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VON NEUMANN SUBALGEBRAS OF A TYPE II_1 FACTOR: CORRESPONDENCES AND PROPERTY T

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INTRODUCTION

Let M be a factor of type II_1 with normalized trace tr_M. Let N be a von Neumann subalgebra of M. In this paper we associate in a canonical way a correspondence $\mathfrak{h}$ from M to M to each inclusion of a von Neumann subalgebra N in M; let ψ be the binormal state defined by $\psi(x \otimes y^0) = \mathrm{tr}_M(E_N(x)y)$ on $M \otimes M^{\mathrm{opp}}$. Alain Connes has defined in his notes [1] a unique correspondence from M to M associated to a binormal state on $M \otimes M^{\mathrm{opp}}$. Using this correspondence we are able to generalize the definition of the index of a von Neumann subalgebra N in M to be equal to the index (right index = left index) of this correspondence $\mathfrak{h}$.

In the first part of this paper we study the properties of this correspondence and of its index.

Let $M_1 = \langle M, e_N \rangle$ as in [2] (e_N is the projection associated to the conditional expectation from M to N). There is a natural semi-finite trace Tr_{M_1} on M_1 such that $\mathfrak{h}$ is equivalent to the correspondence $L^2(M_1, \mathrm{Tr}_{M_1})$ with left (resp. right) action of M given by left (resp. right) multiplication by elements of M.

If the index of $\mathfrak{h}$ is finite, there exists a homomorphism $\rho : M \to M \otimes M_{n+1}(\mathbf{C})$ (n is the integer part of the index) such that $\mathfrak{h}$ is equivalent to $(L^2(M) \oplus \ldots \oplus L^2(M))\rho(1)$ with actions given by $x \cdot \xi \cdot y = x\xi\rho(y)$.

The possible values of the index are $\left\{4\cos^2\frac{\pi}{n}\,;\ n \geqslant 3\right\} \cup [4, +\infty]$.

If $\mathfrak{h}$ associated to the inclusion of N in M is of finite index, then $M_1 \cap M'$ is a von Neumann algebra of finite dimension.

Let M be a II_1 factor; let N and P be von Neumann subalgebras of M such that $M \supset N \supset P$. If $[M:P] < \infty$ then $[M:N] < \infty$.

In the second part of this work we prove one property of rigidity for II_1 factors with property T. We prove that if M has property T, if (M_n) is an increasing sequence of von Neumann subalgebras of M such that $M = \bigcup M_n$, there exists a sequence $(c_n)_{n\in\mathbf{N}}$ of minimal projections $c_n \in M \cap M_n'$ so that $\mathrm{tr}_M(1 - c_n) \to 0$ as $n \to \infty$ and so that $M_{c_n} = (M_n)_{c_n}$ for n large enough. The first step is to prove the existence of projections $\gamma_n \in M \cap M_n'$ so that the index $[M\gamma_n : (M_n)_{\gamma_n}]$ is finite and such that $\mathrm{tr}_M(1 - \gamma_n) \to 0$. (This is done by applying the property T to the sequence $\mathfrak{h}_n$ of correspondences from M to M associated to the inclusion $M_n \subset M$.) The second step is to prove that the γ_n can be chosen in the set of minimal projections of $M \cap M_n'$. And then we get the result by using the rigidity of the index.

We can notice that Alain Connes announced without giving proof, during the Kingston Summer Institute [2] that there exists no non trivial sequence $M_n \subset M_{n+1}$ of subfactors of M with $\bigcup M_n$ dense in M. Here we study the same problem in a more general context: we do not assume that the M_n are subfactors but only von Neumann subalgebras; this hypothesis complicates considerably the proof. In this case we prove the rigidity result as explained above, but in general the sequence M_n itself is not stationnary (we give a counterexample in 2.7). We prove that the sequence is stationnary when $M \cap M_n'$ is a factor or is of finite dimension (we do not know however how to prove it in the case where the M_n are subfactors of M, without any other assumption).

Let M be a factor of type II_1, and N a von Neumann subalgebra of M.

1. CORRESPONDENCE FROM M TO M ASSOCIATED TO THE INCLUSION OF N IN M

Recall that according to [3], a correspondence from M to M is a Hilbert space $\mathfrak{h}$ which is a left M module and a right M module with commuting normal actions. Two correspondences are equivalent if there is a bijective isometry from one to the other which commutes with the actions.

a) DEFINITION OF THE CORRESPONDENCE ASSOCIATED TO THE INCLUSION OF N IN M. Let tr_M be the normal faithful normalized trace on M. Let E_N be the conditional expectation from M to N ($\mathrm{tr}_M(E_N(x)y) = \mathrm{tr}_M(xy)$ $\forall\, x \in M$ $\forall\, y \in N$). Let ψ be the binormal state on $M \otimes M^{\mathrm{opp}}$ (M^{opp} is the opposite of M) given by $\psi(x \otimes y^0) = \mathrm{tr}_M(E_N(x)y)$.

Recall the definition given by Alain Connes in [1] of the correspondence associated to a binormal state on $M \otimes M^{\mathrm{opp}}$.

1.1. DEFINITION. By the Gelfand-Naimark-Segal construction, we get a pair $(\mathfrak{h}, \xi)$ where $\mathfrak{h}$ is a correspondence from M to M and ξ is a cyclic vector $\left(\text{i.e. the finite sums } \sum_{i=1}^{n} x_i\xi y_i \ (x_i, y_i \in M) \text{ are dense in } \mathfrak{h}\right)$, such that the equality $\psi(x \otimes y^0) = \langle x\xi y \mid \xi\rangle$ holds for every $x, y \in M$. And this pair is unique up to unitary equivalence.

$\mathfrak{h}$ will be called the *correspondence associated to the inclusion of N in M*.

We want now to identify this correspondence.

1.2. NOTATIONS. Let $H = L^2(M, \mathrm{tr}_M)$. Let ξ_0 be the canonical vector in H. E_N extends to a projection e_N in $\mathscr{L}(H)$ such that $e_N x\xi_0 = E_N(x)\xi_0 \ \forall\, x \in M$. Let J be the isometric involution defined by $Jx\xi_0 = x^*\xi_0 \ \forall\, x \in M$. As in [2] we note M_1 the von Neumann algebra on $L^2(M, \mathrm{tr}_M)$ generated by M and e_N.

1.3. DEFINITION. In M_1 we consider the unique trace Tr_{M_1} such that $\mathrm{Tr}_{M_1}(x) = \langle x\xi_0, \xi_0\rangle$ for all $x \in (M_1)_{e_N}$.

1.4. REMARK. By [5] Lemma 1.1, the set of finite sums $\sum_{i=1}^{n} x_i e_N y_i$; $x_i, y_i \in M$, is a dense subalgebra of M_1. Then Tr_{M_1} is a well defined semi-finite normal faithful trace on M_1.

1.5. PROPOSITION. *The correspondence* $\mathfrak{h}$ *from M to M associated to the inclusion of N in M is equivalent to the correspondence $L^2(M_1, \mathrm{Tr}_{M_1})$ with left and right actions of M simply given by left and right multiplication, and with cyclic vector e_N (e_N is identified with its image in $L^2(M_1, \mathrm{Tr}_{M_1})$).*

Proof. The finite sums $\sum_{i=1}^{n} x_i e_N y_i$ are dense in M_1 so e_N is cyclic in $L^2(M_1, \mathrm{Tr}_{M_1})$.

Furthermore, if $x, y \in M$

$$\langle x e_N y \mid e_N\rangle_{M_1} = \mathrm{Tr}_{M_1}(e_N x e_N y) =$$

$$= \mathrm{Tr}_{M_1}(e_N E_N(x) y e_N) = \langle e_N E_N(x) y e_N \xi_0 \mid \xi_0\rangle =$$

$$= \mathrm{tr}_M(E_N(x)y) \qquad \text{(because } e_N\xi_0 = \xi_0\text{)}.$$

Q.E.D.

b) PROPERTIES OF THIS CORRESPONDENCE. In all this paragraph we assume that Tr_{M_1} is finite.

1.6. PROPOSITION. *As a left (respectively right) M module, M_1 is projective and finitely generated; more precisely, M_1 is isomorphic as a left (respectively right) module to* $\underbrace{M \oplus \ldots \oplus M}_{n \text{ times}} \oplus Mp$ *(respectively* $\underbrace{M \oplus \ldots \oplus M}_{n \text{ times}} \oplus pM$*) where n is the integer part of $\mathrm{Tr}_{M_1}(1)$, p is a projection, $p \in M$, and $\mathrm{tr}_M(p) = \mathrm{Tr}_{M_1}(1) - n$.*

Let $\tau = \dfrac{1}{\mathrm{Tr}_{M_1}(1)}$, let $\tau_1 = \tau \mathrm{Tr}_{M_1}$. As in the case of factors we can make the basic construction for $M \subset M_1$, and we get M_2 which now is a factor.

1.7. Lemma. 1) (τ_1, M_1) *is a* τ *extension of* M *by* N *(with the definition given in* [5]*).*

2) M_2 *is finite and if we note* τ_2 *the normalized trace on* M_2, (τ_2, M_2) *is a* τ *extension of* M_1 *by* M.

Proof. 1) Let $x \in M$. $\tau_1(e_1 x) = \tau \mathrm{Tr}_{M_1}(e_1 x) = \tau \mathrm{tr}_M(x)$; $\tau_1 | M$ is a normalized trace on the factor M, so $\tau_1 | M = \mathrm{tr}_M$. So $E_M(e_1) = \tau$.

2) Let $a, b \in M$. Let ξ_1 be the canonical vector on $L^2(M_1, \tau_1)$.

$$e_1 e_2 e_1 (a e_1 b)\xi_1 = e_1 e_2 E_N(a) e_1 b \xi_1 = e_1 E_M(E_N(a) e_1 b)\xi_1 = \tau e_1 E_N(a) b \xi_1$$

and

$$e_1(a e_1 b)\xi_1 = e_1 E_N(a) b \xi_1 .$$

Now, using the density of the finite sums $\sum a_i e_1 b_i$ in M_1, we get the equality $e_1 e_2 e_1 = \tau e_1$.

In M_2 we consider the unique faithful semi-finite trace Tr_{M_2} given by $\mathrm{Tr}_{M_2}(x) = \langle x\xi_1, \xi_1 \rangle$, if $x \in (M_2)_{e_2}$ (as we did for M_1). Then

$$\mathrm{Tr}_{M_2}(a e_1 b) = \frac{1}{\tau} \mathrm{Tr}_{M_2}(a e_1 e_2 e_1 b) =$$

$$= \frac{1}{\tau} \langle e_2 e_1 b a e_1 e_2 \xi_1 | \xi_1 \rangle = \frac{1}{\tau} \tau_1(e_1 b a e_1) = \frac{1}{\tau} \tau_1(a e_1 b).$$

So the restriction of $\tau \mathrm{Tr}_{M_2}$ on M_1 is τ_1. It follows that the factor M_2 is finite (cf. $\tau_1(1) = 1$) and that the unique normal faithful normalized trace τ_2 on M_2 is equal to $\tau \mathrm{Tr}_{M_2}$; then $\tau_2(e_2 x) = \tau \mathrm{Tr}_{M_2}(e_2 x) = \tau \tau_1(x) = \tau \tau_2(x)$ for every x in M_1.

Q.E.D.

1.8. Lemma. *There exists a family* $\{m_i\}_{1 \leqslant i \leqslant n+1}$ *of elements in* M_1, *where* n *is the integer part of* $\mathrm{Tr}_{M_1}(1)$ *such that:*

a) $E_M(m_j^* m_k) = 0$ *if* $j \neq k$.

b) $E_M(m_j^* m_j) = 1$ *if* $1 \leqslant j \leqslant n$.

c) $E_M(m_{n+1}^* m_{n+1})$ *is a projection in* M *of trace* $\mathrm{Tr}_{M_1}(1) - n$.

And then each element $x \in M_1$ *has a unique decomposition* $x = \sum_{j=1}^{n+1} m_j x_j$ *with* $x_j \in M$, $x_{n+1} \in E_M(m_{n+1}^* m_{n+1}) M$.

Proof. The proof of this lemma is exactly the same as the proof of Proposition 1.3 in [5], which was given for factors. Indeed, the only properties required to make this proof are:

— the fact that M_2 is a factor which is true (M was a factor and $M_2 = J_1 M' J_1$);

— the fact that (τ_2, M_2) is a τ extension of M_1 by M which has been proved in Lemma 1.7.

Proposition 1.6 is then an easy corollary of Lemma 1.8.

1.9. Definition. Let $\rho : M \to M \otimes M_k(\mathbf{C})$ be a normal $*$-homomorphism; $\rho(1) = e$ is a projection. The Hilbert space $(\underbrace{L^2(M) \oplus \ldots \oplus L^2(M)}_{k \text{ times}})e$ is an M-M bimodule with the representations $\Pi_M(x)\Pi_M^0(y)\zeta = x\zeta\rho(y)$.

We note $L^2(\rho)$ this correspondence.

1.10. Lemma. *Let* $\mathfrak{h}$ *be a correspondence from* M *to* M. *Assume that there exists* $k \in \mathbf{N}$ *and a projection* p *in* $M \otimes M_k(\mathbf{C})$ *such that* $\mathfrak{h}$ *is isomorphic as Hilbert space and left* M *module to* $(\underbrace{L^2(M) \oplus \ldots \oplus L^2(M)}_{k \text{ times}})p$ *with left multiplication by elements of* M. *There exists a normal* $*$*-homomorphism* $\rho : M \to M \otimes M_k(\mathbf{C})$ *such that* $\rho(1) = p$, *so that the correspondence* $\mathfrak{h}$ *is equivalent to* $L^2(\rho)$.

Proof. First determine the commutant of M in $\mathscr{L}((L^2(M) \oplus \ldots \oplus L^2(M))p)$; we note it M'. Let $T \in M'$. Let $T_{ij}(\xi) = P_j(T(0, \ldots, 0, \xi, 0, \ldots, 0))$ $\forall\, \xi \in L^2(M)$ where P_j is the projection on the j component. Then $T_{ij} \in \mathscr{L}(L^2(M))$ and commutes with M so $T_{ij} \in JMJ$; i.e. there exists $x_{ij} \in M$ so that $T_{ij}(\zeta) = \zeta x_{ij}$. So $T(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_n)(x_{ij}) = (\xi_1, \ldots, \xi_n)p(x_{ij})p$. Thus the commutant of M in $\mathscr{L}((L^2(M) \oplus \ldots \oplus L^2(M))p)$ is the algebra of right multiplications by elements of $p(M \otimes M_k(\mathbf{C}))p$. It follows that the right action of M is given by a normal $*$-homomorphism of M in $p(M \otimes M_k(\mathbf{C}))p$.

1.11. Proposition. *Let* M *be a factor of type* II_1; *let* N *be a von Neumann subalgebra of* M. *Suppose that* Tr_{M_1} *is finite; then there exists a homomorphism* $\rho : M \to M \otimes M_{n+1}(\mathbf{C})$ *so that*

$$\rho(1) = \begin{pmatrix} 1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & 1 & \\ 0 & & & & E_M(m_{n+1}^* m_{n+1}) \end{pmatrix}$$

and such that the correspondence $\mathfrak{h}$ *from* M *to* M *associated to the inclusion of* N *in* M *is equivalent to* $L^2(\rho)$.

Proof. This is an immediate consequence of Lemmas 1.8 and 1.10.

We will give now the definition of the index of a von Neumann subalgebra.

1.12. Definition. Let M be a factor of type II_1, and N a von Neumann subalgebra of M.

Let M_1, Tr_{M_1} be as above.

1) If Tr_{M_1} is not finite we define the index of N in M to be infinite.

2) If Tr_{M_1} is finite, then the following numbers are equal:

(i) $\mathrm{Tr}_{M_1}(1)$,

(ii) the left index of the correspondence $\mathfrak{h}$ from M to M associated to the inclusion of N in M ($= \dim_M \mathfrak{h}$),

(iii) the right index of the correspondence $\mathfrak{h}$ ($= \dim_{M^{\mathrm{opp}}} \mathfrak{h}$).

We then call this number the *index of N in M*.

1.13. REMARK. It is trivial that this definition coincides with that of Jones in [4] when N is a factor. We now prove that the possible values of the index are as in the case of subfactors: $\left\{4\cos^2 \frac{\pi}{n}\,;\, n \geqslant 3\right\} \cup [4, +\infty]$.

1.14. LEMMA. *Suppose that* (τ_i, M_i) *is a* τ *extension of* M_{i-1} *by* M_{i-2} $(i \geqslant 1)$ *(we note* $M_0 = M$ *and* $M_{-1} = N$*) then* $e_i e_{i+1} e_i = \tau e_i$ *and* $e_{i+1} e_i e_{i+1} = \tau e_{i+1}$.

Proof. By hypothesis, $E_{M_{i-1}}(e_i) = \tau$. Then $e_{i+1} e_i e_{i+1} = E_{M_{i-1}}(e_i) e_{i+1} = \tau e_{i+1}$. Let ξ_i be the canonical vector in $L^2(M_i)$. Let $a, b \in M_{i-1}$. Then

$$e_i e_{i+1} e_i (a e_i b)\xi_i = e_i e_{i+1} E_{M_{i-2}}(a) e_i b \xi_i = e_i E_{M_{i-1}}(E_{M_{i-2}}(a) e_i b)\xi_i =$$

$$= e_i E_{M_{i-2}}(a) \tau b \xi_i = \tau e_i a e_i b \xi_i .$$

By density of the finite sums $\sum_{j=1}^{n} a_j e_i b_j$ in M_i we get the result.

1.15. COROLLARY. *Let M be a factor of type* II_1. *Let N be a von Neumann subalgebra of M. Then* $[M : N] \in \left\{4\cos^2 \frac{\pi}{n}\,;\, n \geqslant 3\right\} \cup [4, +\infty]$.

Proof. Suppose that $[M : N] < \infty$. As in the case of factors we construct the tower M_j defined by V. Jones in [4]; but here we have to choose on M_{2i+1} a "good trace" (it is defined exactly as we did for M_1). M_{2i} is a factor, and the inclusion $M_{2i-1} \subset M_{2i}$ satisfies the same conditions as the first inclusion $N \subset M$. From Lemma 1.7, at each step, (τ_j, M_j) is a τ extension of M_{j-1} by M_{j-2}; and then from Lemma 1.14 we get a sequence (e_j) $j \geqslant 1$ of projections such that $e_j e_{j+1} e_j = \tau e_j$; $e_{j+1} e_j e_{j+1} = \tau e_{j+1}$ and $e_j e_k = e_k e_j$ if $|j-k| \geqslant 2$. From [4], these relations imply that $\tau \in \left\{4\cos^2 \frac{\pi}{n}\,;\, n \geqslant 3\right\} \cup [4, +\infty]$.

c) Reduced von Neumann algebras, and reduced correspondences.

1.16. Proposition. *Let M be a factor of type* II_1. *Let N be a von Neumann subalgebra of M. Let $f \in N' \cap M$ be a projection. Define* $\mathrm{Tr}_{N'}$ by $\mathrm{Tr}_{N'}(x) = \langle x\xi_0, \xi_0\rangle$ $\forall x \in N'_f$.

If $\mathrm{Tr}_{N'}(f) < \infty$ *then* $[M_f : N_f] \leqslant \dfrac{\mathrm{Tr}_{N'}(f)}{\mathrm{tr}_M(f)}$.

1.17. Notation. We note tr_{M_f} the normalized trace on the factor M_f $\left(\mathrm{tr}_{M_f}(x) = \dfrac{1}{\mathrm{tr}_M(f)}\mathrm{tr}_M(x) \ \forall x \in M_f\right)$. Let $p \in N'_f$ be the projection associated to the conditional expectation from M_f to N_f. We note $\mathrm{Tr}_{N'_f}$ the trace defined by

$$\mathrm{Tr}_{N'_f}(x) = \langle x\xi_f, \xi_f\rangle \quad \forall x \in pN'_fp,$$

where ξ_f is the canonical vector in $L^2(M_f)$.

1.18. Lemma. *Let $b \in M_f$, and $x \in M$. Let $X \in N$ such that $E_{N_f}(bxf) = fX$. Then*

$$E_N(bx) = E_N(f)X = E_N(bxf).$$

Proof. Let $y \in N$;

$$\mathrm{tr}_M(bxy) = \mathrm{tr}_M(bxyf) = \qquad \text{(as } fb = b\text{)}$$

$$= \mathrm{tr}_M(bxfyf) = \qquad \text{(as } y \in N \text{ and } f \in N'\text{)}$$

$$= \mathrm{tr}_M(E_{N_f}(bxf)fyf) = \mathrm{tr}_M(fXfyf) = \mathrm{tr}_M(fXy) = \mathrm{tr}_M(E_N(f)Xy).$$

So $E_N(bx) = E_N(f)X$, and

$$\mathrm{tr}_M(bxfy) = \mathrm{tr}_M(bxy) \quad \text{so} \quad E_N(bxf) = E_N(bx).$$

1.19. Lemma. *Let $x \in M_f$. Then* $\mathrm{Tr}_{N'_f}(pJ_fxJ_f)\mathrm{tr}_M(f) = \mathrm{Tr}_{N'}(e_NJxJf)$; *$J_f$ being the involution on $L^2(M_f)$.*

Proof.

$$\mathrm{Tr}_{N'_f}(pJ_fxJ_f) = \langle pJ_fxJ_fp\xi_f, \xi_f\rangle = \qquad \text{(by definition of } \mathrm{Tr}_{N'_f}\text{)}$$

$$= \langle x^*\xi_f, \xi_f\rangle = \mathrm{tr}_{M_f}(x^*).$$

In a similar way,

$$\mathrm{Tr}_{N'}(e_N JxJf) = \langle e_N JxJfe_N\xi, \xi\rangle =$$

$$= \mathrm{tr}_M(fx^*) = \mathrm{tr}_M(x^*) \qquad \text{because } x \in M_f.$$

And $\mathrm{tr}_{M_f}(x^*) = \dfrac{\mathrm{tr}_M(x^*)}{\mathrm{tr}_M(f)}$; so we get the result.

1.20. LEMMA. *Let* $x_j, y_i \in M_f$. *Let*

$$X = \sum_i J_f x_i J_f p J_f y_i J_f$$

and

$$Y = \sum_i fJx_i Je_N Jy_i Jf.$$

Then

i) $\|Y\| \leqslant \|X\|$,

ii) $|\mathrm{Tr}_{N'_f}(X)| \leqslant \|X\|\ \mathrm{Tr}_{N'}(f) x \dfrac{1}{\mathrm{tr}_M(f)}$.

Proof. i) $\|Y\| = \sup\limits_{\substack{x\in M\\ \mathrm{tr}_M(x^*x)\leqslant 1}} \|Yx\xi_0\|, \quad Y = \sum_i fJx_i e_N y_i Jf \quad (Je_N J = e_N).$

$$\|Yx\xi_0\|^2 = \sum_{i,j} \langle fJx_i Je_N Jy_i Jfx\xi_0 \,|\, fJx_j Je_N Jy_j Jfx\xi_0\rangle =$$

$$= \sum \langle fE_N(fxy_i^*)x_i^*\xi_0 \,|\, fE_N(fxy_j^*)x_j^*\xi_0\rangle$$

as $y_i \in M_f$, we can restrict to the case $x \in M_f$. Furthermore $\mathrm{tr}_{M_f} = \dfrac{1}{\mathrm{tr}_M(f)} \mathrm{tr}_M$, so that

$$(1) \qquad \|Y\| = \sup_{\substack{x\in M_f\\ \mathrm{tr}_{M_f}(x^*x)\leqslant 1}} \|yx\xi_0\| \frac{1}{\sqrt{\mathrm{tr}_M(f)}}.$$

Let $Y_i \in N$ so that $E_{N_f}(y_i x^*) = Y_i f$. From Lemma 1.18 it follows that $E_N(y_i x^*) = E_N(f) Y_i = Y_i E_N(f)$.

So

$$\|Yx\xi_0\|^2 = \sum_{i,j} \langle fE_N(f)Y_i^* x_i^* \xi_0 \,|\, fE_N(f)Y_j^* x_j^* \xi_0\rangle =$$

$$= \sum_{i,j} \mathrm{tr}_M(x_j Y_j E_N(f)^2 Y_i^* x_i^*) \qquad (\text{as } f \in N' \text{ and } x_i \in M_f);$$

$$\|X\| = \sup_{\substack{x\in M_f\\ \mathrm{tr}_{M_f}(x^*x)\leqslant 1}} \|Xx\xi_f\|, \quad X = \sum_i J_f x_i p y_i J_f \qquad (\text{cf. } J_f p J_f = p),$$

and

$$\|Xx\xi_f\|^2 = \sum_{i,j} \langle J_f x_i p y_i J_f x\xi_f,\ J_f x_j p y_j J_f x\xi_f\rangle =$$

$$= \sum_{i,j} \langle J_f x_i E_{N_f}(y_i x^*)\xi_f,\ J_f x_j E_{N_f}(y_j x^*)\xi_f\rangle =$$

$$= \sum_{i,j} \mathrm{tr}_{M_f}(x_j E_{N_f}(y_j x^*) E_{N_f}(x y_i^*) x_i^*) =$$

$$= \sum_{i,j} \mathrm{tr}_{M_f}(x_j Y_j f Y_i^* x_i^*) = \sum_{i,j} \mathrm{tr}_{M_f}(x_j Y_j Y_i^* x_i^*).$$

We then get:

$$\|Yx\xi_0\|^2 = \mathrm{tr}_M((\sum_i x_i Y_i) E_N(f)^2 (\sum_i x_i Y_i)^*) \leqslant$$

$$\leqslant \mathrm{tr}_M((\sum_i x_i Y_i)(\sum_j x_j Y_j)^*) = \|Xx\xi_f\|^2 x \mathrm{tr}_M(f).$$

Using (1) we get: $\|Y\| \leqslant \|X\|$.

ii) As $f \in M$ commutes with $Jy_iJ \in M'$, it is obvious, using Lemma 1.19, that

$$\mathrm{Tr}_{N_f'}(X) = \frac{1}{\mathrm{tr}_M(f)} \mathrm{Tr}_{N'}(Y).$$

And $Y = fY$, so

$$|\mathrm{Tr}_{N_f'}(x)| \leqslant \frac{\|Y\|}{\mathrm{tr}_M(f)} \mathrm{Tr}_{N'}(f) \leqslant \frac{\|X\|}{\mathrm{tr}_M(f)} \mathrm{Tr}_{N'}(f).$$

1.21. *Proof of Proposition 1.16.* From [5], Lemma 1.1, there exists an increasing sequence X_k of elements of the form $\sum_{i=1}^{n} J_f a_i p b_i J_f$ so that $f = \lim X_k$ for the strong topology.

From Lemma 1.20 ii), we obtain:

$$(\mathrm{Tr}_{N_f'}(x_k)) \leqslant \frac{\mathrm{tr}_{N'}(f)}{\mathrm{tr}_M(f)}.$$

So

$$\mathrm{Tr}_{N_f'}(f) \leqslant \frac{\mathrm{tr}_{N'}(f)}{\mathrm{tr}_M(f)}.$$

And $[M_f : N_f] = \mathrm{Tr}_{N_f'}(f)$. So $[M_f : N_f] \leqslant \dfrac{\mathrm{tr}_{N'}(f)}{\mathrm{tr}_M(f)}$.

1.22. PROPOSITION. *Let $\mathfrak{h} = L^2(M_1, \mathrm{Tr}_{M_1})$ be the correspondence from M to M associated to the inclusion of N in M. Let M' be the commutant of the left action of M in $\mathfrak{h}$. Let $p \in M' \cap M_1$ be a projection. Then $p\mathfrak{h}$ is a correspondence from M to M_p. Its left index is $[M : N]\mathrm{tr}_{M'}(p)$.*

Proof. The left index of $p\mathfrak{h}$ is $\dim_{M_p} p\mathfrak{h} = \dim_M \mathfrak{h} \times \mathrm{tr}_{M'}(p)$ (cf. [2]).

1.23. COROLLARY. *With the same notations as above, assume that $[M : N] < \infty$ then the relative commutant $M' \cap M_1$ is of finite dimension.*

Proof. If $M' \cap M_1$ was of infinite dimension, we could find for each $n \in \mathbb{N}^*$ orthogonal projections $(p_i)_{1 \leqslant i \leqslant n}$, $p_i \in M' \cap M_1$ so that $\mathrm{tr}_{M'}(p_i) = \dfrac{1}{n}$ and $\sum_{i=1}^{n} p_i = 1$. Consider then the correspondences $p_i\mathfrak{h}$ from M to M_{p_i}.

Left index $p_i\mathfrak{h}$ $\times$ right index $p_i\mathfrak{h} = \dim_{M_{p_i}} p_i\mathfrak{h} \times \dim_{M^{\mathrm{opp}}} p_i\mathfrak{h}$. If we note π the representation of M_{p_i} in $p_i\mathfrak{h}$ and π^0 the representation of M^{opp} in $p_i\mathfrak{h}$ we get, by definition of the index of a subfactor: left index $p_i\mathfrak{h}$ $\times$ right index $p_i\mathfrak{h} = [\pi(M_{p_i})' : \pi^0(M^{\mathrm{opp}})]$, and $[\pi(M_{p_i})' : \pi^0(M^{\mathrm{opp}})] \geqslant 1$. It follows then from Proposition 1.22 that $[M : N]^2 = \sum_{i=1}^{n} [M : N]^2 \mathrm{tr}_{M'}(p_i) = \sum_{i=1}^{n} [M : N] \times$ left index of $p_i\mathfrak{h}$ and $[M : N] = \dim_{M^{\mathrm{opp}}}\mathfrak{h} \geqslant \dim_{M^{\mathrm{opp}}} p_i\mathfrak{h} =$ right index of $p_i\mathfrak{h}$. So $[M : N]^2 \geqslant n$; and we get a contradiction.

1.24. COROLLARY. *Let M be a factor of type II_1. Let N and P be von Neumann subalgebras of M such that $M \supset N \supset P$. Assume that $[M : P] < \infty$; then $[M : N] < \infty$.*

Proof. Let J be the canonical involution in $L^2(M)$. Let $N_1 = JN'J$ and $P_1 = JP'J$, where N' and P' are the commutants of N and P in $L^2(M)$. Then $N_1 \cap M' \subset P_1 \cap M'$. From Corollary 1.23, $P_1 \cap M'$ is of finite dimension; so the center of N_1 is also of finite dimension. The restriction of Tr_{P_1} to N_1 is a finite trace on N_1, and Tr_{N_1} is a semi-finite trace of N_1. The finite dimension of the center of N_1 implies then that Tr_{N_1} is finite; so $[M : N] < \infty$.

REMARK. This result was not obvious because a priori Tr_{N_1} and Tr_{P_1} are not comparable.

2. ONE RIGIDITY PRORERTY FOR II_1 FACTORS WITH PROPERTY T

From now on M will be a factor of type II_1 with property T, and $(M_n)_{n\in\mathbf{N}}$ an increasing sequence of von Neumann subalgebras of M such that $M = \overline{\bigcup M_n}$.

We note $\mathfrak{h}_n$ the correspondence from M to M associated to the inclusion of M_n in M; ξ_0 is the canonical vector in $L^2(M)$; e_n is given by $e_n x\xi_0 = E_{M_n}(x)\xi_0$ $\forall x \in M$; M_n' is the commutant of M_n in $\mathscr{L}(L^2(M))$ and $\mathrm{Tr}_{M_n'}$ is the semi-finite trace defined by $\mathrm{Tr}_{M_n'}(x) = \langle x\xi_0 \mid \xi_0\rangle$ $\forall x \in (M_n')_{e_N}$.

2.1. LEMMA. *Let* $C = \left\{ \sum_{i=1}^{n} \lambda_i u_i e_n u_i^* \; ; \; u_i \in \mathscr{U}(M),\ \lambda_i \geqslant 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$.

Let α_n be the projection of 0 *on the closure of the convex set C. Then*

(i) *α_n is the orthogonal projection of e_n in $\mathfrak{h}_n$ on* $K_n = \{\xi \in \mathfrak{h}_n \mid x\xi = \xi x\ \forall x \in M\}$.

(ii) $\alpha_n \in JM_n'J \cap M'$ *and* $0 \leqslant \alpha_n \leqslant 1$.

Proof. (i) It is clear that $\alpha_n \in K_n$. Furthermore if $\xi \in K_n$, $\langle e_n - \sum \lambda_i u_i e_n u_i^* \mid \xi\rangle = = \sum_{i=1}^{n} \lambda_i \langle e_n \mid \xi - u_i^*\xi u_i\rangle = 0$; so by continuity of the scalar product we get the result.

(ii) There exists a sequence $(X_k)_{k\in\mathbf{N}}$ of elements of C such that

$$\|X_k - \alpha_n\|_{\mathfrak{h}_n} \to 0 \quad \text{as } k \to \infty.$$

So $\mathrm{Tr}_{JM_n'J}\, |X_k - X_p|^2 \to 0$ as $k, p \to \infty$. For every $a \in M$,

$$\|(X_k - X_p)a\xi_0\|^2 = \mathrm{Tr}_{JM_n'J}(e_n a^* |X_k - X_p|^2 a e_n) \leqslant \|a\|^2 \mathrm{Tr}_{JM_n'J}(|X_k - X_p|^2);$$

and the X_k belong to the unit ball in $\mathscr{L}(L^2(M))$. The sequence (X_k) converges for the strong topology in $\mathscr{L}(L^2(M))$ so $\alpha_n \in JM_n'J$; and $0 \leqslant X_k \leqslant 1$ so $0 \leqslant \alpha_n \leqslant 1$.

Q.E.D.

2.2. LEMMA. i) *There exists a sequence* $(\eta_n)_{n\in\mathbf{N}^*}$, $\eta_n \in M \cap M_n'$, $0 < \eta_n \leqslant 1$ *such that* $\mathrm{tr}_M(1 - \eta_n) \to 0$ *as* $n \to \infty$ *and* $\mathrm{Tr}_{M_n'}(\eta_n^2) \to 1$ *as* $n \to \infty$.

ii) *There exists a sequence* $(\gamma_n)_{n\geqslant n_0}$ *of projections,* $\gamma_n \in M \cap M_n'$, *such that* $\mathrm{tr}_M(1 - \gamma_n) \to 0$ *as* $n \to \infty$ *and* $[M_{\gamma_n} : (M_n)_{\gamma_n}] < \infty$.

Proof. i) Let $\varepsilon > 0$, $y_1, \ldots, y_m \in M$, $K > 0$ associated to the property T of M as in Proposition 1 of [1]

$$\|y_i e_n - e_n y_i\|_{\mathfrak{h}_n}^2 = \mathrm{Tr}_{JM_n'J}\, |y_i e_n - e_n y_i|^2 = 2(\mathrm{tr}_M(y_i^* y_i - y_i^* E_{M_n}(y_i)) \to 0$$

as $n \to \infty$.

Let $\varepsilon_n = \sup\limits_{1\leqslant i\leqslant m} \|y_i e_n - e_n y_i\|_{\mathfrak{h}_n}$; $\varepsilon_n \to 0$ as $n \to \infty$. So for $n \geqslant n_0$, $\varepsilon_n < \varepsilon$ and there exists a vector $\beta_n \in \mathfrak{h}_n$ such that $x\beta_n = \beta_n x \ \forall\ x \in M$ and $\|\beta_n - e_n\| \leqslant K\varepsilon_n$.

Let α_n be as in Lemma 2.1; $\|\alpha_n - e_n\| \leqslant \|\beta_n - e_n\| \leqslant K\varepsilon_n$. Let $\eta_n = J\alpha_n J$; $\eta_n \in M_n' \cap JM'J = M_n' \cap M$;

$$\mathrm{tr}_M(1-\eta_n) = \mathrm{Tr}_{M_n'}(e_n - e_n\eta_n e_n) \leqslant$$

$$\leqslant (\mathrm{Tr}_{JM_n'J}(|e_n - \alpha_n|^2))^{1/2}\mathrm{Tr}_{JM_n'J}(e_n)^{1/2} = \|e_n - \alpha_n\| \leqslant K\varepsilon_n;$$

and

$$\mathrm{Tr}_{M_n'}(|e_n - \eta_n|^2) = \|\alpha_n - e_n\|^2 \leqslant (K\varepsilon_n)^2, \tag{1}$$

i.e. $\mathrm{Tr}_{M_n'}(\eta_n^2 + e_n - e_n\eta_n - \eta_n e_n) \leqslant (K\varepsilon_n)^2$.

Using $\mathrm{Tr}_{M_n'}(e_n - e_n\eta_n e_n) \leqslant K\varepsilon_n$ and $\mathrm{Tr}_{M_n'}(e_n) = 1$, we get $\mathrm{Tr}_{M_n'}(\eta_n^2) \to 1$ as $n \to \infty$.

ii) By functional calculus, $\gamma_n = X_{[\nu_n,1]}(\eta_n)$, for n large enough and $\nu_n > 0$ small enough satisfies $\mathrm{tr}_M(1-\gamma_n) \to 0$ and $\mathrm{Tr}_{M_n'}(\gamma_n) < \infty$. From Lemma 1.20 ii) it is then obvious that $[M\gamma_n : (M_n)\gamma_n] < \infty$.

2.3. Lemma. *There exist $M > 0$, $N \in \mathbb{N}^*$ and minimal projections $c_n \in M \cap M_n'$ such that $\mathrm{tr}_M(c_n) \geqslant M$ for all $n \geqslant N$.*

Proof. Let F_p be the set of all minimal projections in $M \cap M_p'$. Let $\varepsilon_p = \sup\limits_{e\in F_p} \mathrm{tr}_M(e)$ (if $F_p \neq \emptyset$).

$M_{n+1}' \cap M \subset M_n' \cap M$. Let e be a minimal projection in $M_n' \cap M$. Define $\mathscr{P}_e = \{g$ projection in $M_{n+1}' \cap M$ such that $g \geqslant e\}$. By Zorn there is in $\mathscr{P}_e$ a smallest element p. If p is not minimal in $M_{n+1}' \cap M$, there are p_1 and p_2 in $M_{n+1}' \cap M$ such that $p = p_1 + p_2$.

If $p_1 e \neq 0$, as e is minimal in $M_n' \cap M$, $p_1 \geqslant e$ so $p_1 \geqslant p$, i.e. $p_1 = p$. Otherwise $p_2 = p$. So p is a minimal projection of $M_{n+1}' \cap M$. But now $\mathrm{tr}_M(e) \leqslant \mathrm{tr}_M(p) \leqslant \leqslant \varepsilon_{n+1}$. So $\varepsilon_n \leqslant \varepsilon_{n+1}$.

It only remains to prove that for n large enough $F_n \neq \emptyset$. From Lemma 2.2 ii) $[M_{\gamma_n} : (M_n)_{\gamma_n}] < \infty$ so $(M \cap M_n')_{\gamma_n} = M_{\gamma_n} \cap M'_{n\gamma_n}$ is of finite dimension. Let then f_n be a minimal projection in $(M \cap M_n')_{\gamma_n}$. It is obvious that f_n is minimal in $M \cap M_n'$ so $F_n \neq \emptyset$, $\varepsilon_n > 0$.

2.4. Theorem. *Let M be a factor of type II_1. Let $(M_n)_{n\in\mathbb{N}}$ be an increasing sequence of von Neumann subalgebras of M such that $M = \overline{\bigcup M_n}$. There exists a sequence $(c_n)_{n\in\mathbb{N}}$ of minimal projections in $M \cap M_n'$, such that $\mathrm{tr}_M(1-c_n) \to 0$ as $n \to \infty$ and such that $M_{c_n} = (M_n)_{c_n}$ for n large enough.*

Proof. Choose $c_n \in M_n' \cap M$ as in Lemma 2.3; and η_n as in Lemma 2.2. c_n is a minimal projection in $M_n' \cap M$, so there is a scalar λ_n such that $c_n\eta_n c_n = \lambda_n c_n$.

$$\lambda_n \mathrm{tr}_M(c_n) - \mathrm{tr}_M(c_n) = \mathrm{tr}_M(c_n\eta_n c_n - c_n) = \mathrm{tr}_M(c_n(\eta_n - 1)c_n),$$

so

$$|(\lambda_n - 1)\,\mathrm{tr}_M(c_n)| \leqslant \|c_n\|^2 \mathrm{tr}_M(1-\eta_n) \to 0 \quad \text{as } n \to \infty.$$

And

(2) $$\mathrm{tr}_M(c_n) \geqslant M \ \ \forall\, n, \quad \text{so } \lambda_n \to 1 \text{ as } n \to \infty.$$

Let β_n be such that $c_n e_n c_n = \beta_n c_n$; $\beta_n \mathrm{tr}_{M_n'}(c_n) = \mathrm{tr}_M(c_n)$, so $\mathrm{tr}_{M_n'}(c_n) < \infty$.

$$\lambda_n \mathrm{tr}_{M_n'}(c_n) = \mathrm{tr}_{M_n'}(c_n\eta_n) \leqslant \mathrm{Tr}_{M_n'}(c_n)^{1/2}\mathrm{Tr}_{M_n'}(\eta_n^2)^{1/2}.$$

So $\mathrm{Tr}_{M_n'}(c_n)^{1/2} \leqslant \dfrac{\mathrm{Tr}_{M_n'}(\eta_n^2)^{1/2}}{\lambda_n}$. As $\lambda_n \to 1$ (cf. (2)) and $\mathrm{Tr}_{M_n'}(\eta_n^2) \to 1$ (cf. Lemma 2.2), it follows that $\mathrm{Tr}_{M_n'}(c_n) \to 1$ as $n \to \infty$.

And $\mathrm{Tr}_{M_n'}(c_n)(\beta_n - \lambda_n) = \mathrm{Tr}_{M_n'}(c_n e_n - c_n\eta_n)$.

We then get

$$\mathrm{Tr}_{M_n'}(c_n)^{1/2}|\beta_n - \lambda_n| \leqslant \mathrm{Tr}_{M_n'}(|e_n - \eta_n|^2)^{1/2}.$$

From (1) in the proof of Lemma 2.2. i) and (3), it follows that $|\beta_n - \lambda_n| \to 0$ as $n \to \infty$; and $\lambda_n \to 1$ (cf. (2)) so $\beta_n \to 1$ as $n \to \infty$; and $\mathrm{Tr}_M(1 - c_n) \to 0$ as $n \to \infty$.

Using Proposition 1.16, we obtain $[M_{c_n} : (M_n)_{c_n}] \to 1$ as $n \to \infty$. But from Jones [2], we know that the index is either equal to 1 or greater than 2. So, for n large enough, $[M_{c_n} : (M_n)_{c_n}] = 1$, i.e. $M_{c_n} = (M_n)_{c_n}$.

2.5. Lemma. *Let M be a factor of type II_1. Let $(M_n)_{n\in\mathbf{N}}$ be an increasing sequence of von Neumann subalgebras of M such that $M = \overline{\bigcup M_n}$. If $M_n' \cap M$ is a factor or if $M_n' \cap M$ is of finite dimension, the γ_n in Lemma 2.2 can be chosen in $M_n' \cap (\bigcup_{p\in\mathbf{N}} M_p)$.*

Proof. If $M_n' \cap M$ is of finite dimension, it is clear that $M_n' \cap M = M_n' \cap M_p$ for p large enough.

If $M_n' \cap M$ is not of finite dimension, but is a factor, it is a factor of type II_1. Let $\varepsilon > 0$, there exists $g_n \in M_n' \cap M$ such that $\mathrm{tr}_M(\gamma_n) - \varepsilon < \mathrm{tr}_M(g_n) < \mathrm{tr}_M(\gamma_n)$. Then, for p large enough there is a projection $g_{n,p} \in M_n' \cap M_p$ such that $\mathrm{tr}_M(\gamma_n) - \varepsilon <$

$< \mathrm{tr}_M(g_{n,p}) < \mathrm{tr}_M(\gamma_n)$; $M'_n \cap M$ being a factor, $g_{n,p} < \gamma_n$ in $M'_n \cap M$. We then have $\mathrm{Tr}_{M'_n}(g_{n,p}) < \infty$ and $\mathrm{tr}_M(g_{n,p}) > \mathrm{tr}_M(\gamma_n) - \varepsilon$. Q.E.D.

2.6. COROLLARY. *Let M be a factor of type* II_1. *Let $(M_n)_{n\in\mathbf{N}}$ be an increasing sequence of von Neumann subalgebras of M such that $M = \overline{\bigcup M_n}$. If $M'_n \cap M$ is a factor or if $M'_n \cap M$ is of finite dimension, the sequence (M_n) must be stationary.*

Proof. Choose $(\gamma_n)_{n\in\mathbf{N}}$ as in Lemma 2.5. Let n be such that $\mathrm{tr}_M(1-\gamma_n) < 1/2$, we get $(M_k)_{\gamma_n} = M_{\gamma_n}$ for $k \geqslant k_0$, and $\gamma_n \in M'_n \cap M_p$ for $p \geqslant p_0$; $\mathrm{tr}_M(1-\gamma_n) < \mathrm{tr}_M(\gamma_n)$ so there is a unitary $u \in M_p$ such that $1 - \gamma_n = u\gamma_n u^*$. Then each element of M can be written $x = (1-\gamma_n)x(1-\gamma_n) + \gamma_n x \gamma_n + \gamma_n x(1-\gamma_n) + (1-\gamma_n)x\gamma_n$. So $x \in M_j$ for $j \geqslant \sup(k_0, p) = j_0$, i.e. $M = M_j$ $\forall j \geqslant j_0$.

2.7. REMARK. In general, we cannot get the result of Corollary 2.6. Indeed, let M be a II_1 factor, let e_n be a sequence of projections in M such that $\mathrm{tr}_M(e_n) \to 1$. Let $M_n = M_{e_n} + (1-e_n)\mathbf{C}$, then $M = \overline{\bigcup M_n}$.

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