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ON SCHRÖDINGER OPERATORS WITH DISTRIBUTIONAL POTENTIALS

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In memoriam Elvira Herczyńska, Craiova 1922 – Warszawa 1976

INTRODUCTION

The problem of defining self-adjoint Schrödinger operator $l = -\Delta + V$ in $L^2(\mathbf{R}^n)$ for potentials V with severe local singularities has been thoroughly investigated (see for instance [12]). Among the approaches developed we may mention the study of essential self-adjointness of l (or of closability of appropriate form) on $C_0^\infty(\mathbf{R}^n)$ or on smaller domains, see [5, 6, 11, 19] and the references quoted there. This study has to be distinguished from earlier and not less natural perturbation methods, which deal with perturbations of self-adjoint operators and closed symmetric quadratic forms.

Perturbation methods are based on two abstract theorems, the Kato-Rellich theorem for operators and the KLMN theorem for forms [12]. It is easily checked that as applied to locally integrable potentials V both theorems cover for higher dimensions ($n \geqslant 5$) the same class of potentials, essentially $V \in L^{n/2}(\mathbf{R}^n)$ (this is the case $s = 0$ of Theorem 1 below). In lower dimensions, however, the KLMN theorem is much stronger, for instance it covers the case $V \in L^{3/2}(\mathbf{R}^3)$ and $V \in L^1(\mathbf{R})$ while the Kato-Rellich theorem requires (essentially) $V \in L^2(\mathbf{R}^n)$, $n \geqslant 1$. In current monographs [10, 12, 14] the strength of the KLMN theorem is illustrated with the example of the Dirac delta function δ for $n = 1$ (as the quadratic form in $L^2(\mathbf{R})$ associated with δ is not closable, the Kato-Rellich theorem cannot be applied). Such "potentials" are said to describe point interactions, that is interactions of zero range, and have been extensively studied because they provide an exactly solvable physical model (see [2, 3, 8] and the references quoted there). In particular Albeverio *et al.* [2] have proven some results on norm resolvent approximation of Schrödinger operators with point interactions by operators with locally integrable potentials. Relying on a simple bound derived here we are able to strengthen and simplify these results below.

On the other hand, for $n \geqslant 2$, δ does not satisfy the KLMN Theorem. Nevertheless by some limiting procedure (or by closing appropriate forms) a self-adjoint Schrödinger operator with point interactions in $L^2(\mathbf{R}^3)$ (and also in higher dimensions) can be defined and Albeverio *et al.* [3] give a proof, a more subtle one, of norm resolvent approximation by operators with integrable potentials in this case also. We cannot treat this case with our methods.

Quite general results for operators defined as perturbations of $H_0 = -\Delta$ (and of slightly more general H_0) by relatively bounded forms were obtained by Herbst and Sloan [9]. They assume that the perturbation ("the distributional part of the perturbation" in their terminology) satisfies the condition of the KLMN Theorem and make no attempt to establish when this is the case. In a recent paper Tip [18] developed a theory of one dimensional Schrödinger operators perturbed by regular Borel, not necessarily real measures, obtaining among others a unique continuation theorem in this case.

While the example of point interactions is quite striking, the more general problem of which distributions can serve as "potentials" for a self-adjoint Schrödinger operator was left open. The aim of this note is to explore this problem in a systematic way. Given any distribution $T \in \mathscr{D}'(\mathbf{R}^n)$ we define a quadratic form $q_T[\varphi] = \langle T, |\varphi|^2 \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the action of distributions on test functions, with $Q(q_T) = C_0^\infty(\mathbf{R}^n)$ (notation is explained in Section 1). We assume that q_T is symmetric. We treat q_T as a perturbation of the quadratic form $q_{H_0}[\varphi] = \int |\nabla \varphi|^2 dx = (H_0\varphi, \varphi)$, associated with the self-adjoint operator $H_0 = -\Delta$ in $L^2(\mathbf{R}^n)$. The Schrödinger operator with distributional potential T, denoted $H_0 \dotplus T$, is then defined as the unique self-adjoint operator associated with the closure of the quadratic form $q_{H_0} + q_T$, provided the latter is closable. Closability of $q_{H_0} + q_T$ may be established with the use of the KLMN theorem. We say that $T \in \mathscr{D}'(\mathbf{R}^n)$ satisfies the KLMN condition if q_T is relatively form bounded with respect to H_0 with the relative bound less than one (see Section 1 for the definition). A space of distributions satisfies the KLMN condition if all its elements satisfy it.

The theorem below establishes the limits of applicability of the KLMN Theorem to Schrödinger operators with distributional potentials in terms of the scale of Sobolev spaces $H^{s,p}(\mathbf{R})$.

THEOREM 1. *Let $s \in \mathbf{R}$, $1 \leqslant p < \infty$. The Sobolev space $H^{s,p}(\mathbf{R}^n)$ satisfies the* KLMN *condition if and only if $s \geqslant -1$ and, moreover:*

(a) $$s \geqslant \frac{n}{p} - 2 \quad \textit{for } n \geqslant 3;$$

(b) $$s > \frac{2}{p} - 2 \quad \textit{for } n = 2;$$

(c) $$s \geqslant \frac{1}{p} - \frac{3}{2} \ \textit{if}\ p > 1, \quad \textit{and} \quad s > -\frac{1}{2} \ \textit{if}\ p = 1, \quad \textit{for } n = 1.$$

Moreover the space $H^{s,\infty}(\mathbf{R}^n)$ *satisfies the* KLMN *condition if and only if* $s > -1$.

It is not claimed that distributions not belonging to the Sobolev spaces specified in Theorem 1 do not satisfy the KLMN condition; some examples are given in Section 3. The proof of this theorem for the basic spaces, $H^{s,p}(\mathbf{R}^n)$ with p in the open interval $(1, \infty)$ is quite simple, and the same is true of the bound (6), useful for applications (see Section 3). However the limiting cases of $p = 1$ and $p = \infty$ are more subtle and require more elaborate examples.

The main conclusion of Theorem 1 is that in arbitrary dimension n the KLMN condition is satisfied by many singular distributions, with $s < 0$, including a supply of distributions which are not measures. Thus the one-dimensional example of δ is put into a wider setting (note that $\delta \in H^{-\frac{1}{2}-\varepsilon,2}(\mathbf{R})$).

It does not seem likely that distributional potentials other than quite specific measures will be directly used in physical models. The gain lies more in the mathematical simplicity and in the systematic approach, as wittnessed in Section 3.

In Section 1 we recall definitions and basic facts used here. Theorem 1 is proved in Section 2. In Section 3 we illustrate this theorem with some examples and applications. In particular we give some preliminary results toward the spectral and scattering theory of Schrödinger operators with distributional potentials and we give a strengthened version of the theorem of Albeverio *et al.* mentioned above.

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1. PRELIMINARIES

We shall use the following notation. A quadratic form q acting in the Hilbert space $\mathscr{H}$ with the domain $Q(q)$ is a sesquilinear map $q : Q(q) \times Q(q) \to \mathbf{C}$, denoted $\varphi, \psi \mapsto q[\varphi, \psi]$. We write $q[\varphi] = q[\varphi, \varphi]$ (consult [12] for basic properties of quadratic forms).

Suppose q_0 is a positive closed quadratic form with some domain $Q(q_0)$ and with essential domain Q_0 (meaning that q_0 restricted to Q_0 is closable with q_0 as its closure). Suppose moreover that q_1 is another quadratic form which is relatively form bounded with respect to q_0, that is $Q(q_1) \supset Q_0$, and

$$|q_1[\varphi]| \leqslant a q_0[\varphi] + b\|\varphi\|^2 \tag{1}$$

for some constants a, b and for all $\varphi \in Q_0$. Lower bound of such a's is called the q_0-bound of q_1. If q_0-bound of q_1 is 0, we write $q_1 \prec q_0$. This paper is based on the following simple but useful theorem [12].

KLMN THEOREM. *If* q_0*-bound of* q_1 *is less than one, than* $q_0 + q_1$ *is closable on* Q_0 *and the domain of its closure is* $Q(q_0)$. *In particular, if* $Q(q_0) \subset Q(q_1)$, *then* $q_0 + q_1$ *is closed on* $Q(q_0)$.

We shall apply this theorem in the following situation. Let q_{H_0} be as defined in the Introduction, $Q(q_{H_0}) = H^{1,2}(\mathbf{R}^n)$ ($H^{s,p}(\mathbf{R}^n)$ denotes the Sobolev space defined below). Both $C_0^\infty(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$ are essential domains of q_{H_0}. For $T \in \mathcal{D}'(\mathbf{R}^n)$ let q_T be defined as in the Introduction. Theorem 1 establishes conditions on T which guarantee that $q_T \prec q_{H_0}$. We assume that q_T is a symmetric form on $C_0^\infty(\mathbf{R}^n)$. This means that the imaginary part of T, that is the distribution $\operatorname{Im} T$ defined by $\langle \operatorname{Im} T, \varphi \rangle = \frac{1}{2}\langle T, \varphi\rangle - \frac{1}{2}\overline{\langle T, \overline{\varphi}\rangle}$, vanishes.

We shall restrict ourselves to tempered distributions $T \in \mathcal{S}'(\mathbf{R}^n)$. We consider the operator J^s, $s \in \mathbf{R}$, defined by $(J^s T)^\wedge = (1 + |\cdot|^2)^{-s/2}\hat{T}$, where $\hat{}$ denotes the Fourier transform and $(1 + |\cdot|^2)^\alpha$ denotes multiplication by the (smooth) function $k \mapsto (1 + |k|^2)^\alpha$. Formally $J^s = (1 - \Delta)^{-s/2}$, J^s is called *the Bessel potential.* Put $H^{s,p}(\mathbf{R}^n) = J^s L^p(\mathbf{R}^n) = \{T \in \mathcal{S}'(\mathbf{R}^n), J^{-s}T \in L^p(\mathbf{R}^n)\}$ with the norm $\|T\|_{s,p} = \|J^{-s}T\|_p$, $\|\cdot\|_p$ being the norm of $L^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$. For $p = 2$ we have $\|T\|_{s,2}^2 = \int (1 + |k|^2)^s |\hat{T}(k)|^2 dk$. We have $H^{s_1,p}(\mathbf{R}^n) \subset H^{s_2,p}(\mathbf{R}^n)$ for $s_2 < s_1$, with $\|\cdot\|_{s_2,p} \leqslant \|\cdot\|_{s_1,p}$. Moreover the following embedding theorem holds [4]

$$H^{s,p}(\mathbf{R}^n) \hookrightarrow H^{s_1,p_1}(\mathbf{R}^n) \tag{2}$$

if $1 < p \leqslant p_1 \leqslant \infty$, $s, s_1 \in \mathbf{R}$, $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. In particular $H^{s,p}(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ for $q = \frac{np}{n - ps}$, $0 \leqslant s < \frac{n}{p}$. The norm of this embedding we shall denote by $C(n, s, p)$, thus

$$\|f\|_q \leqslant C(n, s, p)\|f\|_{s,p},$$

n, s, p, q related as above. For $1 \leqslant p < \infty$, $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^{s,p}(\mathbf{R}^n)$ for any $s \in \mathbf{R}$ [4].

For $s > 0$, J^s is a convolution operator, $J^s T = G_s * T$, where G_s, the inverse Fourier transform of $(1 + |k|^2)^{-s/2}$, is a positive function vanishing exponentially at infinity, $\int G_s(x)dx = 1$, having for $s < n$ the only singularity at $x = 0$, $G_s(x) \sim |x|^{s-n}$ [16].

The following lemma will be useful for the analysis of the Sobolev spaces $H^{s,p}(\mathbf{R}^n)$, especially for the case $p = \infty$.

LEMMA 1. *Let $s > 0$, $\lambda \in \mathbf{R}^n$, $R > 1$. There exist finite measures μ_s, $\nu_{s,\lambda}$, $\rho_{s,R}$ such that their Fourier transforms are given by*

$$\hat{\mu}_s(k) = \frac{|k|^s}{(1 + |k|^2)^{s/2}}, \quad \hat{\nu}_{s,\lambda}(k) = \frac{(1 + |k - \lambda|^2)^{s/2}}{(1 + |k|^2)^{s/2}}, \quad \hat{\rho}_{s,R}(k) = R^{-s}\frac{(1 + R^2|k|^2)^{s/2}}{(1 + |k|^2)^{s/2}}.$$

Moreover $\|\rho_{s,R}\|$ *is bounded in* $R > 1$ $\left(\text{here } \|\mu\| = \int |\mathrm{d}\mu|\right)$.

Proof. The existence of μ_s is established in [16], we shall use analogous, methods to treat the other two cases. We use the equality $(1 - t)^{s/2} = 1 + \sum_{m=1}^{\infty} A_{m,s} t^m$ where $\sum_{m=1}^{\infty} |A_{m,s}| < \infty$. Thus

$$R^{-s} \frac{(1 + R^2|k|^2)^{s/2}}{(1 + |k|^2)^{s/2}} = 1 + \sum_{m=1}^{\infty} A_{m,s}\left(1 - \frac{1}{R^2}\right)^m \frac{1}{(1 + |k|^2)^m},$$

which implies that

$$\rho_{s,R} = \delta_0 + \sum_{m=1}^{\infty} A_{m,s}\left(1 - \frac{1}{R^2}\right)^m G_{2m}(x)\mathrm{d}x,$$

where the series is convergent in $L^2(\mathbf{R}^n)$-norm, uniformly in $R > 1$, which establishes also the boundedness of the norms $\|\rho_{s,R}\|$. The case of the measure $\nu_{s,\lambda}$ may be treated analogously, although here we have to estimate the derivatives of the functions G_{2m}. We omit the details. The proof is complete.

We shall also require two simple facts concerning the spaces $H^{s,p}(\mathbf{R}^n)$. Firstly, if we prove that for some constant C depending on n, s, p we have

$$|\langle T, |\varphi|^2\rangle| \leqslant C\|T\|_{s,p}\|\varphi\|_{1,2}$$

for all $T \in H^{s,p}(\mathbf{R}^n)$, $\varphi \in C_0^\infty(\mathbf{R}^n)$, then it follows that $H^{s,p}(\mathbf{R}^n)$ satisfies the KLMN condition because of the density of $\mathscr{S}(\mathbf{R}^n)$ in $H^{s,p}(\mathbf{R}^n)$ (note that q_T is a bounded form if $T \in \mathscr{S}(\mathbf{R}^n)$). The second fact will be stated as the following

LEMMA 2. *Let* $1 < p < \infty, s \in \mathbf{R}$, $T \in H^{s,p}(\mathbf{R}^n)$. *Then* $T = G_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j$, *where* $G_0 \in H^{s+2,p}(\mathbf{R}^n)$, $G_j \in H^{s+1,p}(\mathbf{R}^n)$, $j = 1, \ldots, n$, *and* $\frac{\partial}{\partial x_j}$ *is meant in the sense of distributions. Moreover* $G_j, j = 0, \ldots, n$ *may be chosen so that* $\|G_0\|_{s+2,p} = \|T\|_{s,p}$, $\|G_j\|_{s+1,p} \leqslant C_{n,p}\|T\|_{s,p}$, *where* $C_{n,p}$ *depends only on* n, p.

Proof. We put $G_0 = J^2 T \in H^{s+2,p}(\mathbf{R}^n)$ and $G_j = -\frac{\partial}{\partial x_j} G_0$, then clearly $T = G_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j$. We have to check that $G_j \in H^{s+1,p}(\mathbf{R}^n)$. Observe that

$$\hat{G}_j(k) = \frac{ik_j}{|k|} \frac{|k|}{(1 + |k|^2)^{\frac{1}{2}}} (1 + |k|^2)^{\frac{1}{2}} \hat{G}_0(k),$$

hence

$$(J^{-s-1}G_j)^\wedge(k) = \frac{ik_j}{|k|} \frac{|k|}{(1+|k|^2)^{\frac{1}{2}}} (1+|k|^2)^{\frac{1}{2}} (J^{-s-2}G_0)^\wedge(k),$$

thus $J^{-s-1}G_j$ is obtained from $J^{-s-2}G_0$ by successively applying to it commuting operators J^1, a convolution with a finite measure μ_1, given, by Lemma 1, and a Riesz transformation R_j. All of these operators are bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$ (see [16]), so $G_j \in H^{s+1,p}(\mathbf{R})$ and the lemma is proved.

Of course the distributions G_j in the above lemma may be chosen in a variety of ways. Note that if $s \geq -1$ (as in Theorem 1), then all the distributions G_j's appearing in $T = G_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j$ are regular distributions (that is locally integrable functions). It is worth noting that the quadratic form q_T is symmetric if $G_0, G_1, \ldots, G_n$ are real valued.

2. PROOF OF THE MAIN RESULT

We begin with a result illustrating the method of proof in a simple case. Let $L^{-1,\infty}(\mathbf{R}^n)$ denote the space of distributions

$$L^{-1,\infty}(\mathbf{R}^n) = \{T \in \mathscr{D}'(\mathbf{R}^n) : T = G_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j,\ G_j \in L^\infty(\mathbf{R}^n) \text{ for } j = 0, \ldots, n\}.$$

The following lemma holds.

LEMMA 3. *$L^{-1,\infty}(\mathbf{R}^n)$ satisfies the* KLMN *condition. Moreover $q_T \prec q_{H_0}$ for any $T \in L^{-1,\infty}(\mathbf{R}^n)$.*

Proof. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$. Then $q_T[\varphi] = \langle G_0, |\varphi|^2\rangle + \sum_{j=1}^{n} \left\langle G_j, \frac{\partial}{\partial x_j} |\varphi|^2 \right\rangle$. The first term is a bounded form. We have

$$\left| \left\langle G_j, \frac{\partial}{\partial x_j} |\varphi|^2 \right\rangle \right| \leq 2\|G_j\|_\infty \left\| \frac{\partial}{\partial x_j} \varphi \right\|_2 \|\varphi\|_2 \leq \|G_j\|_\infty \left(\varepsilon \left\| \frac{\partial}{\partial x_j} \varphi \right\|_2^2 + \frac{1}{\varepsilon} \|\varphi\|_2^2 \right),$$

for any $\varepsilon > 0$, so

$$|q_T[\varphi]| \leq \varepsilon \sup_{j=1,\ldots,n} \|G_j\|_\infty q_{H_0}[\varphi] + \left(\|G_0\|_\infty + \frac{1}{\varepsilon} \sum_{j=1}^{n} \|G_j\|_\infty \right) \|\varphi\|_2^2,$$

and the result of the lemma follows since ε was arbitrary.

Note that the above lemma does not contradict Theorem 1 for the case $p = \infty$ since the spaces $L^{-1,\infty}(\mathbf{R}^n)$ and $H^{-1,\infty}(\mathbf{R}^n)$ do not coincide.

The proof of Theorem 1 will be first given for $1 < p < \infty$ and then for $p = 1$, $p = \infty$. Finally we will exhibit examples showing that the spaces $H^{s,p}(\mathbf{R}^n)$ not covered by Theorem 1 do not indeed satisfy the KLMN condition.

A. THE CASE $1 < p < \infty$. Suppose first that $n \geqslant 3$. If $1 < p \leqslant n$, then $H^{\frac{n}{p}-2,p}(\mathbf{R}^n) \subset H^{-1,n}(\mathbf{R}^n)$ by the embedding theorem (2). If $n \leqslant p < \infty$, then $L^p(\mathbf{R}^n) \subset L^n(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$, so using Lemma 2 we see that $H^{-1,p}(\mathbf{R}^n) \subset H^{-1,n}(\mathbf{R}^n) + L^{-1,\infty}(\mathbf{R}^n)$. Thus by Lemma 3 we need only show that the space $H^{-1,n}(\mathbf{R}^n)$ satisfies the KLMN condition. Let $T \in H^{-1,n}(\mathbf{R}^n)$, then by Lemma 2 again $T = G_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_j$, where $G_j \in L^n(\mathbf{R}^n)$ for $j = 0, \ldots, n$ so

$$q_T[\varphi] = \int G_0(x)|\varphi(x)|^2 dx - \sum_{j=1}^{n} \int G_j(x) \frac{\partial}{\partial x_j} |\varphi(x)|^2 dx. \tag{3}$$

The Hölder inequality and the embedding theorem (2) yield

$$\left| \int G(x)\psi(x)\varphi(x) dx \right| \leqslant \|G\|_n \|\psi\|_2 \|\varphi\|_{\frac{2n}{n-2}} \leqslant C(n, 1, 2)\|G\|_n \|\psi\|_2 \|\varphi\|_{1,2}. \tag{4}$$

Applying the above inequality $1 + 2n$ times to (3) we obtain

$$|q_T[\varphi]| \leqslant C(n, 1, 2)\|G_0\|_n \|\varphi\|_2 \|\varphi\|_{1,2} + 2C(n, 1, 2) \sum_{j=1}^{n} \|G_j\|_n \left\| \frac{\partial}{\partial x_j} \varphi \right\|_2 \|\varphi\|_{1,2} \leqslant$$

$$\leqslant (1 + 2n)C(n, 1, 2)\, C_{n,n} \|T\|_{-1,n} \|\varphi\|_{1,2}^2,$$

where the constant $C_{n,n}$ comes from Lemma 2. By the remark preceding Lemma 2 the space $H^{-1,n}(\mathbf{R}^n)$ satisfies the KLMN condition.

Suppose now $n = 2$. Arguing as above we see that it is enough to consider the space $H^{-1+\varepsilon,2}(\mathbf{R}^2)$, and arguing further as above we need only an estimate

$$\left| \int G(x)\psi(x)\varphi(x) dx \right| \leqslant \|G\|_{2/(1-\varepsilon)} \|\psi\|_2 \|\varphi\|_{\frac{2}{\varepsilon}} \leqslant$$

$$\leqslant C(2, \varepsilon, 2)C(2, 1 - \varepsilon, 2)\|G\|_{\varepsilon,2} \|\psi\|_2 \|\varphi\|_{1-\varepsilon,2} \leqslant C\|G\|_{\varepsilon,2} \|\psi\|_2 \|\varphi\|_{1,2}.$$

Suppose $n = 1$. Arguing as above we see that we need only to consider the space

$H^{-1,2}(\mathbf{R})$, and the estimate

$$\left|\int G(x)\psi(x)\varphi(x)\mathrm{d}x\right| \leqslant \|G\|_2\|\psi\|_2\|\varphi\|_\infty$$

is sufficient since $\|\cdot\|_\infty \leqslant C\|\cdot\|_{1,2}$ for $n = 1$.

B. THE CASE $p = 1$. For $p = 1$ we can no longer use the embedding theorem (2), nor related inequalities for weak L^p spaces, so we have to approach this problem directly. For $n \geqslant 2$ we will use the following lemma.

LEMMA 4. *Let $n \geqslant 2$, G_α be as in Section 1. If $n-2 \leqslant \alpha$, $0 < \alpha$, then*

$$\|G_\alpha * |\varphi|^2\|_\infty \leqslant C\|\varphi\|_{1,2}^2$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$, where C depends only on α, n.

Proof. Assume without loss of generality that $\alpha < n$. All the constants appearing in this proof depend only on α and n and will not be distinguished. Let K_1 be the unit ball in $\mathbf{R}^n$, $K_1 = \{x \in \mathbf{R}^n, |x| \leqslant 1\}$, and let χ_A denote the indicator function of the set A. Put $G_{\alpha,1} = \chi_{K_1}G_\alpha$, $G_\alpha = G_{\alpha,1} + G_{\alpha,2}$. $G_{\alpha,2}$ is bounded, so

$$\|G_{\alpha,2} * |\varphi|^2\|_\infty \leqslant \|G_{\alpha,2}\|_\infty \||\varphi|^2\|_1 = \|G_{\alpha,2}\|_\infty\|\varphi\|_2^2.$$

We consider now $G_{\alpha,1}$. Taking into account the behaviour of G_α at its singularity (see Introduction) we see that the lemma will follow if we establish that the function $I(x)$,

$$I(x) = \int\limits_{|y|\leqslant 1} |y|^{\alpha-n}|\varphi(x-y)|^2\mathrm{d}y,$$

satisfies $|I(x)| \leqslant C\|\varphi\|_{1,2}^2$. Since both $\|\cdot\|_\infty$ and $\|\cdot\|_{1,2}$ are translation-invariant norms, we can suppress the dependence on x. We find

$$I = \int\limits_{S^{n-1}} \int\limits_0^1 r^{\alpha-1}|\varphi(r\omega)|^2\mathrm{d}r\, \mathrm{d}\omega,$$

where $\mathrm{d}\omega$ is the appropriately normalized Lebesgue measure on $S^{n-1} = \partial K_1$. Clearly, I is finite. Since $\alpha > 0$ we can integrate by parts to obtain

$$I = \int\limits_{S^{n-1}} \left[\frac{1}{\alpha}|\varphi(w)|^2 - \frac{1}{\alpha}\int\limits_0^1 r^\alpha\left\{\overline{\varphi}\frac{\partial}{\partial r}\varphi + \varphi\frac{\partial}{\partial r}\overline{\varphi}\right\}\mathrm{d}r\right]\mathrm{d}\omega.$$

Now observe that S^{n-1} is a manifold of codimension 1 regularly embedded in $\mathbf{R}^n$ [12], so for $s > \frac{1}{2}$ we have

$$\|\mathrm{tr}_{S^{n-1}}\varphi\|_{L^2(S^{n-1})} \leqslant C\|\varphi\|_{s,2},$$

where C depends on s, n, and $\mathrm{tr}_{S^{n-1}}$ is the trace operator (restricting the function φ to S^{n-1}). We choose $s = 1$. We have

$$I \leqslant C\|\varphi\|_{1,2}^2 + \frac{2}{\alpha} \int_{S^{n-1}} \left(\int_0^1 r^{2\alpha-n+1}|\varphi(r\omega)|^2 \,\mathrm{d}r\right)^{\frac{1}{2}} \left(\int_0^1 r^{n-1}\left|\frac{\partial}{\partial r}\varphi(r\omega)\right|^2 \mathrm{d}r\right)^{\frac{1}{2}} \mathrm{d}\omega.$$

We have $2\alpha - n + 1 \geqslant \alpha - 1$, so using the inequality $2AB \leqslant \frac{\alpha}{2}A^2 + \frac{2}{\alpha}B^2$ we find

$$I \leqslant C\|\varphi\|_{1,2}^2 + \frac{1}{2}I + \frac{2}{\alpha^2}\int_{K_1}\left|\frac{\partial}{\partial r}\varphi(x)\right|^2 \mathrm{d}x,$$

which completes the proof since $\left|\frac{\partial}{\partial r}\varphi(x)\right| \leqslant |\nabla \varphi(x)|$, so the integral above is dominated by $\|\varphi\|_{1,2}^2$. The lemma is proved.

Suppose that $s = n - 2$ for $n \geqslant 3$ and $s > 0$ for $n = 2$. Then $T \in H^{s,1}(\mathbf{R}^n)$ is a regular distribution and $T = J^s g$, $g \in L^1(\mathbf{R}^n)$, $\|T\|_{s,1} = \|g\|_1$. We have

$$q_T[\varphi] = \int g(x)(G_s * |\varphi|^2)(x)\mathrm{d}x$$

so applying Lemma 4 we obtain

$$|q_T[\varphi]| \leqslant C\|T\|_{s,1}\|\varphi\|_{1,2}^2,$$

which ends the proof of the theorem for $p = 1$, $n \geqslant 2$.

Suppose now that $n = 1$, $p = 1$, and $s > -\frac{1}{2}$. We write $s = -\frac{1}{2} + 2\varepsilon$, $\varepsilon > 0$. In order to show that

$$|q_T[\varphi]| \leqslant C\|T\|_{-\frac{1}{2}+2\varepsilon,1}\|\varphi\|_{1,2}^2$$

it is enough to establish an analogue of Lemma 4, namely

$$\|J^{-\frac{1}{2}+2\varepsilon}|\varphi|^2\|_\infty \leqslant C\|\varphi\|_{1,2}^2.$$

Since $(1+|k|^2) \leqslant 2(1+|k+k'|^2)(1+|k'|^2)$ for every $k, k' \in \mathbf{R}$, we have

$$\|J^{-\frac{1}{2}+2\varepsilon}|\varphi|^2\|_\infty \leqslant C\|(J^{-\frac{1}{2}+2\varepsilon}|\varphi|^2)^\wedge\|_1 \leqslant$$

$$\leqslant C\iint (1+|k+k'|^2)^{\frac{1}{4}-\varepsilon}(1+|k'|^2)^{\frac{1}{4}-\varepsilon}|\hat{\varphi}(k+k')|\,|\hat{\varphi}(k')|\,dk'dk =$$

$$= C\|(1+|\cdot|^2)^{\frac{1}{4}-\varepsilon}\hat{\varphi}\|_1^2,$$

so we only need an inequality $\|(1+|\cdot|^2)^{\frac{1}{4}-\varepsilon}\hat{\varphi}\|_1 \leqslant C\|(1+|\cdot|^2)^{\frac{1}{2}}\hat{\varphi}\|_2$, which is obvious since $(1+|\cdot|^2)^{-\frac{1}{4}-\varepsilon} \in L^2(\mathbf{R})$.

C. THE CASE $p = \infty$. We have to show that the space $H^{-1+\varepsilon,\infty}(\mathbf{R}^n)$, $\varepsilon > 0$, satisfies the KLMN condition. Let $T = J^{-1+\varepsilon}f$, $f \in L^\infty(\mathbf{R}^n)$. Since $\langle T, |\varphi|^2\rangle = \langle f, J^{-1+\varepsilon}|\varphi|^2\rangle$, we only need to prove that for any $\eta > 0$ there exists a constant C depending only on ε, η and n, such that the following inequality holds

$$\|J^{-1+\varepsilon}|\varphi|^2\|_1 \leqslant \eta q_{H_0}[\varphi] + C\|\varphi\|_2^2 \tag{5}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$. Observe that

$$J^{-1+\varepsilon}|\varphi|^2 = G_{1+\varepsilon} * J^{-2}|\varphi|^2 = G_{1+\varepsilon} * |\varphi|^2 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} G_{1+\varepsilon} * \frac{\partial}{\partial x_j} |\varphi|^2,$$

where the functions $G_{1+\varepsilon}$, $\frac{\partial}{\partial x_j} G_{1+\varepsilon}$ are in $L^2(\mathbf{R}^n)$. Using the inequality $\left\|\frac{\partial}{\partial x_j}|\varphi|^2\right\|_1 \leqslant \eta\left\|\frac{\partial}{\partial x_j}\varphi\right\|_2^2 + \frac{1}{\eta}\|\varphi\|_2^2$ for any $\eta > 0$ we deduce (5) immediately.

We have thus established that all $H^{s,p}(\mathbf{R}^n)$ spaces claimed by Theorem 1 do satisfy the KLMN condition.

D. COUNTEREXAMPLES. We will now exhibit a series of examples showing that the Sobolev spaces not covered by Theorem 1 contain distributions T giving rise to quadratic forms q_T which are not relatively form bounded with respect to q_{H_0} (this is more than saying that they do not satisfy the KLMN condition).

EXAMPLE 1. $s < \frac{n}{p} - 2$, $p \geqslant 1$, $n \geqslant 3$. Observe that $G_\alpha \in H^{\frac{n}{p}-2-\varepsilon,p}(\mathbf{R}^n)$ if $\alpha = n - 2 - \frac{1}{2}\varepsilon$, $1 > \varepsilon > 0$. Moreover $G_\beta \in H^{1,2}(\mathbf{R}^n)$ if $\beta = \frac{n}{2} + 1 + \frac{1}{8}\varepsilon$. The integral

$$\int G_\alpha(x)|G_\beta(x)|^2 \mathrm{d}x$$

diverges at $x = 0$ for α, β as above. If we approximate G_β in $H^{1,2}(\mathbf{R}^n)$ by $\varphi_k \in C_0^\infty(\mathbf{R}^n)$ with, for instance, $\varphi_k \to G_\beta$ monotonically, we find that $\langle G_\alpha, |\varphi_k|^2\rangle \to \infty$ as $k \to \infty$ despite boundedness of $\|\varphi_k\|_{1,2}$. This shows that G_α is not relatively form bounded with respect to q_{H_0}.

EXAMPLE 2. $s = \frac{2}{p} - 2$, $p \geqslant 1$, $n = 2$. It is well known that the space $L^1(\mathbf{R}^2)$ does not satisfy the KLMN condition, so we need to discuss only the case $p > 1$. As above, note that $\frac{\partial}{\partial x_1} G_1 \in H^{\frac{2}{p}-2,p}(\mathbf{R}^2)$. If φ is smooth outside $x = 0$, has compact support, and in some neighbourhood of $x = 0$ behaves like $(-\log|x|)^{1/3}$, then $\varphi \in H^{1,2}(\mathbf{R}^2)$. The divergence of the integral

$$\int_{\mathbf{R}^2} G_1(x)\varphi(x)\frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x$$

implies, as above, that $H^{\frac{2}{p}-2,p}(\mathbf{R}^2)$ does not satisfy the KLMN condition.

EXAMPLE 3. $s < \frac{1}{p} - \frac{3}{2}$, $p > 1$, $n = 1$. Observe that $\frac{\mathrm{d}}{\mathrm{d}x} G_{\frac{1}{2}} \in H^{\frac{1}{p}-\frac{3}{2}-\varepsilon,p}(\mathbf{R})$. If φ has compact support, is smooth outside $x = 0$, $\varphi(x) \geqslant 1$ for $|x| \leqslant 1$ and in the vicinity of $x = 0$ behaves as $1 + |x|^{\frac{1}{2}}(-\log|x|)^{-\frac{2}{3}}$, then $\varphi \in H^{1,2}(\mathbf{R})$. The integral

$$\int G_{\frac{1}{2}}(x)\frac{\mathrm{d}}{\mathrm{d}x}|\varphi(x)|^2\mathrm{d}x$$

is divergent, which shows that $\frac{\mathrm{d}}{\mathrm{d}x} G_{\frac{1}{2}}$ is not relatively form bounded with respect to q_{H_0}.

EXAMPLE 4. $s = -\frac{1}{2}$, $p = 1$, $n = 1$. Let $f \in L^1(\mathbf{R})$ be such that in the vicinity of $x = 0$, $f(x) = |x|^{-1}(-\log|x|)^{-1-\varepsilon}$ and, moreover, $\operatorname{supp}\hat{f} \subset [0, \infty)$ Then $\frac{1}{i} I^1 \frac{d}{dx} f = f$, where $(I^\alpha f)\hat{}(k) = |k|^{-\alpha}\hat{f}(k)$. Moreover, by Lemma 1, $J^1 I^{-1}$ is the operator of convolution with the finite measure μ_1, so is a bounded operator in $L^1(\mathbf{R})$, and $J^1 \frac{d}{dx} f = g \in L^1(\mathbf{R})$. We thus have $J^{\frac{1}{2}} \frac{d}{dx} f = J^{-\frac{1}{2}} g \in H^{-\frac{1}{2},1}(\mathbf{R})$. Observe now that

$$\langle J^{-\frac{1}{2}} g, |\varphi|^2 \rangle = \left\langle f_1, G_{\frac{1}{2}} * \frac{d}{dx} |\varphi|^2 \right\rangle =$$

$$= \int f(x) \left(G_{\frac{1}{2}} * \frac{d}{dx} |\varphi|^2 \right)(x) dx.$$

Choosing φ as in Example 3 we find that $G_{\frac{1}{2}} * \frac{d}{dx} |\varphi|^2$ has at $x = 0$ a singularity of the form $(-\log|x|)^{1/3}$ so the above integral is divergent. Hence $J^{-\frac{1}{2}} g$ is not relatively form bounded with respect to q_{H_0}.

EXAMPLE 5. $s = -1$, $p = \infty$. In this example we exploit a bounded function f such that $J^{-1} f = \frac{\partial}{\partial x_j} g$, where g has some singularities. A careful examination of the sequence $f(x/k)$, $k = 1, 2, \ldots$ yields the example needed. As f we take $\frac{\partial}{\partial x_j} G_{n+1}$, which is a bounded function. We have $J^{-1} f = \frac{\partial}{\partial x_j} G_n$, where G_n has a weak, logarithmic singularity at $x = 0$. For any $k \in \mathbf{N}$ let S_k be defined by $S_k \varphi(x) = \varphi(x/k)$, and let $f_k = S_k f = S_k \frac{\partial}{\partial x_j} G_{n+1} \in L^\infty(\mathbf{R}^n)$, $\sup_{k \in \mathbf{N}} \|f_k\|_\infty < \infty$. Observe now that the operator $\frac{1}{k} J^1 S_k J^{-1} S_{1/k}$ is exactly the convolution with the finite measure $\rho_{1,k}$ established in Lemma 1, so it is a bounded operator in $L^\infty(\mathbf{R}^n)$, uniformly in $k \in \mathbf{N}$. Put $\tilde{g}_k = \frac{1}{k} J^1 S_k J^{-1} S_{1/k} f_k$, then $\tilde{g}_k = \frac{1}{k} J^1 S_k \frac{\partial}{\partial x_j} G_n = J^1 \frac{\partial}{\partial x_j} S_k G_n$, $\sup_{k \in \mathbf{N}} \|\tilde{g}_k\|_\infty < \infty$. Fix $\lambda \in \mathbf{R}^n$, to be determined later, and let $g_k = \nu_{1,\lambda/k} * \tilde{g}_k$, where $\nu_{1,\lambda/k}$ is also a finite measure established in Lemma 1. Thus

$\|g_k\|_\infty$ is also bounded in k and, moreover,

$$g_k = J^1\exp(\mathrm{i}x\cdot\lambda/k)\frac{\partial}{\partial x_j}S_kG_n = -\frac{\mathrm{i}\lambda_j}{k}J^1S_k\exp(\mathrm{i}x\cdot\lambda)G_n + J^1\frac{\partial}{\partial x_j}S_k\exp(\mathrm{i}x\cdot\lambda)G_n$$

where the first term is bounded uniformly in k and $x\cdot y = \sum x_jy_j$. Thus $\left\|\frac{\partial}{\partial x_j}S_k\exp(\mathrm{i}x\cdot\lambda)G_n\right\|_{-1,\infty}$ is bounded in k, and using the fact that G_n decreases exponentially fast at infinity we find that $T = \sum_{k\in\mathbf{N}}\frac{\partial}{\partial x_j}S_k\exp(\mathrm{i}(x + k^2x_0)\cdot\lambda)G_n(x + k^2x_0)$, where $x_0 \in \mathbf{R}^n$ is an arbitrary fixed vector, is in $H^{-1,\infty}(\mathbf{R}^n)$. We shall show that T is not relatively form bounded with respect to q_{H_0}. Choose any $\varphi \in C_0^\infty(\mathbf{R}^n)$ and put $\varphi_k(x) = \varphi(x + k^2x_0)$. Then $\langle T, |\varphi_k|^2\rangle - \left\langle\frac{\partial}{\partial x_j}S_k\exp(\mathrm{i}x\cdot\lambda)G_n, |\varphi|^2\right\rangle$ tends to zero as $k\to\infty$, while

$$\left\langle S_k\exp(\mathrm{i}x\cdot\lambda)G_n, \frac{\partial}{\partial x_j}|\varphi|^2\right\rangle = C\int\left(\frac{1}{k^2} + |\xi-\lambda|^2\right)^{-n/2}\left(\frac{\partial}{\partial x_j}|\varphi|^2\right)^\wedge(\xi)\mathrm{d}\xi,$$

so we need only to choose $\lambda \in \mathbf{R}^n$ such that $\left(\frac{\partial}{\partial x_j}|\varphi|^2\right)^\wedge(\lambda) > 0$ to obtain that $\langle T, |\varphi_k|^2\rangle \to \infty$ as $k\to\infty$.

EXAMPLE 6. $s < -1$. In view of the examples given above we need only consider the cases $2 \leqslant p < \infty$, $n \geqslant 1$. Observe that $H^{s,p}(\mathbf{R}^n) = (H^{-s,p}(\mathbf{R}^n))'$ (here X' denotes the Banach space conjugate of X), $\frac{1}{p} + \frac{1}{q} = 1$. In order to prove the existence of a distribution $T \in H^{-1-\varepsilon,p}(\mathbf{R}^n)$, $p \geqslant 2$, which is not relatively form bounded with respect to q_{H_0}, it is enough to find a sequence $\varphi_k \in C_0^\infty(\mathbf{R}^n)$ with $\|\varphi_k\|_{1,2} = 1$, $\|\,|\varphi_k|^2\|_{1+\varepsilon,q} \to \infty$, q as above, $1 < q \leqslant 2$. Indeed, one can then define $T \in (H^{1+\varepsilon,q}(\mathbf{R}^n))'$ by putting $\langle T, |\varphi_k|^2\rangle = \|\,|\varphi_k|^2\|_{1+\varepsilon,q}$, $k = 1, 2, \ldots$, and by extending T to a functional on the whole of $H^{1+\varepsilon,q}(\mathbf{R}^n)$ by linearity (this procedure is allowed because $|\varphi_k|^2$ constructed below, as elements of $H^{1+\varepsilon,q}(\mathbf{R}^n)$, are linearly independent). Observe that $\|\,|\varphi_k|^2\|_{1+\varepsilon,q} = \|J^{-1-\varepsilon}|\varphi_k|^2\|_q \geqslant C\|(J^{-1-\varepsilon}|\varphi_k|^2)^\wedge\|_p$. Now let $\psi \in C_0^\infty(\mathbf{R}^n)$ be such that $(|\psi|^2)^\wedge \geqslant \chi_{K_1}$ where χ_{K_1} is the indicator function of the set $K_1 = \{x \in \mathbf{R}^n, |x| \leqslant 1\}$. Put $\varphi_k(x) = \psi(x)\left[1 + \frac{1}{k}\exp(\mathrm{i}kx_1)\right]$. Clearly, $\|\varphi_k\|_{1,2}$ is bounded in k and, moreover

$$|\varphi_k(x)|^2 = |\psi(x)|^2\left(1 + \frac{1}{k^2} + \frac{1}{k}\exp(\mathrm{i}kx_1) + \frac{1}{k}\exp(-\mathrm{i}kx_1)\right),$$

thus $(|\varphi_k|^2)^{\wedge} = \left(1 + \frac{1}{k^2} + \frac{1}{k} T_k + \frac{1}{k} T_{-k}\right)(|\psi|^2)^{\wedge}$, where T_α is a translation operator along the first coordinate axis. Since $(|\psi|^2)^{\wedge}$ vanishes exponentially at infinity we have, for sufficiently large k,

$$|(|\varphi_k|^2)^{\wedge}| \geqslant \frac{1}{2k} T_k \chi_{K_1}.$$

Thus $|(J^{-1-\varepsilon}|\varphi_k|^2)^{\wedge}| \geqslant Ck^{\varepsilon} T_k \chi_{K_1}$ so we obtain $\|(J^{-1-\varepsilon}|\varphi_k|^2)^{\wedge}\|_p \geqslant Ck^{\varepsilon}$, as required.

The proof of Theorem 1 is completed. Note that a slight extension of the above proof establishes the existence of constants $C_{n,s,p}$ such that

$$|q_T[\varphi]| \leqslant C_{n,s,p}\|T\|_{s,p}\|(H_0 + 1)^{\frac{1}{2}}\varphi\|_2^2 \tag{6}$$

for all $T \in H^{s,p}(\mathbf{R}^n)$, $\varphi \in C_0^\infty(\mathbf{R}^n)$ whenever the space $H^{s,p}(\mathbf{R}^n)$ satisfies the conditions of Theorem 1. Indeed, for $p \leqslant n$ this has in fact been proved, while for $p \geqslant n$ we need only use

$$\left|\int G(x)\psi(x)\varphi(x)\mathrm{d}x\right| \leqslant \|G\|_p\|\psi\|_2\|\varphi\|_{\frac{2p}{p-2}} \leqslant C\left(n, \frac{n}{p}, 2\right)\|G\|_n\|\psi\|_2\|\varphi\|_{\frac{n}{p},2},$$

in place of (4).

3. EXTENSIONS, EXAMPLES AND APPLICATIONS

In this section we give some simple examples showing how the theory of Schrödinger operators with distributional potentials may be developed, and relate it to some known results. These examples are applications of the inequality (6) and of a related inequality established below. We begin, however, with useful extensions of Theorem 1.

A minor modification of the proof given in the preceding section yields the following result.

THEOREM 2. *Suppose* $1 < p < \infty$, $s \geqslant -1$ *and, moreover*

(a) $$s > \frac{n}{p} - 2 \quad \textit{for } n \geqslant 2,$$

(b) $$s > \frac{1}{p} - \frac{3}{2} \quad \textit{for } n = 1.$$

Then for any $\varepsilon > 0$ *there exists a constant* $A_\varepsilon > 0$ *such that*

$$|q_T[\varphi]| \leqslant \|T\|_{s,p}(\varepsilon q_{H_0}[\varphi] + A_\varepsilon\|\varphi\|_2^2)$$

for all $T \in H^{s,p}(\mathbf{R}^n)$, $\varphi \in C_0^\infty(\mathbf{R}^n)$.

Observe that Strichartz's argument [17] may be used to treat the case of distributional potentials not necessarily vanishing at infinity, for instance of periodic potentials, with singularities much more severe that those allowed by $H^{-1+\varepsilon,\infty}(\mathbf{R}^n)$. Indeed, let $H^{s,p}_{u,loc}(\mathbf{R}^n)$ be the space of uniformly locally $H^{s,p}(\mathbf{R}^n)$ distributions, that is of distributions $T \in \mathscr{D}'(\mathbf{R}^n)$ for which

$$\sup_{\tau\in\mathbf{R}^n} \|\psi_\tau T\|_{s,p} < \infty$$

for any $\psi \in C_0^\infty(\mathbf{R}^n)$, $\psi_\tau(x) = \psi(x-\tau)$. The following result holds.

THEOREM 3. *Let* s, p *be as in Theorem* 2, $T \in H^{s,p}_{u,loc}(\mathbf{R}^n)$. *Then* $q_T \prec q_{H_0}$.

Proof. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be a non-negative function such that $\{\psi_m^2\}_{m\in\mathbf{Z}^n}$ is a partition of unity on $\mathbf{R}^n$, $\psi_m(x) = \psi(x-m)$. Moreover, let $\tilde{\psi} \in C_0^\infty(\mathbf{R}^n)$ be such that $\tilde{\psi}(x) = 1$ for $x \in \operatorname{supp}\psi$. Put $M = \sup_{m\in\mathbf{Z}^n} \|\tilde{\psi}_m T\|_{s,p}$ and fix $\varepsilon > 0$. Then by Theorem 2

$$|\langle\tilde{\psi}_m T, |\varphi|^2\rangle| \leqslant M(\varepsilon q_{H_0}[\varphi] + A_\varepsilon\|\varphi\|_2^2)$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$ (A_ε depends on n, s, p and ε). We have

$$\langle T, |\varphi|^2\rangle = \sum_{m\in\mathbf{Z}^n} \langle\tilde{\psi}_m T, \psi_m^2|\varphi|^2\rangle$$

so by the above

$$|\langle T, |\varphi|^2\rangle| \leqslant M(\varepsilon \sum_{m\in\mathbf{Z}^n} q_{H_0}[\psi_m\varphi] + A_\varepsilon\|\psi_m\varphi\|_2^2). \tag{7}$$

Observe now that $\sum_{m\in\mathbf{Z}^n} \|\psi_m\varphi\|_2^2 = \|\varphi\|_2^2$, while

$$q_{H_0}[\psi_m\varphi] = \int |\nabla(\psi_m\varphi)|^2 dx \leqslant 2\int \{|\psi_m|^2|\nabla\varphi|^2 + |\nabla\psi_m|^2|\varphi|^2\}dx.$$

Using the inequality $\left\|\sum_{m\in\mathbf{Z}^n} |\nabla\psi_m|^2\right\|_\infty < \infty$ and summing over $m \in \mathbf{Z}^n$ in (7) we obtain

$$|\langle T, |\varphi|^2\rangle| \leqslant 2M(\varepsilon q_{H_0}[\varphi] + B_\varepsilon\|\varphi\|_2^2)$$

where B_ε depends on A_ε and on the choice of ψ. The proof is complete.

Note that Theorem 3 cannot be proved by using Theorem 1 only because some uniformity with respect to $\|\cdot\|_{s,p}$ is needed. Indeed, it is not the case that for s, p as in Theorem 1 the space $H^{s,p}_{u,loc}(\mathbf{R}^n)$ satisfies the KLMN condition (this is the case only if some additional "uniformity" condition is imposed, for instance if T is periodic). Counterexamples may be easily given for $s = 0, p = \frac{n}{2}$, $n \geqslant 3$.

Recall now that the first monograph [15] which emphasized the use of Hamiltonians defined as quadratic forms (and thus the use of the KLMN theorem) deals with the class $\mathfrak{R}$ of Rollnick potentials, that is of measurable functions on $\mathbf{R}^3$ for which the Rollnick norm

$$\|V\|_{\mathfrak{R}}^2 = \frac{1}{4\pi}\iint_{\mathbf{R}^6} \frac{V(x)V(y)}{|x-y|^2}\, dx\, dy$$

is finite. The following example shows that $\mathfrak{R}$ is covered by Theorem 1.

EXAMPLE 1. $\mathfrak{R} \subset H^{-\frac{1}{2},2}(\mathbf{R}^3)$. Indeed, observe that $\iint \frac{V(v)V(y)}{|x-y|^2}\, dx dy = C\int \frac{1}{|k|}|\hat{V}(k)|^2 dk$, see [16], so

$$\|V\|_{\mathfrak{R}}^2 \geqslant C\int (1+|k|^2)^{-\frac{1}{2}}|\hat{V}(k)|^2 dk = C\|V\|^2_{-\frac{1}{2},2}.$$

Similar results hold in lower dimension (but not in higher dimension).

Another class of locally integrable potentials satisfying the KLMN condition, introduced in Aizenman, Simon [1], Cycon *et al.* [7], is defined as

$$K_n = \left\{ V : \lim_{\varepsilon \to 0} \sup_{x \in \mathbf{R}^n} \int_{|x-y|<\varepsilon} \frac{1}{|x-y|^{n-2}} |V(y)| dy = 0 \right\}$$

for $n \geqslant 3$ (for $n = 1, 2$ and for related Stummel classes S_n see [1, 7]). There exist weird potentials in K_n, of the form $W(x) = |x|^{-\alpha} \sum_{k \in \mathbf{N}} \chi_{[a_k, b_k]}(|x|)$, where χ is the indicator function, $\alpha > 0$, $a_k, b_k \to 0$, $a_k - b_k$ very small for k large, which are not in $H^{-1,n}(\mathbf{R}^n)$. On the other hand, it is easy to check that $V_\delta(x) = |x|^{-2}|\ln|x||^{-\delta}$ is in $H^{-1,n}(\mathbf{R}^n)$ for $\delta > \frac{1}{n}$ and is in K_n for $\delta > 1$. Since V_δ satisfies the KLMN condition for $\delta > 0$, this provides another example, besides W as above, of relatively form bounded perturbations of H_0 not covered by Theorem 1.

One of the reasons to introduce the Rollnick class $\mathfrak{R}$ was that if V is a Rollnick potential, then $|V|^{\frac{1}{2}}(H_0+1)^{-1}|V|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, and in particular $(H+\mathrm{i})^{-1}-(H_0+\mathrm{i})^{-1}$ is Hilbert-Schmidt. A similar weaker result holds for general classes of distributional potentials also.

THEOREM 4. *Suppose s, p as in Theorem 1, $p<\infty$, $T\in H^{s,p}(\mathbf{R}^n)$. Let $H=H_0\dot{+}T$. Then $(H+\mathrm{i})^{-1}-(H_0+\mathrm{i})^{-1}$ is compact. In particular $\sigma_{\mathrm{ess}}(H)=[0,\infty)$.*

Proof. Putting $R=(H_0+\mathrm{i})^{-1}-(H+\mathrm{i})^{-1}$ we find

$$(R\varphi,\psi)=q_T[(H+\mathrm{i})^{-1}\varphi,(H_0-\mathrm{i})^{-1}\psi],$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbf{R}^n)$. Thus for $f\in\mathscr{S}(\mathbf{R}^n)$ we have

$$(R\varphi,\psi)=q_{T-f}[(H+\mathrm{i})^{-1}\varphi,(H_0-\mathrm{i})^{-1}\psi]+q_f[(H+\mathrm{i})^{-1}\varphi,(H_0-\mathrm{i})^{-1}\psi]=$$

$$=(R_1\varphi,\psi)+(R_2\varphi,\psi)$$

where $R_2=(H_0+\mathrm{i})^{-1}f(H+\mathrm{i})^{-1}$ is compact. Using (6) we obtain

$$|(R_1\varphi,\psi)|\leqslant C_{n,s,p}\|T-f\|_{s,p}\|(H_0+\mathrm{i})^{\frac{1}{2}}(H+\mathrm{i})^{-1}\varphi\|_2\|(H_0+\mathrm{i})^{-\frac{1}{2}}\psi\|_2$$

which shows that $\|R_1\|\to 0$ as $\|T-f\|_{s,p}\to 0$. Thus R is a norm limit of compact operators, so is compact. The theorem is proved.

It may be observed that the simplicity of the above proof depends on the assumed symmetry of q_T. For more general potentials, giving rise to closed, m-sectorial operators, the boundedness of $(H_0+z)^{\frac{1}{2}}(H+z)^{-1}$ is not guaranteed. In an interesting paper Tip [18] obtained, in one dimension, the above theorem for a class of perturbations of H_0 by regular Borel, not necessarily real measures on $\mathbf{R}$ (this class is contained in $H^{-1,2}(\mathbf{R})$).

As a consequence of the proof of Theorem 4 we obtain that distributional potentials vanish at infinity in the following sense.

COROLLARY 1. *Suppose T and H as in Theorem 4. For $a\in\mathbf{R}^n$ let Tr_a denote the translation by a. Then*

$$\mathrm{Tr}_a[(H_0+\mathrm{i})^{-1}-(H+\mathrm{i})^{-1}]\mathrm{Tr}_{-a}\to 0$$

strongly as $|a|\to\infty$.

We note that in one dimension Tip [18] obtained this result for his class of potentials.

It is well known that the wave operators

$$\Omega^{\pm} = \text{s-}\lim_{t \to \mp\infty} e^{itH} e^{-itH_0}$$

exist and are complete if V is the Dirac delta function δ or a finite combination of δ's $(n = 1)$, see [8]. The following theorem generalizes this result.

THEOREM 5. *Let $T \in \mathscr{D}'(\mathbf{R})$ be such that for some $\alpha > \frac{1}{4}$, $(1 + |\cdot|^2)^\alpha T \in H^{-1,2}(\mathbf{R})$, and put $H = H_0 \dot{+} T$. Then the wave operators $\Omega^{\pm}$ exist and are complete.*

Proof. We shall show that $R = (H_0 + k)^{-1} - (H + k)^{-1}$ is a trace class operator, $k > -\inf \sigma(H)$. The result of the theorem follows then upon applying the Birman-Rozenblyum theory [13]. As in the proof of the preceding theorem we have $(R\varphi, \psi) = q_T[(H + k)^{-1}\varphi, (H_0 + k)^{-1}\psi]$. Obviously $T = G + \frac{d}{dx} G$, $G \in L^2(\mathbf{R})$ and, moreover, $(1 + |\cdot|^2)^\alpha G \in L^2(\mathbf{R})$. Thus we have

$$(R\varphi, \psi) = (G(H + k)^{-1}\varphi, (H_0 + k)^{-1}\psi) +$$

$$+ (G(H + k)^{-1}\varphi, D(H_0 + k)^{-1}\psi) + (GD(H + k)^{-1}\varphi, (H_0 + k)^{-1}\psi),$$

where $D = \frac{d}{dx}$. Since $D(H + k)^{-1}$, $D(H_0 + k)^{-\frac{1}{2}}$ and $(H_0 + k)^{\frac{1}{2}}(H + k)^{-\frac{1}{2}}$ are bounded operators we need only to show that $(H_0 + k)^{-1}G$ and $(H_0 + k)^{-\frac{1}{2}}G(H_0 + k)^{-\frac{1}{2}}$ are trace class operators, which follows directly from [13], Theorem XI.20 and Theorem XI.21. The theorem is proved. ○

Note that in the above theorem the condition on α is optimal, as for the Coulomb potential the wave operators $\Omega^{\pm}$ do not exist. It seems likely that analogous results are true in higher dimensions, but since Birman-Rozenblyum theory gives poor results for $n > 1$, they are probably more difficult to prove.

As mentioned in the Introduction, Albeverio *et al.* [2] proved a result on approximation of one-dimensional Schrödinger operators with point interactions by operators with locally integrable potentials. Their result, appearing as Corollary 2 below, is an immediate consequence of the following theorem.

THEOREM 6. *Suppose $H^{s,p}(\mathbf{R}^n)$ satisfies the* KLMN *condition, $p < \infty$. Let $T_j, T \in H^{s,p}(\mathbf{R}^n)$, $H_j = H_0 \dot{+} T_j$, $H = H_0 \dot{+} T$. If $T_j \to T$ in $\|\cdot\|_{s,p}$ norm, then $H_j \to H$ in uniform resolvent convergence.*

Proof. Let $f \in \mathscr{S}(\mathbf{R}^n)$ be such that $C_{n,s,p}\|T - f\|_{s,p} < \frac{1}{4}$, where $C_{n,s,p}$ is the constant appearing in (6). Choose N so that $C_{n,s,p}\|T_j - f\|_{s,p} < \frac{1}{2}$ for $j \geqslant N$. We have then

$$|q_{T_j}[\varphi]| \leqslant \frac{1}{2}(H_0\varphi, \varphi) + C_1\|\varphi\|_2^2, \tag{8}$$

where $C_1 = \|f\|_\infty + \frac{1}{2}$. Put $k = C_1 + 1$, then $H + k \geqslant 0$, $H_j + k \geqslant 0$. We will show that

$$\|(H_j + k)^{-1} - (H + k)^{-1}\| \leqslant 2C_{n,s,p}\|T_j - T\|_{s,p} \tag{9}$$

for $j \geqslant N$. This inequality implies the result of the theorem immediately. To show (9) observe that

$$((H_j + k)^{-1}\varphi, \psi) - ((H + k)^{-1}\varphi, \psi) = q_{T-T_j}[(H_j + k)^{-1}\varphi, (H + k)^{-1}\psi]$$

so using the inequality (6) we find

$$|(\{(H_j + k)^{-1} - (H + k)^{-1}\}\varphi, \psi)| \leqslant C_{n,s,p}\|T_j - T\|_{s,p}\|(H_0 + 1)^{\frac{1}{2}}(H_j + k)^{-1}\varphi\|_2 \times$$
$$\times \|(H_0 + 1)^{\frac{1}{2}}(H + k)^{-1}\psi\|_2.$$

The operators $(H_0 + 1)^{\frac{1}{2}}(H_j + k)^{-\frac{1}{2}}$ and $(H_0 + 1)^{\frac{1}{2}}(H + k)^{-\frac{1}{2}}$ are bounded; we have to show that their norms are bounded in j. We have, using (8) and the definition of k,

$$\|(H_j + k)^{\frac{1}{2}}\varphi\|_2^2 = ((H_0 + k)\varphi, \varphi) + q_{T_j}[\varphi] \geqslant$$
$$\geqslant ((H_0 + 1)\varphi, \varphi) - \frac{1}{2}(H_0\varphi, \varphi) \geqslant \frac{1}{2}\|(H_0 + 1)^{\frac{1}{2}}\varphi\|_2^2,$$

so

$$\|(H_0 + 1)^{\frac{1}{2}}(H_j + k)^{-\frac{1}{2}}\varphi\|_2 \leqslant \sqrt{2}\,\|\varphi\|_2.$$

Analogously, $\|(H_0 + 1)^{\frac{1}{2}}(H + k)^{-\frac{1}{2}}\| \leqslant \sqrt{2}$. The formula (9), and consequently the theorem, is proved.

The above theorem may be extended to potentials which do not vanish at infinity. Using Theorem 2 in place of (6) we can prove

THEOREM 7. *Suppose s, p are as in Theorem 2, $T_j, T \in H^{s;p}_{u,loc}(\mathbf{R}^n)$, $H_j = H_0 \dot{+} T_j$, $H = H_0 \dot{+} T$. If*

$$\lim_{j\to\infty} \sup_{\tau\in\mathbf{R}^n} \|\varphi_\tau(T_j - T)\|_{s,p} = 0$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$, then $H_j \to H$ in norm resolvent convergence.

The above two theorems allow us to restate the main result of Albeverio *et al.* [2] in a somewhat generalized form.

COROLLARY 2. *Suppose that $a_j \in \mathbf{R}$, $V_j \in L^1(\mathbf{R})$, V_j real-valued, $\int V_j(x)dx = 1$ $j = 1, \ldots, N$. Moreover let $\lambda_j(\varepsilon)$ be real-valued on $[0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, continuous at $\varepsilon = 0$. Put*

$$V^\varepsilon(x) = \varepsilon^{-1} \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j\left(\frac{1}{\varepsilon}(x - a_j)\right)$$

and let $H^\varepsilon = H_0 \dot{+} V^\varepsilon$ in the sense of forms. Then $H^\varepsilon \to H$ in norm resolvent convergence, where $H = H_0 \dot{+} \sum_{j=1}^{N} \lambda_j(0)\delta_{a_j}$ (here δ_{a_j} denotes the Dirac delta function supported by $\{a_j\}$).

Proof. Without loss of generality we consider the case $N = 1$, $a_1 = 0$. Put $T^\varepsilon = \varepsilon^{-1}V\left(\frac{1}{\varepsilon}\cdot\right)$, $T = \lambda_1(0)\delta$. In view of Theorem 6 we need only to check that $T^\varepsilon \to T$ in $H^{s,2}(\mathbf{R})$, $s \geq -1$. We have $\hat{T}^\varepsilon(k) = \lambda_1(\varepsilon)\hat{V}(\varepsilon k)$, $\hat{T}(k) = \lambda_1(0)\hat{V}(0)$, $\hat{T}^\varepsilon - \hat{T}$ is uniformly bounded and converges pointwise to zero as $\varepsilon \to 0$. Hence

$$\|T^\varepsilon - T\|_{s,2}^2 = \int (1 + |k|^2)^s |\hat{T}^\varepsilon(k) - \hat{T}(k)|^2 dk \to 0$$

for any $s < -\frac{1}{2}$. The corollary now follows upon applying Theorem 6.

Albeverio *et al.* [2] treat also the case of infinitely many δ's. The following result generalizes [2] considerably since it allows the supports of δ's to have accumulation points (we omit the proof since it is analogous to the proof of Corollary 2 except that we use Theorem 7 in place of Theorem 6).

COROLLARY 3. *Suppose* $a_j, \alpha_j \in \mathbf{R}$, $i \in \mathbf{N}$ *are such that* $\sup_{\tau\in\mathbf{R}} \sum_{\{j: a_j \in (\tau, \tau+1)\}} |\alpha_j| < \infty$. *Let* $V_j \in L^1(\mathbf{R})$ *be real-valued,* $\int V_j(x)\mathrm{d}x = 1$, *and such that there exists* $W \in L^1(\mathbf{R})$ *with* $|V_j(x)| \leqslant W(x)$ a.e. *Moreover let* $\lambda_j(\varepsilon)$ *be real-valued functions on* $[0, \varepsilon_0)$ *for some* $\varepsilon_0 > 0$, *equicontinuous at* $\varepsilon = 0$, $\lambda_j(0) = 1$. *Then the function*

$$V^\varepsilon(x) = \varepsilon^{-1} \sum_{j\in\mathbf{N}} \lambda_j(\varepsilon)\alpha_j V_j\left(\frac{1}{\varepsilon}(x - a_j)\right)$$

is uniformly $L^1(\mathbf{R})$ *for any* $0 < \varepsilon < \varepsilon_0$, *and the distribution* $T = \sum_{j\in\mathbf{N}} \alpha_j \delta_{a_j}$ *is in* $H^{s,2}_{\mathrm{u,loc}}(\mathbf{R})$ *for* $-1 < s < -\frac{1}{2}$. *Put* $H^\varepsilon = H_0 \dot{+} V^\varepsilon$, $H = H_0 \dot{+} T$. *Then* $H^\varepsilon \to H$ *in norm resolvent convergence as* $\varepsilon \to 0$.

Our final example concerns singular measures in $\mathbf{R}^n$, $n \geqslant 2$. Let $M \subset \mathbf{R}^n$ be a regularly embedded submanifold of codimension 1 and let μ be the Lebesgue measure on M (see IX.9 of [12]). For $f \in L^p(M, \mathrm{d}\mu)$ let $f\,\mathrm{d}\mu$ denote the distribution defined as

$$\langle f\mathrm{d}\mu, \varphi\rangle = \int_M \varphi(m)f(m)\mathrm{d}\mu(m)$$

for $\varphi \in C_0^\infty(\mathbf{R}^n)$. Theorem IX.39 of [12] may be strengthened to the following.

LEMMA 5. *If* $p \geqslant 2$, $sp < -1$, *then* $f\,\mathrm{d}\mu \in H^{s,p}(\mathbf{R}^n)$ *for any* $f \in L^p(M, \mathrm{d}\mu)$.

EXAMPLE 2. The space $L^p(M, \mathrm{d}\mu)$, treated as the space of distributions of the form $f\,\mathrm{d}\mu$ satisfies the KLMN condition if $p \geqslant 2$, $p > \frac{n+1}{2}$. Indeed, the second condition on p implies that $\frac{n}{p} - 2 < -\frac{1}{p}$, so we need only set $s = \max\left(\frac{n}{p} - 2, -1\right)$ and apply the above lemma and Theorem 1.

Observe that if M is a hyperplane, then for $p \geqslant n$ we need not use Lemma 5. Assume for simplicity that $M = \{x : x_1 = 0\}$, and let $f \in L^p(M, \mathrm{d}\mu)$. Put $G(x) = f(x_2, \ldots, x_n)\varphi(x_1)$, where φ satisfies

$$\varphi(x_1) = \begin{cases} 0 & \text{if } x_1 < 0, \\ 1 & \text{if } 0 < x_1 < 1, \\ 0 & \text{if } 2 < x_1, \end{cases}$$

and is smooth on $(0, \infty)$, then $G \in L^p(\mathbf{R}^n)$, so $\frac{\partial}{\partial x_1} G \in H^{-1,p}(\mathbf{R}^n)$. We need only observe that $\frac{\partial}{\partial x_1} G - f\,\mathrm{d}\mu \in L^p(\mathbf{R}^n)$ and apply Theorem 1. The same trick works if, M still being a hyperplane, f is bounded (or essentially bounded, $f \in L^\infty(M, \mathrm{d}\mu)$), and we use Lemma 3 in place of Theorem 1. This covers the often discussed case of f being the indicator function of an arbitrary measurable subset of M.

Finally note that if M has codimension 2 (or more), $f\,\mathrm{d}\mu$ will generally violate the KLMN condition.

EXAMPLE 3. Let $n \geqslant 3$, $M = \{x \in \mathbf{R}^n,\ x_1 = x_2 = 0\}$, $f \in C_0^\infty(\mathbf{R}^{n-2})$, $f \neq 0$ $T = f\,\mathrm{d}\mu$, that is

$$\langle T, \varphi \rangle = \int \varphi(0, 0, x_3, \ldots, x_n) f(x_3, \ldots, x_n)\, \mathrm{d}x_3 \ldots \mathrm{d}x_n .$$

Let $\psi_k \in C_0^\infty(\mathbf{R}^2)$ be a sequence of functions converging in $\|\cdot\|_{1,2}$-norm to $\tilde{\psi}$ which in the vicinity of $x = 0$ satisfies $\tilde{\psi}(x) = (-\ln |x|)^{\frac{1}{4}}$ and is smooth and of compact support outside $x = 0$, $\tilde{\psi} \in H^{1,2}(\mathbf{R}^2)$. Fix any $\varphi \in C_0^\infty(\mathbf{R}^{n-2})$ such that $\int_{\mathbf{R}^{n-2}} \varphi(x) f(x) \mathrm{d}x > 0$. Then putting $\varphi_k(x) = \psi_k(x_1, x_2)\varphi(x_3, \ldots, x_n)$ we have $\sup_k \|\varphi_k\|_{1,2} < \infty$ and $\langle T, |\varphi_k|^2 \rangle \to \infty$, which shows that T does not satisfy the KLMN condition.

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