J. OPERATOR THEORY
21(1989), 255–271

© Copyright by INCREST, 1989

APPROACHING INFINITY IN C^*-ALGEBRAS

CHARLES A. AKEMANN, JOEL ANDERSON and GERT K. PEDERSEN

1. INTRODUCTION

If X is a locally compact but non-compact Hausdorff space, there is a well-defined and useful notion of "approaching infinity". Thus a sequence (x_n) in X approaches infinity if $\lim \varphi(x_n) = 0$ for every φ in $C_0(X)$, i.e., if (x_n) is eventually outside each compact subset of X. More generally we say that a sequence (X_n) of subsets of X approaches infinity if $\lim\|\varphi|X_n\| = 0$ for every φ in $C_0(X)$, i.e., if $X_n \cap K = \emptyset$, eventually, for each compact subset K of X.

Regarding a non-commutative, non-unital C^*-algebra A as a generalization of $C_0(X)$, we ask here how the notions of approaching infinity translate best and what consequences the notions have. As applications we prove an infinite version of Kadison's transitivity theorem (Theorem 4.7), and a result (Corollary 4.4) that shows when, given a sequence $\{f_n\}$ of pure states of A, a maximal abelian C^*-subalgebra C of A can be found, such that each $f_n|C$ is a pure state on C with unique state extension to A.

The first difficulty one encounters in the non-commutative case is the break-down of the easy relation between points and sets. Baldly put, a set is no longer a collection of points. The reason for this is that, following Gelfand, we like to think of the pure states of A as the analogues of points; whereas, following von Neumann, we like to think of projections as the analogues of sets. To clarify this last point in a C^*-algebraic context, we first identify A with its image in the enveloping von Neumann algebra A^{**}. Then, in order to maintain both generality and a close analogy with the commutative case, we consider the class $\mathcal{U}(A)$ of universally measurable elements in A^{**}. Recall from [12, 4.5.15 and 4.3.15] that $\mathcal{U}(A)$ is a strong* sequentially closed subspace of A^{**} containing A, and that $\mathcal{U}(A)$ is faithfully represented in the atomic representation of A. Thus every projection from $\mathcal{U}(A)$ may be regarded as the non-commutative analogue of a universally measurable subset of X.

Following the notation from [12] we let $S(A)$ denote the states of A, $Q(A)$ the quasi-states of A [12, p. 44], and $P(A)$ the pure states of A [12, p. 69]. Unless otherwise specified we endow $Q(A)$ and its subsets $S(A)$ and $P(A)$ with the weak* topology. Thus $Q(A)$ is a compact, convex space which is metrizable if and only if A is separable. Other notation is from [12]. The symbol ▨ denotes the end of a proof.

With these concepts we can formulate our main definitions. We say that a net (f_α) of pure states *approaches infinity* if $\lim f_\alpha(a) = 0$ for every a in A, i.e. if $\lim f_\alpha = 0$. We say that a net (p_α) of projections in $\mathscr{U}(A)$ *approaches infinity* if $\lim \|ap_\alpha\| = 0$ for every a in A.

2. APPROACHING INFINITY, THE BASICS

In the commutative case all possible notions of approaching infinity coincide. However, if A is a non-commutative, non-unital C^*-algebra, there are several distinct ways of approaching infinity, apart from the one we have chosen. In this section we look at some of them and prove the easier implications. It will be increasingly clear along the way that approaching infinity almost inevitably requires us to choose a direction in A in the form of an (maximal) abelian subalgebra (MASA). Presumably the complications arising from the structure of the MASA's in A are the really hard part of the theory of approaching infinity.

We have defined the universally measurable elements $\mathscr{U}(A)$ as the complexification of what is called $\mathscr{U}(A)$ in [12, 4.3.11]. Since $\mathscr{U}(A)$ has not been shown to be a C^*-algebra, and since products do occur in our results, we include (somewhat out of context) a lemma that saves the situation.

2.1. Lemma. *If $\mathscr{U}(A)$ denotes the complex vector space of universally measurable elements in A^{**}, then the set*

$$\mathscr{C} = \{a \in \mathscr{U}(A) \mid a\mathscr{U}(A) + \mathscr{U}(A)a \subset \mathscr{U}(A)\}$$

is a strong sequentially closed C^*-subalgebra of A^{**} containing the multiplier algebra $M(A)$.*

Proof. If $a \in M(A)$, $\|a\| \leqslant 1$, and $x \in \mathscr{U}(A)_{sa}$, we claim that $axa^* \in \mathscr{U}(A)_{sa}$. Indeed, if $f \in Q(A)$ then $f(a \cdot a^*) \in Q(A)$, so by definition there exist, for each $\varepsilon > 0$, elements y and z in $(A_{sa})^m$ such that

$$-y \leqslant x \leqslant z \quad \text{and} \quad f(a(z + y)a^*) < \varepsilon.$$

Since aya^* and aza^* both belong to $(A_{sa})^m$, and

$$-aya^* \leqslant axa^* \leqslant aza^*, \quad f(aza^* + aya^*) < \varepsilon,$$

it follow that $axa^* \in \mathscr{U}(A)_{sa}$. The case of an ordinary product follows by polarization: if $a \in M(A)$ and $x \in \mathscr{U}(A)$ then

$$4ax^* = \sum_{k=0}^{3} i^k(1 + i^k a)x(1 + i^k a)^* \in \mathscr{U}(A).$$

Finally, if $\{a_n\}$ is a strong* convergent sequence in $\mathscr{C}$, then the limit point belongs to $\mathscr{C}$ because $\mathscr{U}(A)$ is strong* sequentially closed by [12, 4.5.15]. It is easy to check that $\mathscr{C}$ is a C^*-algebra. ▨

If $f \in Q(A)$ and $a \in A^{**}$ we say that f is *definite* on a if $|f(a)|^2 = \|f\| f(a^*a)$. Using the Cauchy-Schwarz inequality, this is seen to imply that $f(ba)\|f\| = f(b)f(a)$ for every b in A^{**}. We note that if $f \in S(A)$, $a \in A^{**}$ with $0 \leqslant a \leqslant 1$, and $f(a) = 1$, then f is definite on a.

2.2. PROPOSITION. *If (p_α) is a net of projections in $\mathscr{U}(A)$ the following conditions are equivalent.*

(1) *(p_α) approaches infinity.*

(2) $\lim f_\alpha = 0$ *for every net (f_α) in $S(A)$ such that $f_\alpha(p_\alpha) = 1$ for each α.*

(3) $\lim f_\alpha = 0$ *for every net (f_α) in $P(A)$ such that $f_\alpha(p_\alpha) = 1$ for each α.*

Proof. The estimate

$$|f_\alpha(a)| = |f_\alpha(ap_\alpha)| \leqslant \|ap_\alpha\|,$$

which is valid for every a in A^{**} and $(f_\alpha) \subset S(A)$ such that $f_\alpha(p_\alpha) = 1$ for each α, shows that (1) implies (2), and evidently (2) implies (3).

To show that (3) implies (1), fix a in A and choose (by [12, 4.3.15] and Lemma 2.1) for each α some g_α in $P(A)$ such that

$$g_\alpha(p_\alpha a^* a p_\alpha) \geqslant \frac{1}{2}\|p_\alpha a^* a p_\alpha\|.$$ ▨

Set $f_\alpha \in g_\alpha(p_\alpha)^{-1} g_\alpha(p_\alpha \cdot p_\alpha)$, and note that $f_\alpha \in P(A)$ since it is equivalent to g_α [12, 3.13.2]. Since $f_\alpha(p_\alpha) = 1$ for every α we have by assumption (3) that

$$\|ap_\alpha\|^2 \leqslant 2g_\alpha(p_\alpha)f_\alpha(a^*a) \to 0.$$

If A is separable, $Q(A)$ has a metric d. Thus the closed subsets of $Q(A)$ can be given the Hausdorff metric d_H, viz.

$$d_H(E, F) = \inf\{\varepsilon > 0 \mid E \subset F_\varepsilon,\ F \subset E_\varepsilon\},$$

where F_ε denotes the set of elements f in $Q(A)$ for which there exists a g in F with $d(f, g) < \varepsilon$ (and similarly for E_ε).

We wish to apply this to the *facial supports* $F(p)$ of certain projections p in $\mathcal{U}(A)$. Here

$$F(p) = \{f \in Q(A) \mid f(p) = \|f\|\};$$

and $F(p)$ is closed in $Q(A)$ if and only if p is a *closed projection* in A^{**}. This by definition means that $1 - p$ is *open*, i.e. supports a hereditary C^*-subalgebra B of A. Thus

$$B = (1 - p)A^{**}(1 - p) \cap A \quad \text{and} \quad 1 - p \in B^{**},$$

[12, 3.11.9-10]. Note that since $1 - p \in (A_+)^m$, both p and $1 - p$ belong to $\mathcal{U}(A)$.

2.3. PROPOSITION. *If A is separable and (p_n) is a sequence of closed projections in A^{**}, then (p_n) approaches infinity if and only if the corresponding sequence $(F(p_n))$ of facial supports converges to $\{0\}$ in the Hausdorff metric on Q.*

Proof. If $(F(p_n))$ converges to $\{0\}$ and (f_n) is a sequence in $S(A)$ with $f_n(p_n) = 1$ for each n, then $f_n \in F(p_n)$, so that $f_n \to 0$. Hence (p_n) approaches infinity by Proposition 2.2.

Conversely, assume that (p_n) approaches infinity. If $d_H(F(p_n), \{0\}) > \varepsilon$ for some $\varepsilon > 0$ and infinitely many n, there are f_n in $F(p_n)$ with $d(f_n, 0) > \varepsilon$. However, $f_n(p_n)^{-1}f_n \to 0$ by Proposition 2.2, whence $f_n \to 0$, a contradiction. Thus $\lim d_H(F(p_n), \{0\}) = 0$, as desired. ▨

2.4. LEMMA. *For a projection p in A^{**} the following conditions are equivalent.*

(1) *$F(p) \cap S(A)$ is closed $Q(A)$.*

(2) *p is a closed projection in $(\tilde{A})^{**}$, where $\tilde{A} = A + \mathbf{C}1$.*

(3) *p is a closed projection in A^{**}, and $p = ep$ for some positive, norm one element e in A.*

Proof. Each f in $Q(\tilde{A})$ has the uniquely determined form $f = g + \alpha f_\infty$, where $g \in Q(A)$, $g = f|A$, $\|f\| = \|g\| + \alpha$, and f_∞ is the unique state of $\tilde{A}$ annihilating A. Since $p \in A^{**}$, $f_\infty(p) = 0$, so that $f(p) = g(p)$. With

$$K = \{f \in Q(\tilde{A}) \mid f(p) = 1\}$$

it follows that $f \in K$ if and only if $g(p) = 1$, whence $\alpha = 0$ and $g \in S(A)$. Consequently

$$K = F(p) \cap S(A).$$

On the other hand it is easy to see that the facial support $\tilde{F}(p)$ of p, regarding p as an element in $\tilde{A}^{**}$, via the inclusion $A \subseteq \tilde{A}$, is the convex hull of $\{0\}$ and K.

If (1) is satisfied, then K is compact, whence also $\tilde{\mathrm{F}}(p)$ is compact in $Q(\tilde{A})$. Thus p is closed in $(\tilde{A})^{**}$. Conversely, if (2) holds we know that $\tilde{\mathrm{F}}(p)$ is compact, and thus

$$K = \tilde{\mathrm{F}}(p) \cap S(\tilde{A})$$

is compact; which means that (1) is satisfied.

Given (1) and (2) we first note that p is closed in A^{**}, because $\mathrm{F}(p) = \mathrm{F}(p) \cap \cap Q(A)$ is closed. To find the element e we apply the generalized Urysohn Lemma [4, Theorem 1.1] to the orthogonal projections p and p_∞, where p_∞ is the support projection of f_∞ in $(\tilde{A})^{**}$, i.e. the minimal projection p_∞ for which $f_\infty(p_\infty) = 1$. Since f_∞ is a pure state, p_∞ is a closed and minimal projection. Thus we obtain e in $\tilde{A}_+$ of norm one with $ep_\infty = 0$ and $ep = p$. Since $f_\infty(e) = f_\infty(ep_\infty) = 0$, we see that $e \in A$, so that (3) follows.

Assuming that (3) holds, let (g_α) be a net in $\mathrm{F}(p) \cap S(A)$ converging to some g in $Q(A)$. Since

$$g(e) = \lim g_\alpha(e) = \lim g_\alpha(ep) = \lim g_\alpha(p) = 1,$$

we see that $g \in S(A)$; and since p is closed in A^{**}, $\mathrm{F}(p)$ is closed, so $g \in \mathrm{F}(p)$. Thus (1) holds. ▨

A (closed) projection in A^{**} satisfying the conditions of the previous lemma is called a *compact* projection. It follows from [3, II.8] that projections of finite rank in A^{**} are compact, and we shall use this fact.

The following easy lemma (2.5) is the non-commutative analogue of the trivial fact that a compact set contained in an open subset of a topological space is compact in the subset. The next result (2.6), however, corresponds to Urysohn's Lemma, as it applies to a pair of closed, disjoint subsets of a locally compact (not necessarily normal) space, one of which is compact.

2.5. LEMMA. *If p is a compact projection in A^{**} and r is an open projection in A^{**} supporting the hereditary C^*-subalgebra B (*i.e. $B = rA^{**}r \cap A$*), such that $p \leqslant r$, then $p \in B^{**}$ and p is compact in B^{**}.*

Proof. As usual we identify B^{**} with the weak* closure of B in A^{**}, so that $B^{**} = rA^{**}r$, cf. [12, 3.11.9]. Since

$$\mathrm{F}(p) \cap S(B) = \mathrm{F}(p) \cap S(A)$$

by the unique state extension property [12, 3.1.6], the compactness of p relative to B follows immediately from Lemma 2.4 and the fact that the $\sigma(B^*, B)$-topology coincides with the $\sigma(A^*, A)$-topology on compact subsets of B^*.

2.6. PROPOSITION. *Suppose that p and q are closed, orthogonal projections in A^{**}, with p compact, and that $\|ap\| < \varepsilon$ for some a in A. There are then orthogonal, open projections r and s in A^{**} with $p \leqslant r$, $q \leqslant s$ and $\|ar\| < \varepsilon$.*

Proof. Since $pq = 0$, the open projection $1 - q$ dominates p. If B denotes the hereditary C^*-subalgebra of A supporting $1 - q$, then Lemma 2.5 shows that p is compact in B^{**}. By Lemma 2.4 we can find a positive, norm one element e in B such that $ep = p$ (and, of course, $eq = 0$).

Let $(u_\lambda)_{\lambda \in \Lambda}$ be a positive, increasing approximate unit for the hereditary C^*-subalgebra of A supported by the open projection $1 - p - q$. Then (u_λ) converges weakly to $1 - p - q$ in A^{**}, so that $(e(1 - u_\lambda)e)$ converges weakly down to $e(p + q)e = epe = p$. Since $\|ap\| < \varepsilon$, the closed sets

$$F_\lambda = \{f \in Q(A) \mid f(ae(1 - u_\lambda)ea^*) \geqslant \varepsilon^2\}$$

decrease to $\emptyset$. By compactness we can find λ such that $F_\lambda = \emptyset$, which means that $\|ae(1 - u_\lambda)ea^*\| < \varepsilon^2$. Let r and s be the spectral projections of $e(1 - u_\lambda)e$ corresponding to the relatively open intervals $]\delta^2, 1]$ and $[0, \delta^2[$, respectively, where δ is chosen so near to 1 that $\|ae(1 - u_\lambda)ea^*\| < \delta^2\varepsilon^2$. It follows from spectral theory that r and s are open, orthogonal projections in A^{**}. Since $e(1 - u_\lambda)ep = p$, we have $rp = p$, i.e. $p \leqslant r$. And since $e(1 - u_\lambda)eq = 0$ we have $q \leqslant s$. Finally,

$$\|ar\|^2 = \|ara^*\| \leqslant \delta^{-2}\|ae(1 - u_\lambda)ea^*\| < \varepsilon^2,$$

completing the proof. ∎

We now specialize the hypotheses of Proposition 2.2 by assuming that A is separable and that the projections $\{p_n\}$ are minimal and pairwise orthogonal. There is then a uniquely determined sequence (f_n) in $P(A)$ such that $f_n(p_n) = 1$ for each n.

Regarding A as an essential ideal in its multiplier algebra $M(A)$ [12, 3.12], each f_n has a unique extension $\tilde{f}_n$ in $P(M(A))$. Since $1 \in M(A)$, $S(M(A))$ is compact, and one may ask whether the limit points of $(\tilde{f}_n)$ in $S(M(A))$ are pure, assuming that (f_n) approaches infinity. Our failure in establishing this result prompted the extra condition (I) in the next theorem, with which we could prove that all limit points in $S(M(A))$ were pure [5, Proposition 4.10].

2.7. THEOREM. *Suppose that A is separable and (p_n) is a sequence of pairwise orthogonal, minimal projections in A^{**} supporting the pure states (f_n). Then for the following five conditions we have the implications*

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Leftrightarrow (5).$$

(1) *There is a strictly positive element b in A such that each f_n is definite on b and $\sum f_n(b) < \infty$.*

(2) (f_n) *approaches infinity and* $\sum_{n=m}^{\infty} p_n$ *is a closed projection for every* m.

(3) (f_n) *approaches infinity and there exist pairwise orthongonal, positive, norm one elements* (b_n) *in* A *such that* $f_n(b_n) = 1$ *for every* n.

(4) (f_n) *approaches infinity.*

(5) (p_n) *approaches infinity.*

Proof. Assume that (1) holds and that $\|b\| = 1$. Since each f_n is definite on b we have for every a in A and every continuous function φ on [0,1] that

$$f_n(a\varphi(b)) = f_n(a)f_n(\varphi(b)) = f_n(a)\varphi(f_n(b)). \tag{$*$}$$

Moreover,

$$\varphi(b)p_n = f_n(\varphi(b))p_n = \varphi(f_n(b))p_n. \tag{$**$}$$

Let (φ_k) be an increasing sequence of positive continuous functions on [0,1], such that $\varphi_k(0) = 0$ for all k, but $\lim \varphi_k(t) = 1$ for every $t > 0$. The strict positivity of b implies (as in the proof of [12, 3.10.5]) that $(\varphi_k(b))$ is an approximate unit for A. Since $\sum f_n(b) < \infty$ it follows that $\lim f_n(\varphi_k(b)) = 0$ for each k. These observations together with $(*)$ imply that $\lim f_n = 0$, i.e. (f_n) approaches infinity.

For the second assertion in (2), fix m and put $p = \sum_{n=m}^{\infty} p_n$. As noted before, p is closed if its facial support $\mathrm{F}(p)$ is closed in $Q(A)$, [12, 3.11.9]. Assume therefore that (g_α) is a net in $\mathrm{F}(p)$ with a limit g. Since $\sum f_n(b) < \infty$, there are, for each $\varepsilon > 0$, only finitely many n for which $f_n(b) > \varepsilon$. We can therefore arrange the sequence (φ_k) above, such that $\varphi_k(f_n(b))$ is either 0 or 1, and such that for each k there is only a finite set $N(k)$ of n's for which $\varphi_k(f_n(b)) = 1$. By $(**)$ this implies that

$$\varphi_k(b)p = \sum_{n=m}^{\infty} \varphi_k(f_n(b))p_n = \sum_{n \in N(k)} p_n,$$

which is a projection of finite rank in A^{**}, and therefore closed. Thus if we define

$$g_k(a) = g(\varphi_k(b)a\varphi_k(b)), \quad a \in A^{**},$$

then

$$g_k = \lim g_\alpha(\varphi_k(b) \cdot \varphi_k(b)) \in \mathrm{F}(\varphi_k(b)p) \subset \mathrm{F}(p).$$

However, since $(\varphi_k(b))$ is an approximate unit for A, it converges strongly to 1 in A^{**}. Therefore $g = \lim g_k$ in the $\sigma(A^*, A^{**})$-topology. As $\mathrm{F}(p)$ is norm closed and convex, it is also $\sigma(A^*, A^{**})$-closed. Consequently $g \in \mathrm{F}(p)$, as desired.

Assume now that (2) holds. Put $q = \sum_{n=2}^{\infty} p_n$ and note that q is closed and orthogonal to p_1. Since p_1 is compact in A^{**} it is also compact in B_1^{**} by Lemma 2.5, where

$$B_1 = (1-q)A^{**}(1-q) \cap A.$$

Thus $p_1 = e_1 p_1$ for some positive, norm one element e_1 in B_1 by (3) in Lemma 2.4. Let φ be the monotone, piecewise linear function on [0,1] such that $\varphi(t) = 0$ if $t \leqslant 1/2$ and $\varphi(t) = 1$ if $t \geqslant 3/4$. Take s_1 to be the spectral projection of e_1 corresponding to the relatively open interval [0,1/2], and let $b_1 = \varphi(e_1)$. Clearly $f_1(b_1) = 1$, $b_1 s_1 = 0$, and $s_1 p_n = p_n$ for all $n \geqslant 2$. By spectral theory s_1 is an open projection in A^{**}, and we let

$$A_1 = s_1 A^{**} s_1 \cap A.$$

Then $p_n \in A_1^{**}$ for $n \geqslant 2$ and $f_n \mid A_1$ is a pure state of A_1, again for $n \geqslant 2$. Thus a similar construction will allow us to choose a positive, norm one element b_2 in A_1 such that $f_2(b_2) = 1$, and a routine induction produces the sequence (b_n) required by condition (3). Thus (2) implies (3).

To show that (2) implies (1), let us first choose the elements (b_n) as above. Then put

$$A_0 = \{a \in A \mid f_n(a^*a) = f_n(aa^*) = 0 \quad \text{for all } n\}.$$

Let b_0 be a strictly positive element of norm one in the separable algebra A_0 and set $b = \sum_{n=0}^{\infty} 2^{-n-1} b_n$. Clearly each f_n is definite on b and $\sum f_n(b) = 1/2$. To show that b is strictly positive in A we merely note that its range projection $[b]$ is 1 in A^{**}, because

$$[b] = \vee [b_n] \geqslant [b_0] \vee \left(\sum_{n=1}^{\infty} p_n \right),$$

and $[b_0] \geqslant 1 - \sum p_n$. Thus (2) implies (1).

That (3) implies (4) is trivial, and the equivalence of (4) and (5) follows from Proposition 2.2 and the fact that for each n, f_n is the unique state that satisfies $f_n(p_n) = 1$. ▨

3. COUNTEREXAMPLES

In this section we show that the missing implications in Theorem 2.7 are false in general. We shall see in Section 4 that there are reasonable conditions under which they do hold.

First we need a result inspired by a theorem of Glimm [7, Theorem 2]. Recall that a C^*-algebra is *σ-unital* if it possesses a strictly positive element or, equivalently, if it has a countable approximate unit [12, 3.10.5]. All separable C^*-algebras are σ-unital, and commutative algebras of the form $C_0(X)$ are σ-unital precisely when X is σ-compact.

3.1. PROPOSITION. *If A is a non-unital, σ-unital C^*-algebra acting on a separable Hilbert space $\mathscr{H}$, there is an orthonormal basis (ξ_n) for $\mathscr{H}$ such that* $\lim(a\xi_n \mid \xi_n) = 0$ *for every a in A.*

Proof. We may evidently assume that A is acting non-degenerately on $\mathscr{H}$. If b is a strictly positive element in A of norm one, this implies that $b\zeta \neq 0$ for every unit vector ζ in $\mathscr{H}$. Since A is non-unital, $0 \in \mathrm{sp}(b)$, but we see from above that 0 cannot be an isolated point in the spectrum. We can therefore find a sequence of pairwise orthogonal, non-zero spectral projections for b corresponding to disjoint intervals approaching 0. This in turn produces an orthonormal sequence (ζ_n) in $\mathscr{H}$ such that $\lim\|b\zeta_n\| = 0$. Passing to a subsequence we may assume that $\|b\zeta_n\| \leqslant 2^{-n-1}$ for all n.

If $\mathrm{span}\{\zeta_n\}$ has finite co-dimension in $\mathscr{H}$ we are done. Otherwise choose an orthonormal sequence (ζ_n') such that $\{\zeta_n\} \cap \{\zeta_n'\}$ forms an orthonormal basis for $\mathscr{H}$. We now decompose $\mathscr{H}$ as an orthogonal sum of subspaces $\mathscr{H}_n$, such that $\dim\mathscr{H}_n = 2^n$, and such that each $\mathscr{H}_n$ is spanned by one vector ζ' from $\{\zeta_n'\}$ and $2^n - 1$ vectors ζ_k from $\{\zeta_n\}$, each with $k \geqslant 2^{n-1}$. A new basis $\{\xi_{nk}\}$ for $\mathscr{H}_n$ is obtained by transforming the original basis with a Hadamard matrix u of order 2^n. This u is a unitary each of whose entries is plus or minus $(\sqrt{2})^{-n}$. One construction, see [9], goes as follows: let

$$v = (\sqrt{2})^{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and define u as the n-fold tensor product of v with itself. We estimate

$$\|b\xi_{nk}\| \leqslant (\sqrt{2})^{-n}\left(\|b\zeta'\| + \sum_{m=1}^{2^n-1} \|b\zeta_{k(m)}\|\right) \leqslant$$

$$\leqslant (\sqrt{2})^{-n}(1 + (2^n - 1)2^{-n}) \leqslant (\sqrt{2})^{-n+2}.$$

Thus (dictionary) ordering all the ξ_{nk}'s into a single sequence (ξ_n), we have an orthonormal basis for $\mathscr{H}$ with $\lim\|b\xi_n\| = 0$. Since Ab is dense in A, it follows from the Cauchy-Schwarz inequality that $\lim(a\xi_n \mid \xi_n) = 0$ for every a in A. ▨

REMARK. Although it is not needed here, with a bit more work we can combine a slight strengthening of Glimm's idea [7, Theorem 2] with the Hadamard matrix technique used in Proposition 3.1 to prove the following result.

Given a separable C^*-algebra A acting on a separable Hilbert space H and a state f of A which annihilates each compact operator in A, there exists an orthonormal basis $\{\xi_n\}$ in H such that

$$f(a) = \lim(a\xi_n \mid \xi_n)$$

for every a in A.

3.2. EXAMPLE. Let $\mathscr{K}$ denote the C^*-algebra of compact operators on the separable, infinite-dimensional Hilbert space $\mathscr{H}$, and take A to be the C^*-algebra generated by $\mathscr{K}$ and q, where q is a projection on $\mathscr{H}$ such that both q and $1-q$ have infinite rank. By Proposition 3.1 we can find an orthonormal basis (ξ_n) for $\mathscr{H}$, such that the corresponding sequence (f_n) of pure vector states on A approaches infinity. If p_n denotes the projection along ξ_n, then (p_n) is a sequence of pairwise orthogonal, minimal projections in A^{**} contained in A and supporting the pure states (f_n). Thus condition (3) in Theorem 2.7 is satisfied.

We claim that $p = \sum p_n$ computed in A^{**} (which is isomorphic to $\mathbf{B}(\mathscr{H})\oplus\mathbf{C}$) is not a closed projection. If it were, then p, which is obviously an open projection in A^{**}, being a sum of elements from A_+, would be a multiplier of A by [12, 3.12.9]. Now $p = 1$ on $\mathscr{H}$, but $p \neq 1$ in A^{**}, since otherwise the partial sums of the p_n's would be an approximate unit for A. Therefore $pq \notin A$, a contradiction. Thus p is not closed and condition (2) in Theorem 2.7 fails.

3.3. EXAMPLE. With $\mathscr{H}$ and $\mathscr{K}$ as in 3.2, let A be any separable, non-unital C^*-algebra with a faithful irreducible representation on $\mathscr{H}$ such that $A \cap \mathscr{K} = \{0\}$. Again by Proposition 3.1 we can find an orthonormal basis (ξ_n) for $\mathscr{H}$ such that the corresponding sequence (f_n) of pure vector states on A approaches infinity. Thus (f_n) and the corresponding sequence (p_n) of mutually orthogonal, minimal projection in A^{**} satisfy conditions (4) and (5) of Theorem 2.7. However, condition (3) is not satisfied. For if (b_n) were a sequence of pairwise orthogonal, norm one elements in A with $f_n(b_n) = 1$ for every n, then

$$f_n(b_m) = f_n(b_n b_m) = 0$$

for $n \neq m$. Thus each b_n would be a rank one projection on $\mathscr{H}$, in contradiction with $A \cap \mathscr{K} = \{0\}$.

3.4. EXAMPLE. Let I be the (restricted) direct sum of matrix algebras M_{2^n}, $n \in \mathbf{N}$, regarded as a C^*-algebra of operators on $\mathscr{H} = \oplus\, \mathbf{C}^{2^n}$; and let A be the C^*-algebra on $\mathscr{H}$ generated by I and q, where is the projection on $\oplus\, \mathbf{C}$, and each $\mathbf{C}$ is identified with the subspace of $\mathbf{C}^{2^n}$ obtained by projecting on the first coordinate. Thus we are repeating the construction from Example 3.2 with $\mathscr{K}$ replaced by I. Using the Hadamard matrices as in the proof of Proposition 3.1 we can find an orthonormal basis $\{\zeta_{nk} \mid 1 \leqslant k \leqslant 2^n\}$ for $\mathbf{C}^{2^n}$ such that $\|q\zeta_{nk}\| = 2^{-n}$ for every k. Arranging all these in a sequence we obtain an orthonormal basis (ξ_n) for $\mathscr{H}$ such that $\lim(a\xi_n \mid \xi_n) = 0$ for every a in A. Thus the corresponding sequences (f_n) and (p_n) of pure states and rank one projections both approach infinity. However, exactly as in Example 3.2 we see that $p = \sum p_n$ is not a closed projection in A^{**}, because it is not a multiplier of A. This is an example of an algebra in which condition (3) from Theorem 2.7 is satisfied while conditions (1) and (2) are not. It differs from the previous one in that the pure states are not all unitarily equivalent; in fact each equivalence class has 2^n elements.

4. THE NEARLY INEQUIVALENT CASE AND INFINITE TRANSITIVITY

In Exemples 3.2 and 3.3 all of the pure states f_n are unitarily equivalent, and in Example 3.4 the equivalence classes are finite but unbounded. This suggests that we consider the case in which the sequence (f_n) consists of *nearly inequivalent* pure states, which by definition means that there is a uniform bound on the size of the equivalence classes. In this case all five conditions of Theorem 2.7 are equivalent. The same idea will yield both a generalization of Kadison's transitivity theorem [8, Corollary 7] and a method, applicable to a C^*-algebra A with an orthogonal sequence (f_n) of pure states satisfying certain conditions, which produces a maximal abelian C^*-subalgebra C of A such that each $f_n \mid C$ is pure on C and has a unique state extension to A. The latter generalizes [1, Theorem 1].

STANDING ASSUMPTIONS. From now through Theorem 4.3 we shall assume that A is a non-unital C^*-algebra, (f_n) an orthogonal sequence of pure states of A which are approaching infinity, and (p_n) the corresponding sequence of minimal support projections in A^{**}. We shall further assume that the unitary equivalence classes $\{\Gamma_j\}$ of $\{f_n\}$ each have cardinality less than or equal to some number N.

4.1. LEMMA. *Under our standing assumptions, if* $q_j = \sum_{n \in \Gamma_j} p_n$, *then the sequence* (q_j) *approaches infinity.*

Proof. If $a \in A$ then

$$\|aq_j\|^2 = \|aq_j a^*\| \leqslant \sum_{n \in \Gamma_j} \|ap_n a^*\| =$$

$$= \sum_{n \in \Gamma_j} \|p_n a^* a p_n\| = \sum_{n \in \Gamma_j} f_n(a^* a) \leqslant N f_{n(j)}(a^* a)$$

for some $n(j)$ in Γ_j. Since the subsequence $(f_{n(j)})$ approaches infinity, it follows that $\lim \|aq_j\| = 0$, as desired. ▨

4.2. PROPOSITION. *Let (q_n) be a sequence of closed projections in A^{**} with pairwise orthogonal central covers. If (q_n) approaches infinity, then $q = \sum q_n$ is a closed projection in A^{**}.*

Proof. Let c_n denote the central cover of q_n. By [12, 3.11.9] we must show that the facial support $\mathrm{F}(q)$ of q is closed in $Q(A)$, so we consider a net (g_α) in $\mathrm{F}(q)$ converging to some g in $Q(A)$. For each n and α set $g_{n\alpha} = g_\alpha(c_n \cdot)$, and write $K_n = \mathrm{F}(q_n)$. By assumption each K_n is closed ($=$ compact), and $g_{n\alpha} \in K_n$ for each α because

$$g_{n\alpha}(1-q_n) = g_\alpha(c_n(1-q_n)) = g_\alpha(c_n(1-q_n)q) =$$

$$= g_\alpha(\sum_m c_n(1-q_n)q_m) = g_\alpha((1-q_n)q_n) = 0,$$

as $c_n q_m = 0$ for $n \neq m$. Thus for each fixed n, $(g_{n\alpha})$ converges to some h_n in K_n. If (e_λ) denotes an increasing approximate unit for A, then for each m we have

$$\sum_{n=1}^{m} \|h_n\| = \sup_\lambda \sum_{n=1}^{m} h_n(e_\lambda) = \sup_\lambda \lim_\alpha \sum_{n=1}^{m} g_{n\alpha}(e_\lambda) \leqslant$$

$$\leqslant \sup_\lambda \lim_\alpha g_\alpha(e_\lambda) = \sup_\lambda g(e_\lambda) = \|g\|.$$

It follows that $\sum h_n$ is norm convergent to an element h in $Q(A)$. Since $\|h_n\|^{-1}h_n \in K_n$ whenever $h_n \neq 0$, and $K_n \subset \mathrm{F}(q)$ for all n, we see that

$$h = \sum \|h_n\|(\|h_n\|^{-1}h_n) \in \mathrm{F}(q),$$

because $\mathrm{F}(q)$ is convex and norm closed.

If $a \in A_+$ then

$$h(a) = \sum h_n(a) = \sum \lim g_{n\alpha}(a) \leqslant \lim \sum g_{n\alpha}(a) =$$

$$= \lim g_\alpha(a) = g(a),$$

so that $h \leqslant g$. On the other hand, since (q_n) approaches infinity we may for each a in A_+ and $\varepsilon > 0$ choose m such that $\|aq_n\| < \varepsilon$ for $n > m$. This means that

$$g(a) = \lim g_\alpha(a) = \lim \sum g_{n\alpha}(a) = \lim \sum g_{n\alpha}(aq_n) =$$

$$= \lim \left(\sum_{n=1}^{m} g_{n\alpha}(a) + \sum_{n>m} \varepsilon \|g_{n\alpha}\| \right) \leqslant \sum_{n=1}^{m} h_n(a) + \varepsilon.$$

Consequently $g(a) \leqslant h(a) + \varepsilon$, whence $g \leqslant h$. Taken all together we have $g = h \in \mathrm{F}(q)$, as desired. ∎

4.3. THEOREM. *Under our standing assumptions,* $\sum_{n=k}^{\infty} p_n$ *is closed in* A^{**} *for every* k. *Thus all five conditions in Theorem* 2.7 *are equivalent when* A *is separable.*

Proof. By Lemma 4.1 the sequence (q_j) approaches infinity, and since the q_j's have orthogonal central covers it follows from Proposition 4.2 that $\sum_{j=m}^{\infty} q_j$ is closed in A^{**} for every m.

Since each Γ_j is finite there is for every k only a finite number m of Γ_j's that contain elements p_m with $m \leqslant k$. This means that the projection

$$q = \sum_{n=k}^{\infty} p_n - \sum_{j=m+1}^{\infty} q_j$$

has finite rank in A^{**}. By [3, II.9] the orthogonal sum of a closed projection and a projection with finite rank $\left(\text{namely } \sum_{j=m+1}^{\infty} q_j \text{ and } q\right)$ is closed in A^{**}. ▨

Given a sequence (f_n) of pure states of a (separable) C^*-algebra A, consider the question of finding a maximal abelian C^*-subalgebra C of A, such that each $f_n \mid C$ is pure, i.e. multiplicative, on C and has a unique state extension to A. Clearly a necessary condition will be that the f_n's are pairwise orthogonal (even when restricted to C), i.e. the support projections (p_n) are pairwise orthogonal (and minimal) in A^{**}. Thus the question has the equivalent formulation: can we find a MASA C such that $\{p_n\} \subset C^{**}$? As noted in [4, Theorem II.9] the answer is "yes" whenever there is an orthogonal sequence (r_n) of open projections such that $p_n \leqslant r_n$ for every n. This can be seen from the fact that the separability of A allows us to find, for each n, a positive, norm one element e_n in A such that $p_n \leqslant e_n \leqslant r_n$ and such that p_n is the spectral projection of e_n corresponding to the eigenvalue 1, cf. [1, Theorem 1.1]. Any MASA C of A containing $\{e_n\}$ will have the desired properties. These observations suggest the following corollary to Theorem 4.3.

4.4. COROLLARY. *Let* (f_n) *be a sequence of pairwise orthogonal, nearly inequivalent pure states of a separable* C^**-algebra* A. *Denote by* K *the set of all accumulation points of* $\{f_n\}$ *in* $Q(A)$, *and let* B *denote the hereditary* C^**-subalgebra of elements* a *in* A *such that* $f(a^*a + aa^*) = 0$ *for every* f *in* K. *If each* $f_n \mid B$ *is pure state of* B, *there exists a maximal abelian* C^**-subalgebra* C *of* A *such that each* $f_n \mid C$ *is pure and has a unique state extension to* A.

Proof. As noted above, it suffices to find pairwise orthogonal, open projections (r_n) such that $f_n(r_n) = 1$ for each n. Since

$$B = \{a \in A \mid \lim f_n(a^*a + aa^*) = 0\},$$

we see that Theorem 4.3 and Theorem 2.7 (3) applies to B and the sequence $(f_n \mid B)$ to give a sequence (b_n) of pairwise orthogonal, positive norm one elements in B such that $f_n(b_n) = 1$ for every n. Taking r_n to be the range projection of b_n we get the desired sequence of open projections. ▨

4.5. EXAMPLE. With $\mathscr{H}$ and $\mathscr{K}$ as in Example 3.2, let (q_n) be a sequence of pairwise orthogonal projections in $\mathbf{B}(\mathscr{H})$ with infinite rank, and let A be the C^*-algebra generated by $\mathscr{K}$ and (q_n). By Proposition 3.1 we can find an orthonormal basis (ξ_n) for $\mathscr{H}$ such that $\lim(a\xi_n \mid \xi_n) = 0$ for every a in A. Since $\mathscr{K}$ is an ideal in A with $A/\mathscr{K}$ isomorphic to c_0, there is a sequence (g_n) of pairwise orthogonal pure states of A, each of which satisfies $g_n \mid \mathscr{K} = 0$ and $g_n(q_n) = 1$. Now define f_n to be g_k if $n = 2k$ and $f_n(a) = (a\xi_k \mid \xi_k)$, $a \in A$, if $n = 2k - 1$. Clearly (f_n) is a sequence of pairwise orthogonal pure states of A approaching infinity; but the conclusions of Corollary 4.4 do not hold. Indeed, if we could find a maximal abelian C^*-algebra C as specified, then C^{**} would contain each of the projections p_k along ξ_k. Hence $\{p_k\} \subset C^{**} \cap A = C$, which means that C must be the diagonal operators in A, i.e. the C^*-algebra generated by $\{p_k\}$. But then $C \subset \mathscr{K}$, which means that $f_n \mid C = 0$ for all even n, a contradiction. This shows that the condition of near inequivalence cannot be omitted from Corollary 4.4.

We have used an example with a sequence (f_n), where one of the equivalence classes, $\{f_{2k}\}$, is infinite. An elaborated version of Example 3.4 could be constructed (replacing the projection q with an orthogonal sequence (q_n) as in the present example), in which each equivalence class would be finite — but unbounded.

4.6. REMARK. In the proof of Theorem 4.3 the assumption of near inequivalence was only used to show that the sequence (q_j) of support projections for the equivalence classes approaches infinity (by Lemma 4.1). Thus we could establish the equivalence of the five conditions in Theorem 2.7 under the slightly weaker hypo theses that the equivalence classes are finite (but maybe unbounded), and the sequence (q_j) of sums of equivalent p_n's approaches infinity. We shall use this observation in our next result, which generalizes Kadison's transitivity theorem (in the form given by Glimm and Kadison in [8, Corollary 7]).

4.7. THEOREM. *Let (π_n) be a sequence of pairwise inequivalent, irreducible representations of a separable non-unital C^*-algebra A, and suppose that for each n we are given a projection q_n (on the representation space $\mathscr{H}_n$) of finite rank, such that $\lim\|\pi_n(a)q_n\| = 0$ for each a in A. For every bounded sequence (a_n) of operators (= matrices) on $(q_n\mathscr{H}_n)$* (i.e. $a_n = q_n a_n q_n$), *there is an element b in $M(A)$ such that $\pi_n(b)q_n = a_n$ for every n.*

We can take $\|b\| = \sup\|a_n\|$; and if all the a_n's are self-adjoint (resp. positive, resp. unitary) on $q_n\mathscr{H}_n$ we can take b to be self-adjoint (resp. positive, resp. unitary) in $M(A)$. Finally, if $\lim\|a_n\| = 0$ we can take b in A.

Proof. Using once more that the (reduced) atomic representation of A is faithful on $\mathscr{U}(A)$, which contains all closed projections, we may identify (q_n) with a sequence of pairwise centrally orthogonal projections of finite rank (hence closed) in A^{**} approaching infinity. Thus by Proposition 4.2 we know that $p_k = \sum_{n=k}^{\infty} q_n$ is closed in A^{**} for every k.

Let e_0 be a strictly positive element in the separable, hereditary C^*-subalgebra

$$B_0 = (1 - p_1)A^{**}(1 - p_1) \cap A$$

[12, 3.11.10]. (Incidentally, this is the only place in the proof where the separability of A is used.) Now proceed by induction. Assume that for all $j < n$ we have chosen pairwise orthogonal, open projections r_j, open projections s_j, positive, norm one elements e_j in A and elements b_j in A satisfying the conditions:

(1) $b_j q_j = a_j, \quad \|b_j\| = \|a_j\|$.

(2) $b_j r_j = r_j b_j = b_j$.

(3) $q_j \leqslant e_j \leqslant r_j$.

(4) $\|e_0 r_j\| < 2^{-j}$.

(5) $s_{j-1} \geqslant s_j$ (taking $s_0 = 1$).

(6) $p_{j+1} \leqslant s_j, \quad r_j \leqslant s_{j-1}, \quad r_j s_j = 0$.

Let $A_n = s_{n-1}A^{**}s_{n-1} \cap A$, and apply Proposition 2.6 with A, p, q and ε replaced by A_n, q_n, p_{n+1}, e_0 and 2^{-n}. Note that this is legitimate since q_n is compact in A_n^{**} by Lemma 2.5, and since $e_0 q_n = 0$. We obtain orthogonal, open projections r_n and s_n in A_n^{**}, i.e. $r_n + s_n \leqslant s_{n-1}$, such that $q_n \leqslant r_n$, $p_{n+1} \leqslant s_n$ and $\|e_0 r_n\| < 2^{-n}$. Thus (4), (5) and (6) are satisfied for n. By Lemma 2.4, applied to $r_n A^{**} r_n \cap A$, courtesy of Lemma 2.5, we find a positive, norm one element e_n in A, such that $e_n q_n = q_n$ and $e_n r_n = e_n$. Thus $q_n \leqslant e_n \leqslant r_n$, so that (3) is satisfied for n. We now apply Kadison's theorem (in the Lusin type form [12, 2.7.5]) to the finite rank projection q_n in $r_n A^{**} r_n$ to obtain an element b_n in $r_n A^{**} r_n \cap A$ such that $b_n q_n = a_n$ and $\|b_n\| = \|a_n\|$. For future use we note that if $a_n = a_n^*$ (resp. $a_n \geqslant 0$) we may assume that $b_n = b_n^*$ (resp. $b_n \geqslant 0$), which in particular means that b_n commutes with q_n. Thus also (1) and (2) are satisfied for n, and we have completed the induction.

Put $e = \sum_{n=0}^{\infty} 2^{-n} e_n$. Clearly the range projection of e in A^{**} is 1, because $e_n \geqslant q_n$ for all $n \geqslant 1$, and the range projection of e_0 is $1 - p_1$ by definition. Thus e is strictly positive in A. Set $b = \sum b_n$, where strong convergence in A^{**} is assured by the orthogonality of the bounded sequence (b_n), cf. (2). Note that $\|b\| = \sup\|b_n\| = \sup\|a_n\|$ by (1). From (2) and (4) we see that $\|b_n e_0\| < 2^{-n}$ and $\|e_0 b_n\| < 2^{-n}$ for all n, from which we conclude that $be_0 \in A$ and $e_0 b \in A$. Since $be_n = b_n e_n$ and $e_n b = e_n b_n$ by (2) and (3) it follows that

$$be = \sum b2^{-n}e_n = be_0 + \sum 2^{-n}b_n e_n \in A;$$

and similarly $eb \in A$. Since eA and Ae are dense in A this implies that $b \in M(A)$. Furthermore, by (1), (2) and (3)

$$bq_n = (\sum b_k)r_n q_n = b_n q_n = a_n$$

for every n.

If all the a_n's are self-adjoint (resp. positive) we can choose the b_k's similarly; whence $b = b^*$ (resp. $b \geqslant 0$) in $M(A)$. If each a_n is a unitary matrix, it has the form $\exp ic_n$ for some self-adjoint matrix c_n with $\|c_n\| \leqslant \pi$. Choose $b_n = b_n^*$ in A with $b_n q_n = c_n$ and form $b = \sum b_n$ in $M(A)$ as above. Then the element $u = \exp ib$ is unitary in $M(A)$, and since b commutes with each q_n we have

$$uq_n = (\exp ic_n)q_n = (\exp ic_n q_n)q_n = a_n,$$

as desired. Finally, if $\lim\|a_n\| = 0$ it is immediate that the orthogonal sum $\sum b_n$ is norm convergent by (1), so that $b \in A$. ▨

4.8. Corollary. *Let A be a separable C^*-algebra with Hausdorff primitive ideal space $\check{A}$. Let (P_n) be a convergent sequence in $\check{A}$ with limit P, where $P \neq P_n$ for all n, and let π_n (resp. π) be an irreducible representation of A on $\mathscr{H}_n$ with kernel P_n (resp. P). For every sequence (a_n) of finite rank operators on $(\mathscr{H}_n)$ with corresponding range projections (q_n) and with $\lim\|a_n\| = 0$, there is an element b in P such that $\pi_n(b)q_n = a_n$ for every n.*

Proof. If $a \in P$, then

$$\lim\|\pi_n(a)q_n\| \leqslant \lim\|\pi_n(a)\| = \|\pi(a)\| = 0,$$

because the norm varies continuously over $\check{A}$ by Kaplansky's theorem [12, 4.4.5] Thus the projections (q_n) approaches infinity in P^{**}, so that Theorem 4.7 applies to P. ▨

The authors wish to thank the referee for a critical reading of an earlier version of this article. His constructive suggestions were very helpful for the rewriting.

Partially supported by NSF and MSRI.

REFERENCES

1. Akemann, C. A., Approximate units and maximal abelian C^*-subalgebras, *Pacific J. Math.*, 33(1970), 543–550.
2. Akemann, C. A., A Gelfand representation theory for C^*-algebras, *Pacific J. Math.*, 39(1971), 1–11.

3. AKEMANN, C. A., The general Stone-Weierstrass problem, *J. Funct. Anal.*, 4(1969), 277–294.
4. AKEMANN, C. A., Left ideal structure of C^*-algebras, *J. Funct. Anal.*, 6(1970), 305–317.
5. AKEMANN, C. A.; ANDERSON, J.; PEDERSEN, G. K., Excising states of C^*-algebras, *Canad. J. Math.*, **38**(1986), 1239–1260.
6. ANDERSON, J., Extensions, restrictions and representations of states of C^*-algebras, *Trans. Amer. Math. Soc.*, **249**(1979), 303–340.
7. GLIMM, J., A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.*, 72(1960), 216–244.
8. GLIMM, J.; KADISON, R. V., Unitary operators in C^*-algebras, *Pacific J. Math.*, **10**(1960), 547–556.
9. HADAMARD, J., Résolution d'un question relative aux déterminants, *Bull. Sci. Math.*, **17**(1893), 240–246.
10. KADISON, R. V.; RINGROSE, J. R., *Fundamentals of the theory of operator algebras. I–II*, Academic Press, New York, 1986.
11. KELLEY, J. L., *General topology*, Van Nostrand, Princeton, 1955.
12. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, London–New York, 1979.
13. SAKAI, S., *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

CHARLES A. AKEMANN
Mathematics Department,
University of California,
Santa Barbara, CA 93106,
U.S.A.

JOEL ANDERSON
Mathematics Department,
Penn. State University,
University Park, PA 16802,
U.S.A.

GERT K. PEDERSEN
Mathematics Institute,
Universitetsparken 5,
2100 Copenhagen
Denmark.

Received February 25, 1987; revised June 5, 1988.