R(\zeta - \varepsilon(U, u))f(\zeta)\left(\int R(\omega - \zeta)g(\omega)\xi\, d\bar{\omega}\right) \wedge d\bar{\zeta} =$$

$$= \frac{1}{(2\pi i)^{n}}\int R(\zeta - \varepsilon(U, u))fg\,\xi \wedge d\bar{\zeta} = (fg)(u)\xi,$$

where we have used Proposition 4.24, which concludes the proof of the theorem.

5.3. Corollary. *Let $\mathcal{O}(\sigma(u, X/Y))$ denote the algebra of germs of analytic functions defined in neighbourhoods of the set $\sigma(u, X/Y)$. Then the mapping* (5.1) *can be defined as a unital algebra homomorphism of $\mathcal{O}(\sigma(u, X/Y))$ into $\mathcal{L}(X/Y)$.*

Proof. Indeed, by Proposition 4.14, the mapping (5.1) commutes with restrictions.

5.4. Proposition. *Let X_1/Y_1, X_2/Y_2 be* qF-*spaces and let $b \in \mathcal{L}(X_1/Y_1, X_2/Y_2)$. Let also $u = (u_1, \ldots, u_n) \subset \mathcal{L}(X_1/Y_1)$ and $v = (v_1, \ldots, v_n) \subset \mathcal{L}(X_2/Y_2)$ be* c.r.m. *such that $bu_j = v_jb$ ($j = 1, \ldots, n$). If f is analytic in a neighbourhood of the set $\sigma(u, X_1/Y_1) \cup \sigma(v, X_2/Y_2)$, then $bf(u) = f(v)b$.*

Proof. This is a consequence of Remarks 4.13.

We shall prove in the following that the spectral mapping theorem still holds in our context (see Theorem 5.11 below). To this end we need some auxiliary results.

5.5. LEMMA. *Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a* c.r.m. *and let V be an analytic manifold. Then*

i) $\sigma(\mathscr{O}(V, u), \mathscr{O}(V, X/Y)) = \sigma(u, X/Y)$

and

(ii) $f(\mathscr{O}(V, u)) = \mathscr{O}(V, f(u))$

for every f analytic in a neighbourhood of $\sigma(u, X/Y)$.

Proof. The complex $K(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u))$ is exact for some polydisc $D \subset \mathbf{C}^n$ if and only if the complex $K(\mathscr{O}(D, \mathscr{O}(V, X/Y)), \zeta - \mathscr{O}(D, \mathscr{O}(V,u)))$ is exact, by Proposition 2.19. From this remark we easily infer (i).

Let us deal with (ii). Let $U \supset \sigma(u, X/Y)$ be an open set on which the function f is defined. The mapping

$$X/Y \ni \xi \to f\xi \in \mathscr{O}(U, X/Y) \tag{5.4}$$

will be denoted by μ_f. Let also

$$s^0 : \mathscr{O}(U, X/Y) \to H^n(\mathscr{O}(U, X/Y), \delta(\zeta - \varepsilon(U, u)) + \bar{\partial}), \tag{5.5}$$

which is given by (4.1).

We shall use in the following some mappings which appeared in the proof of Proposition 4.10. Namely, let L, M be compact neighbourhoods of $\sigma(u, X/Y)$ in U such that L is contained in the interior of M. Then there exists an isomorphism

$$j^n_{L,U} : Z^n_L/N^n_L \to H^n(\varepsilon(U, X/Y), \delta(\zeta - \varepsilon(U, u)) + \bar{\partial}), \tag{5.6}$$

which occurs in (4.14). Let also

$$\pi^n_{\sigma,M} : Z^n_L/N^n_L \to H^n(\varepsilon(M, X/Y), \bar{\partial}), \tag{5.7}$$

which is given by (4.16), and let

$$\rho^n_M : H^n(\varepsilon(M, X/Y), \bar{\partial}) \to X/Y, \tag{5.8}$$

which is an "integration" defined by (4.12).

According to (4.18) and (5.1), we must have

$$f(u) = (2\pi i)^{-n}\rho_M^n \pi_{\sigma,M}^n (j_{L,U}^n)^{-1} s^0 \mu_f. \tag{5.9}$$

In order to prove (ii), we can apply the functor $\mathcal{O}(V, \cdot)$ to (5.9) and compare the result with the version of (5.9) when $f(u)$ is replaced by $f(\mathcal{O}(V, u))$. Using Lemmas 2.11, 2.17, Corollary 2.18 and Proposition 2.19, it is easily seen that the mappings $\mathcal{O}(V, \mu_f)$, $\mathcal{O}(V, s^0)$, $\mathcal{O}(V, j_{L,U}^n)$, $\mathcal{O}(V, \pi_{\sigma,M}^n))$ and $\mathcal{O}(V, \rho_M^n)$ can be respectively identified with the mappings (5.4)—(5.8) that correspond to $f(\mathcal{O}(V, u))$, which shows that (ii) holds.

5.6. REMARK. Let $u = (u_1, \ldots, u_n) \subset \mathcal{L}(X/Y)$ be a c.r.m., let $U \supset \sigma(u, X/Y)$ be an open set and let V be an analytic manifold. In virtue of preceding lemma, for every $h \in \mathcal{O}(U, \mathcal{O}(V, X/Y))$ we may define the integral

$$C(h) = \frac{1}{(2\pi i)^n}\int R(\zeta - \varepsilon(\mathcal{O}(V, u)))h \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n,$$

which is an element of $\mathcal{O}(V, X/Y)$. An argument used in the last part of the proof of Theorem 5.2 shows that

$$C(fh) = f(\mathcal{O}(V, u))C(h) = \mathcal{O}(V, f(u))C(h)$$

for each $f \in \mathcal{O}(U)$, that is, C is an $\mathcal{O}(U)$-module homomorphism (a similar assertion occurs in [16], Chapter III, equation (9.2)).

5.7. LEMMA. *Let $u = (u_1, \ldots, u_n) \subset \mathcal{L}(X/Y)$ be a* c.r.m. *and let f be analytic in a neighbourhood of $\sigma(u, X/Y)$. Then the operator $f(u)$ is regular.*

Proof. As a matter of fact we shall prove the inclusion $\sigma(f(u), X/Y) \subset \subset f(\sigma(u, X/Y))$. Let $w_0 \in \hat{\mathbf{C}} \setminus f(\sigma(u, X/Y))$. Then we can choose two open sets $W \subset \hat{\mathbf{C}}$ and $U \subset \mathbf{C}^n$ such that $w_0 \in W$, $\sigma(u, X/Y) \subset U$, f is defined in U and $W \cap f(U) = \emptyset$. Let $G' \subset W$ be an arbitrary open set and let ω be the coordinate function on $G = G' \cap \mathbf{C}$. The function $g(z, w) = (w - f(z))^{-1}$ is analytic on $U \times G'$ (we set $g(z, \infty) = 0$ if $\infty \in G'$). Similarly, $g_1(z, w) = wg(z, w)$ (with $g_1(z, \infty) = 1$) and $g_2(z, w) = -f(z)g(z, w)$ (with $g_2(z, \infty) = 0$) are analytic in $U \times G'$.

Let

$$\theta(g)h = \frac{1}{(2\pi i)^n}\int R(\zeta - \varepsilon(U, \mathcal{O}(G, u)))\, gh \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n, \tag{5.10}$$

where $h \in \mathcal{O}(G, X/Y)$ is arbitrary. Then $\theta(g)$ is an operator on $\mathcal{O}(G, X/Y)$. Moreover

$$(\omega - f(\mathcal{O}(G, u)))\theta(g)h = \theta(g)(\omega - f(\mathcal{O}(G, u)))h = h, \tag{5.11}$$

by Remark 5.6.

If $G' \ni \infty$ and $\theta'(g)$ is defined by (5.10) with G replaced by G', then $\theta'(g)$ is an operator on $\mathcal{O}(G', X/Y)$ whose restriction to $\mathcal{O}(G, X/Y)$ is precisely $\theta(g)$, by Remarks 4.13. Note that $\theta'(g)$ takes values in ${}_0\mathcal{O}(G', X/Y)$. Indeed, if $h \in \mathcal{O}(G', X/Y)$, then

$$h = \theta'(g_1)h + \theta'(g_2)h.$$

But

$$h|G = \omega\theta(g)(h|G) + \theta(g_2)(h|G),$$

which shows that $\theta(g)(h|G) = (\theta'(g)h)|G$ is the restriction of an element of ${}_0\mathcal{O}(G', X/Y)$. Therefore

$$(\omega - \mathcal{O}(G', f(u)))\theta'(g)h = h.$$

Since $\omega - \mathcal{O}(G', f(u))$ is injective by (5.11), it follows that

$$\omega - \mathcal{O}(G', f(u)) \colon {}_0\mathcal{O}(G', X/Y) \to \mathcal{O}(G', X/Y)$$

is bijective, and so $\sigma(f(u), X/Y) \subset f(\sigma(u, X/Y))$.

The next result is a version of Proposition 4.6 from [14].

5.8. PROPOSITION. *Let $(u_1, \ldots, u_n, v_1, \ldots, v_n) \subset \mathscr{L}(X/Y)$ be a* c.r.m. *and set $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$. If f is analytic in a neighbourhood of the set $\sigma(u, X/Y) \cup \sigma(v, X/Y)$, then $\mathcal{O}(G, f(u) - f(v))$ acts as the zero operator on $H^p(\mathcal{O}(G, X/Y), \mathcal{O}(G, u - v))$ for every open set $G \subset \mathbf{C}^n$ and each $p \geqslant 0$.*

Proof. Let $U \supset \sigma(u, X/Y) \cup \sigma(v, X/Y)$ be an open set in the domain of definition of f. Let also $\eta \in \Lambda^p[\sigma, \mathcal{O}(G, X/Y)]$ be such that $\delta(\mathcal{O}(G, u - v))\eta = 0$, where σ is a system of indeterminates associated with $u - v$. By Corollary 1.15 from [14], we infer that

$$(R(\zeta - \varepsilon(U, \mathcal{O}(G, u))) - R(\zeta - \varepsilon(U, \mathcal{O}(G, v))))f\eta =$$

$$= \delta(\varepsilon(U, \mathcal{O}(G, u - v)))\xi,$$

where $\xi \in \Lambda^{p-1}[\sigma, H^n(\varepsilon_0(U, \mathcal{O}(G, X/Y)))]$. Therefore

$$(\mathcal{O}(G, f(u)) - \mathcal{O}(G, f(v)))\eta =$$

$$= (2\pi i)^{-n}\int (R(\zeta - \varepsilon(U, \mathcal{O}(G, u))) - R(\zeta - \varepsilon(U, \mathcal{O}(G, v)))f\eta \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n =$$

$$= (2\pi i)^{-n}\delta(\mathcal{O}(G, u - v))\int \xi d\zeta_1 \wedge \ldots \wedge d\zeta_n,$$

which is the desired conclusion.

5.9. COROLLARY. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* $U \supset \supset \sigma(u, X/Y)$ *be open. Then for every* $f \in \mathcal{O}(U)$ *and each polydics* $D \subset \bar{D} \subset U$ *the operator* $\mathscr{O}(D, f(u))$ *acts as the operator* $f(\zeta)$ *on* $H^p(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u))$ *for all* $p \geqslant 0$.

Proof. The assertion is a direct consequence of the previous proposition. We only observe that $\sigma(\zeta, \mathscr{O}(D, X/Y)) \subset \bar{D} \subset U$ (as in (4.32)).

5.10. LEMMA. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m., *let* $U \supset \supset \sigma(u, X/Y)$ *be open and let* $f \in \mathcal{O}(U)$. *If* $0 \in \sigma(u, X/Y)$, *we have* $0 \notin \sigma((u, f(u)), X/Y)$ *if and only if* $f(0) \neq 0$.

Proof. Let $D \subset \mathbf{C}^{n+1}$ be a polydisc and let $(\zeta, \zeta_{n+1}) = (\zeta_1, \ldots, \zeta_n, \zeta_{n+1})$ be the coordinate functions in $\mathbf{C}^{n+1}$. We have an exact complex

$$
\begin{aligned}
&\ldots \to H^{p-1}(\mathscr{O}(D, X/Y), \alpha) \to H^p(\mathscr{O}(D, X/Y), \beta) \to \\
(5.12)\qquad &\to H^p(\mathscr{O}(D, X/Y), \alpha) \xrightarrow{\theta^p} H^p(\mathscr{O}(D, X/Y), \alpha) \to \ldots,
\end{aligned}
$$

where $\alpha = \zeta - \mathscr{O}(D, u)$, $\beta = (\zeta - \mathscr{O}(D, u), \zeta_{n+1} - \mathscr{O}(D, f(u)))$, and the connecting homomorphism θ^p is induced by $\zeta_{n+1} - \mathscr{O}(D, f(u))$ (see [13], Theorem 3.1 or [19], Proposition 1). By virtue of Corollary 5.8, the actions of the operators $\zeta_{n+1} - \mathscr{O}(D, f(u))$ and $\zeta_{n+1} - f(\zeta)$ are equal on the spaces $H^p(\mathscr{O}(D, X/Y), \alpha)$ for all $p \geqslant 0$.

If $f(0) \neq 0$, then we can choose the polydisc D such that $\zeta_{n+1} - f(\zeta) \neq 0$ on D. Therefore θ^p is an isomorphism for every p. Then the exactness of (5.12) shows that $H^p(\mathscr{O}(D, X/Y), \beta) = \{0\}$ for all $p \geqslant 0$, so that $0 \notin \sigma((u, f(u)), X/Y)$.

Conversely, assume that $0 \notin \sigma((u, f(u)), X/Y)$. Then the hypothesis $f(0) = 0$ leads to a contradiction. Indeed, if $f(0) = 0$, then we may assume that the polydisc D in (5.12) is of the form $D = G \times D'$, where $G \subset \mathbf{C}^n$, $D' \subset \mathbf{C}$ and $f(G) \subset D'$. Moreover, $H^p(\mathscr{O}(D, X/Y), \beta) = \{0\}$ and the mapping θ^p is an isomorphism for all $p \geqslant 0$. Let $\theta_1 \colon \mathcal{O}(G) \to \mathcal{O}(D)$ be given by $(\theta_1 h)(z, z_{n+1}) = h(z)$ $(h \in \mathcal{O}(G))$ and let $\theta_2 \colon \mathcal{O}(D) \to \mathcal{O}(G)$ be given by $(\theta_2 k)(z) = k(z, f(z))$ $(k \in \mathcal{O}(D))$. Then $\theta_2\theta_1$ is the identity on $\mathcal{O}(G)$. We denote also by θ_1 (resp. θ_2) the mapping induced by $\theta_1 \hat{\otimes} 1_X$ (resp. $\theta_2 \hat{\otimes} 1_X$) from $\Lambda[\sigma, \mathscr{O}(G, X/Y)]$ (resp. $\Lambda[\sigma, \mathscr{O}(D, X/Y)]$) into $\Lambda[\sigma, \mathscr{O}(D, X/Y)]$ (resp. $\Lambda[\sigma, \mathscr{O}(G, X/Y)]$), where σ is a system of indeterminates associated with $\zeta - \mathscr{O}(G, u)$ (or with $\zeta - \mathscr{O}(D, u)$).

Let $\eta \in \Lambda^p[\sigma, \mathscr{O}(G, X/Y)]$ be such that $\delta(\zeta - \mathscr{O}(G, u))\eta = 0$. Then $\delta(\zeta - \mathscr{O}(D, u))\theta_1\eta = 0$. Since θ^p acts as $\zeta_{n+1} - f(\zeta)$, we can find elements $\xi \in \Lambda^p[\sigma, \mathscr{O}(D, X/Y)]$ and $\lambda \in \Lambda^{p-1}[\sigma, \mathscr{O}(D, X/Y)]$ such that

$$\theta_1\eta = (\zeta_{n+1} - f(\zeta))\xi + \delta(\zeta - \mathscr{O}(D, u))\lambda.$$

Notice that $\theta_2((\zeta_{n+1} - f(\zeta))\xi) = 0$. Hence

$$\theta_2\theta_1\eta = \eta = \delta(\zeta - c(G, u))\theta_2\lambda.$$

This shows that $H^p(c(G, X/Y), \zeta - c(G, u)) = \{0\}$, from which we derive easily that $0 \notin \sigma(u, X/Y)$. This contradiction shows that $f(0) \neq 0$, and the proof of the lemma is completed.

We can now prove the desired version of the spectral mapping theorem (see also [14], Theorem 4.8 or [16], Theorem III.10.4).

5.11. THEOREM. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m., *let* $U \supset \supset \sigma(u, X/Y)$ *be an open set in* $\mathbf{C}^n$ *and let* $f = (f_1, \ldots, f_m) \in \mathcal{O}(U, \mathbf{C}^m)$. *If* $f(u) = (f_1(u), \ldots, f_m(u))$, *then* $\sigma(f(u), X/Y) = f(\sigma(u, X/Y))$.

Proof. In virtue of Theorem 3.11, it suffices to prove that for every $f \in \mathcal{O}(U, \mathbf{C}^m)$ one has

$$\sigma((u, f(u)), X/Y) = \{(z, w) \in \mathbf{C}^{n+m} : z \in \sigma(u, X/Y),\ w = f(z)\}.$$

This equality is a simple consequence of the following statement:

(P_m) *If* $z_0 \in \sigma(u, X/Y)$, *we have* $(z_0, 0) \notin \sigma((u, f(u)), X/Y)$ *if and only if* $f(z_0) \neq 0$.

Let us remark that (P_1) is a direct consequence of Lemma 5.10.

Assume that (P_m) is satisfied for some $m \geq 1$. The statement (P_{m+1}) is then implied by the following condition:

(P'_{m+1}) *If* $(z_0, 0) \in \sigma((u, f(u)), X/Y)$, *then* $(z_0, 0) \notin \sigma((u, f'(u)), X/Y)$ *if and only if* $f_{m+1}(z_0) \neq 0$, *where* $f_{m+1} \in \mathcal{O}(U)$ *is arbitrary and* $f' = (f, f_{m+1})$.

Indeed, if $(z_0, 0) \notin \sigma((u, f'(u)), X/Y)$ and $(z_0, 0) \in \sigma((u, f(u)), X/Y)$, then $f_{m+1}(z_0) \neq 0$ by (P'_{m+1}), and hence $f'(z_0) \neq 0$. If $(z_0, 0) \notin \sigma((u, f(u)), X/Y)$, then $f(z_0) \neq 0$ from (P_m), and thus $f'(z_0) \neq 0$.

Conversely, if $f'(z_0) \neq 0$, then either $f(z_0) \neq 0$ or $f_{m+1}(z_0) \neq 0$. In the first case, that $(z_0, 0) \notin \sigma((u, f'(u)), X/Y)$ follows from (P_m), via Lemma I.2.3 from [16], and in the second case the same thing follows from (P'_{m+1}).

Note that condition (P'_{m+1}) is a consequence of Lemma 5.10. Indeed, if $G(z, w) = f_{m+1}(z)$ $(z \in U, w \in \mathbf{C}^m)$, then $G(u, f(u)) = f_{m+1}(u)$, by Theorem 4.17, and therefore Lemma 5.10 applies. This completes the proof of the theorem.

5.12. REMARK. The above proof of Theorem 5.11 can be somehow simplified if one takes into consideration the inclusion $\sigma(f(u), X/Y) \subset f(\sigma(u, X/Y))$, already obtained in the proof of Lemma 5.7.

We end this work with an extension of J. L. Taylor's version of Shilov's idempotent theorem (see [14], Theorem 4.9 and [16], Theorem III.13.5). We prove first an auxiliary result.

5.13. LEMMA. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *and let* $j \in \mathscr{L}(X/Y)$ *be an idempotent that commutes with* $u_1, \ldots, u_n$. *Then* $j(X/Y)$ *is a* qF-*subspace of* X/Y *that is invariant under* $u_1, \ldots, u_n$. *Moreover,* $\sigma(u, j(X/Y)) \subset \sigma(u, X/Y)$.

Proof. It is clear that $j(X/Y)$ is a qF-subspace of X/Y that is invariant under $u_1, \ldots, u_n$.

To prove the spectral inclusion, let $D \subset \mathbf{C}^n$ be a polydicsc, let σ be a system of indeterminates associated with $\zeta - \mathscr{O}(D, u)$ (or with $\zeta - \mathscr{O}(D, u \mid j(X/Y))$), and let $\eta \in \Lambda^p[\sigma, \mathscr{O}(D, j(X/Y))]$ be such that $\delta(\zeta - \mathscr{O}(D, u \mid j(X/Y)))\eta = 0$. Since $j(X/Y)$ is a qF-subspace of X/Y, we may regard η as an element of $\Lambda^p[\sigma, \mathscr{O}(D, X/Y)]$. Moreover, $\delta(\zeta - \mathscr{O}(D, u))\eta = 0$.

Now, assume that $H^r(\mathscr{O}(D, X/Y), \zeta - \mathscr{O}(D, u)) = \{0\}$ for all $r \geqslant 0$. Then, in particular, there is a form $\xi \in \Lambda^{p-1}[\sigma, \mathscr{O}(D, X/Y)]$ such that $\delta(\zeta - \mathscr{O}(D, u))\xi = \eta$. It follows from the results in the second section that $\mathscr{O}(D, j(X/Y)) = \mathscr{O}(D, j)((\mathscr{O}(D, X/Y))$ (by Lemma 2.17) and that

$$\mathscr{O}(D, u \mid j(X/Y)) = \mathscr{O}(D, u) | \mathscr{O}(D, j) (\mathscr{O}(D, X/Y))$$

(see Remarks 2.4, 2.6 and Lemma 2.17). Therefore $\mathscr{O}(D, j)\eta = \eta$ and

$$\delta(\zeta - \mathscr{O}(D, u \mid j(X/Y)))\mathscr{O}(D, j)\xi = \eta.$$

Since p is arbitrary, this shows that

$$H^p(\mathscr{O}(D, j(X/Y)), \zeta - \mathscr{O}(D, u \mid j(X/Y))) = \{0\}$$

for all $p \geqslant 0$, from which we infer easily that $\sigma(u, j(X/Y)) \subset \sigma(u, X/Y)$.

5.14. THEOREM. *Let* $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ *be a* c.r.m. *such that* $\sigma(u, X/Y) = K_1 \cup K_2$, *where* K_1, K_2 *are compact and disjoint sets. Then there exist* qF-*subspaces* X_1/Y, X_2/Y *of* X/Y *that are invariant under* $u_1, \ldots, u_n$, *and with the following properties:*

(1) $X/Y = X_1/Y + X_2/Y$, $(X_1/Y) \cap (X_2/Y) = \{0\}$.

(2) $\sigma(u, X_r/Y) = K_r$, $r = 1, 2$.

Proof. With no loss of generality we may assume that both K_1 and K_2 are non-empty (otherwise the assertion is trivial). Let $U_r \supset K_r$ $(r = 1, 2)$ be open sets such that $U_1 \cap U_2 = \emptyset$. If χ_1 is the characteristic function of U_1 in $U_1 \cup U_2$, then $\chi_1 \in \mathscr{O}(U_1 \cup U_2)$. Then the operator $j = \chi_1(u)$ makes sense and is an idempotent on X/Y, by Theorem 5.2. We set $X_1 = \mathrm{Im}_0(j)$ and $X_2 = \mathrm{Im}_0(1 - j)$. Since j commutes with $u_1, \ldots, u_n$, the spaces X_1/Y and X_2/Y are invariant under $u_1, \ldots, u_n$. The properties $X/Y = X_1/Y + X_2/Y$ and $(X_1/Y) \cap (X_2/X) = \{0\}$ are obvious.

Let us prove that $\sigma(u, X_r/Y) = K_r$ $(r = 1,2)$. We have $\sigma(u, X_r/Y) \subset \sigma(u, X/Y)$, by Lemma 5.13. From Proposition 5.4 we deduce that $\chi_1(u|(X_1/Y)) = \chi_1(u)|(X_1/Y)$. If $\sigma(u, X_1/Y) \cap K_2 \neq \emptyset$, then $k = (1 - \chi_1)(u \mid (X_1/Y))$ is a non-null idempotent in $\mathscr{L}(X_1/Y)$, since $\sigma(k, X_1/Y) = \{0,1\}$, in virtue of Theorem 5.11. On the other hand, $k = jk = \chi_1(u)(1 - \chi_1(u))|(X_1/Y) = 0$, which is a contradiction. Therefore $\sigma(u, X_1/Y) \subset K_1$. Similarly, $\sigma(u, X_2/Y) \subset K_2$. These inclusions are actually equalities. Indeed, since X_2/Y is isomorphic to X/X_1, it follows from Proposition 3.13 that

$$\sigma(u, X/Y) = K_1 \cup K_2 \subset \sigma(u, X_1/Y) \cup \sigma(u, X_2/Y),$$

so that $\sigma(u, X_r/Y) = K_r$, which completes the proof of the theorem.

5.15. REMARKS. 1° Let $u = (u_1, \ldots, u_n) \subset \mathscr{L}(X/Y)$ be a c.r.m. and let $\xi \in X/Y$. Assume that there is a form

$$\varphi \in \Lambda^{n-1}[(\sigma, d\bar{\zeta}), \varepsilon(\mathbf{C}^n, X/Y)]$$

such that

$$(\delta(\zeta - \varepsilon(\mathbf{C}^n, u)) + \bar{\partial})\varphi = \xi\sigma_1 \wedge \ldots \wedge \sigma_n,$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ are indeterminates associated with u. Then we must have $\xi = 0$.

Indeed, in this case the projection on $\Lambda[d\bar{\zeta}, \varepsilon_0(\mathbf{C}^n, X/Y)]$ of the form

$$(-1)^n(\xi\sigma_1 \wedge \ldots \wedge \sigma_n - (\delta(\zeta - \varepsilon(\mathbf{C}^n, u)) + \bar{\partial})\varphi)$$

represents the coset $R(\zeta - \varepsilon(\mathbf{C}^n, u))\xi$ and is null in $\mathbf{C}^n$. Hence

$$\xi = (2\pi i)^{-n} \int R(\zeta - \varepsilon(\mathbf{C}^n, u))\, \xi \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = 0.$$

2° The previous observation can be used to prove that the pair $(X_1/Y, X_2/Y)$ with the properties (1) and (2) from Theorem 5.14 is uniquely determined. Indeed, if $(X_1'/Y, X_2'/Y)$ is a pair with similar properties and if $\xi_1 \in X_1/Y$, we can write $\xi_1 = \xi_1' + \xi_2'$, with $\xi_r' \in X_r'/Y$ $(r = 1,2)$. Then we can find forms

$$\varphi_r \in \Lambda^{n-1}[(\sigma, d\bar{\zeta}), \varepsilon(V_r, X/Y)]$$

such that

$$(\delta(\zeta - \varepsilon(V_r, u)) + \bar{\partial})\varphi_r = \xi_2' = \xi_1 - \xi_1',$$

where $V_r = \mathbf{C}^n \setminus K_r$ $(r = 1, 2)$. Since $V_1 \cap V_2 \cap \sigma(u, X/Y) = \emptyset$, we may assume $\varphi_1|V_1 \cap V_2 = \varphi_2|V_1 \cap V_2$, which shows that ξ_2' has the property of the vector ξ from the preceding remark. Therefore $\xi_2' = 0$, that is $X_1/Y \subset X_1'/Y$. Similarly, $X_2/Y \subset X_2'/Y$. That $X_1'/Y \subset X_1/Y$ and $X_2'/Y \subset X_2/Y$ follows analogously.

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