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DIFFERENTIABLE STRUCTURE OF SIMILARITY ORBITS

E. ANDRUCHOW and D. STOJANOFF

INTRODUCTION

It is well known that for any $A \in L(\mathbf{C}^n)$ the similarity orbit of A, $S(A) = \{UAU^{-1} : U \in \mathrm{Gl}_n(\mathbf{C})\}$, is a holomorphic submanifold of $L(\mathbf{C}^n)$. This is no longer the case when, instead of $\mathbf{C}^n$, we consider an infinite dimensional separable complex Hilbert space H. In this paper we characterize the elements $T \in L(H)$ for which $S(T) = \{UTU^{-1} : U \in G(H)\}$ is a holomorphic submanifold of $L(H)$ (these are the same operators whose orbits have any kind of differentiable structure). The main theorem of this paper establishes that they are the operators that are similar to nice Jordan operators. This class was studied by Fialkow and Herrero in [3], and is characterized, for example, as the class of operators $T \in L(H)$ such that

$$\pi_T : G(H) \to S(T), \quad \pi_T(U) = UTU^{-1}$$

has continuous local cross sections.

In Section 1 we prove that $S(T)$ is a holomorphic submanifold of $L(H)$ when T is similar to a nice Jordan nilpotent operator.

In Section 2 we prove the main theorem (2.3), which establishes the mentioned equivalence.

In Section 3 the canonical decomposition of a nilpotent operator $\varphi : S(T) \to P_n^+(H)$, is studied (see [1]). We show that when continuous, φ is also C^∞. With the help of it, we are able to exhibit explicit expressions of C^∞ local cross sections for π_T (although the holomorphic structure of $S(T)$ assures the existence of holomorphic cross sections). This is done generalizing a formula obtained by Fialkow [6] for the case when T is similar to the ampliation of the nilpotent $n \times n$ Jordan cell. Also we include a remark considering the analogous problem in the Calkin algebra.

In Section 4 we consider the case of unitary orbits. In Theorem 4.5 we prove that the unitary orbit $U(T)$ of $T \in L(H)$ is a C^∞ submanifold of $L(H)$ if and only if T satisfies one of the equivalent conditions of Theorem 2.1 in [5], i.e., $C^*(T)$ is finite dimensional.

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NOTATION

Let H be a separable complex Hilbert space and let $L(H)$ be the set of all bounded linear operators acting on H. $G(H)$ will denote the group of invertible operators of $L(H)$. If M is a closed linear manifold of H, we write P_M for the orthogonal projection onto M. Let $q_n \in L(\mathbf{C}^n)$ be the n-dimensional Jordan cell. $B \in L(H)$ is a nice Jordan operator if it has the form

$$B = \bigoplus_{i=1}^{r} (\lambda_i + Q_i),$$

where the Q_i, $1 \leqslant i \leqslant r$, are nice Jordan nilpotent operators, that is

$$Q_i = \bigoplus_{j=1}^{n_i} q_j^{(\alpha_{ij})}$$

where $0 \leqslant \alpha_{ij} < \infty$ for every j but one index j_0. For the definitions of holomorphic submanifold, differential of a map and other geometric concepts used in this paper, we refer the reader to [8].

1. NICE JORDAN NILPOTENT OPERATORS

Let A be a complex Banach algebra, $1 \in A$, and let G be the group of invertible elements of A. $S(a) = \{vav^{-1} : v \in G\}$ is called the *similarity orbit* of $a \in A$. For a fixed $a \in A$, we have the holomorphic map

$$\pi_a : G \to A, \quad \pi_a(v) = vav^{-1}.$$

Let $\delta_a = d(\pi_a)_1$. Observe that $\delta_a(x) = xa - ax$, $x \in A$.

Troughout the work we will use the following corollary of the Implicit Function Theorem.

PROPOSITION 1.1. *If $a \in A$ verifies*

1) $\ker \delta_a$ *is a complemented subspace of* A,

2) $R\delta_a$ *is a complemented subspace of* A, *and*

3) $\pi_a : G \to S(a)$ *is an open map,*

then $S(a)$ *is a holomorphic submanifold of* A *and* π_a *is a submersion.*

For a complete proof of the preceding result see [10].

REMARK 1.2. If 1.1 holds, then the tangent space $T[S(a)]_a$ can be identified with the subspace $\{xa - ax : x \in A\}$ of A.

Also, $\pi_a : G \to S(a)$ defines a holomorphic homogeneous space.

We will use this result to characterize the elements $T \in L(H)$ such that $S(T)$ is a submanifold of $L(H)$. We can restrict ourselves to operators which are similar to nice Jordan operators. Indeed, in [3] Fialkow and Herrero proved (Theorem 16.1, p. 352) that if $S(T)$ is locally closed (i.e. given $Q \in S(T)$ there exists $\varepsilon > 0$ such that $\{N : \|N - Q\| \leqslant \varepsilon\} \cap S(T)$ is a closed subset of $L(H)$), then T mus be similar to a nice Jordan operator.

So, if $S(T)$ is a submanifold of $L(H)$ (of any type), then T must be similar to a nice Jordan operator.

In the same theorem, Fialkow and Herrero also showed that π_T is open if and only if T is similar to a nice Jordan. And in a previous lemma ([3], 16.10, p. 362) they proved that if T is similar to a Jordan operator, then $\{x : xT = Tx\} = \ker \delta_T$ is a complemented complex linear manifold of $L(H)$. Therefore, in view of 1.1, to prove our statement it only remains to see that $R\delta_T$ is complemented.

We will end this section proving the latter for the case when T is nilpotent.

DEFINITION 1.3. J is a *Jordan nilpotent* if $J = \bigoplus_{i=1}^{n} q_i^{(\alpha_i)}$, $0 \leqslant \alpha_i \leqslant \infty$.

PROPOSITION 1.4. *If J is Jordan and $J^n = 0$, then $\delta_J : L(H) \to L(H)$ has complemented range.*

Proof. Let $J = q_1^{(\alpha_1)} \oplus q_2^{(\alpha_2)} \oplus \ldots \oplus q_n^{(\alpha_n)}$, $0 \leqslant \alpha_i \leqslant \infty$. Put $J_i = q_i^{(\alpha_i)}$, and H_i the subspace of H associated to J_i, $1 \leqslant i \leqslant n$, and let $H_{ij} = \ker J_i^j \ominus \ker J_i^{j-1}$, $1 \leqslant j \leqslant i$. Observe that $\dim H_{ij} = \alpha_i$. So we can suppose that all H_{ij} are equal to H_i' for a fixed i and $H_i = H_{i1} \oplus H_{i2} \oplus \ldots \oplus H_{ii} = H_i'^{(i)}$. Set $P_{ij} = P_{H_{ij}}$. Then

$$P_{ij} J P_{i,j-1} = P_{ij} J_i P_{i,j-1} : H_{i,j+1} \to H_{ij}$$

is the identity map of H_i', and $P_{ij} J P_{ik} = 0$ if $k \neq j - 1$:

$$= \left[\begin{array}{c|cc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & I & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & I & 0 & \\ 0 & 0 & 0 & 0 & 0 & I & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline \vdots & \vdots & \vdots & & \vdots & & \ddots \end{array}\right] \begin{array}{l} H_1 \\ \left.\begin{array}{l} H_{21} \\ H_{22} \end{array}\right\} H_2 = H_2'^{(2)} \\ \left.\begin{array}{l} H_{31} \\ H_{32} \\ H_{33} \end{array}\right\} H_3 = H_3'^{(3)} \\ \vdots \end{array}$$

.

With these decompositions we can represent the elements of $L(H)$ as matrices of two types:

Using $H_1, \ldots, H_n$ we get $A = (A_{r,s})_{r,s=1}^n$, and using $H_{11}, H_{21}, H_{22}, \ldots, H_{n,n-1}, H_{nn}$ we get

$$A = ((A_{rk,sl})_{r,s=1}^n : 1 \leqslant k \leqslant r;\ 1 \leqslant l \leqslant s;\ 1 \leqslant r, s \leqslant n).$$

Therefore, each block $A_{r,s}$ of the first representation can be seen as a matrix whose blocks are

$$A_{r,k,s,l} : H'_s \to H'_r.$$

For example if $n = 3$, $B = (B_{r,k,s,l})$, then

$$\delta_J(B) =$$

$$(1)\qquad = \left[\begin{array}{c|cc|ccc}
0 & 0 & -B_{11,12} & 0 & -B_{11,31} & -B_{11,32} \\
\hline
B_{22,11} & B_{22,21} & B_{22,22} - B_{21,21} & B_{22,31} & B_{22,32} - B_{21,31} & B_{22,33} - B_{21,32} \\
0 & 0 & -B_{22,21} & 0 & -B_{22,31} & -B_{22,32} \\
\hline
B_{32,11} & B_{32,21} & B_{32,22} - B_{31,21} & B_{32,31} & B_{32,32} - B_{31,31} & B_{32,33} - B_{31,32} \\
B_{33,11} & B_{33,21} & B_{33,22} - B_{32,21} & B_{33,31} & B_{33,22} - B_{32,31} & B_{33,33} - B_{32,32} \\
0 & 0 & -B_{33,21} & 0 & -B_{33,31} & -B_{33,32}
\end{array}\right].$$

Observe that if $C = \delta_J(B)$, then C_{rs} depends only on the entries of the block B_{rs}. This can be verified in a straightforward way using (1). Thus, it will be enough to study each (r, s)-block of $\delta_J(B)$ separately. Fix r and s, and put $m = \min\{r, s\} - 1$.

The operators $C \in \mathrm{R}\delta_J$ will be characterized by the following linear equations:

$$C_{rr,s1} = 0$$

$$C_{rr-1,s1} + C_{rr,s2} = 0$$

$$C_{rr-2,s1} + C_{rr-1,s2} + C_{rr,s3} = 0$$

$$\vdots$$

$$C_{rr-m,s1} + C_{rr-m+1,s2} + \ldots + C_{rr,sm} = 0,$$

and the entries $C_{rk,sl}$ can be freely chosen if $k < l + r - m - 1$, or, equivalently $k - l < r - \min\{r, s\}$.

The $m + 1$ equations mean that the $m + 1$ diagonals under the main diagonal of C_{rs} have "null trace". Now we can exhibit a supplement for $\mathrm{R}\delta_J$, which we will denote $\mathrm{M}(\delta_J)$:

$$D = (D_{rk,sl}) \in \mathrm{M}(\delta_J)$$

if and only if for every r and s, we have

$$D_{rk,sl} = 0 \quad \text{if } k - l < r - m - 1$$

and

$$D_{rr-1,s1} = D_{rr-i+1,s2} = \dots = D_{rr,si}, \quad 0 \leqslant i \leqslant m.$$ ▨

REMARK 1.5. Reasoning analogously we can obtain a supplement for $\ker \delta_J$, in such a way that if we consider the involution $*: L(H) \to L(H)$ then

$$*(\mathrm{M}(\delta_J)) = \ker \delta_J$$

and

$$*(\mathrm{R}(\delta_J)) \text{ is a supplement for } \ker \delta_J.$$

COROLLARY 1.6. *Let T be a nilpotent operator. $S(T) \subseteq L(H)$ is a holomorphic submanifold if and only if T is similar to a nice Jordan nilpotent operator. In that case the map $\pi_T : G(H) \to S(T)$ defines a holomorphic homogeneous space.*

REMARK 1.7. Let $N_n(H) = \{T \in L(H) : T^n = 0,\ T^{n-1} \neq 0\}$ and put $V_n(H) = \bigcup_{j=0}^{n-1} S(q_j \oplus q_n^{(\infty)})$; then $V_n(H)$ is open and dense in $N_n(H)$ (see [7], Chapter 7 and 8) and furthermore, $V_n(H)$ is a holomorphic submanifold of $L(H)$.

2. NICE JORDAN OPERATORS

We will study now the case when T is similar to a (not necessarily nilpotent) nice Jordan operator. To prove that $S(T)$ is a submanifold we will use the following statement, which is a special case of a general geometric fact.

Let A be a complex Banach algebra, G the group of invertible elements.

Suppose that G acts holomorphically on A and for a fixed $a \in A$, consider the holomorphic map

$$\pi_a(g) = g * a, \quad g \in G$$

where $*$ denotes the action of G into A. Let $S \subset A$ be the orbit of a.

PROPOSITION 2.1. *With the above notations, the following are equivalent:*

1) *There exists an open neighborhood U_a of a in A, and a holomorphic map*

$$\omega_a : U_a \to G, \quad \text{such that } \omega_a(a) = 1$$

and $\omega_a | U_a \cap S$ is a local cross section for π_a.

2) *$\pi_a : G \to S$ defines a holomorphic homogeneous space.*

Proof. First we check that 1) implies 2). It is enough to prove that the conditions required by 1.1 hold. Clearly, π_a is open.

We will see that

$$d(\pi_a)_1 \circ d(\omega_a)_a \circ d(\pi_a)_1 = d(\pi_a)_1$$

or equivalently

$$d(\pi_a \circ \omega_a)_a \mid Rd(\pi_a)_1 = \mathrm{id}_{Rd(\pi_a)_1}.$$

Fix $V \in A$. Let $\alpha(t) = \pi_a(1 + tV)$, $t \in (-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ sufficiently small so that $1 + tV \in G$.

Then $\alpha'(0) = d(\pi_a)_1(V)$, $\alpha(0) = a$ and $\alpha(t) \in S$ for $t \in (-\varepsilon, \varepsilon)$.

Therefore

$$d(\pi_a \circ \omega_a)_a(d(\pi_a)_1(V)) = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \circ \omega_a \circ \alpha(t))\big|_{t=0} = \alpha'(0) = d(\pi_a)_1(V).$$

Then $d(\pi_a)_1 \circ d(\omega_a)_a$ and $d(\omega_a)_a \circ d(\pi_a)_1$ are idempotent operators acting on A, and

$$\ker d(\pi_a)_1 = \ker[d(\omega_a)_a \circ d(\pi_a)_1],$$

$$Rd(\pi_a)_1 = R[d(\pi_a)_1 \circ d(\omega_a)_a].$$

On the other hand, it is well known that if $\pi_a : G \to S$ is a submersion, then it has holomorphic local cross sections. Let V be a coordinated neighborhood of a, i.e., there is a diffeomorphism $\mu : U \to V$, where U is a neighborhood of 0 in the Banach space $E = E_1 \oplus E_2$, where E_1 and E_2 are supplements of $\ker d(\pi_a)_1$ and $Rd(\pi_a)_1$ in A respectively, such that

$$\mu(E_1 \cap U) = V \cap S.$$

Put $\omega : U \to G$, $\omega(e_1, e_2) = \exp(e_1)$ and define $\omega_a : V \to G$, $\omega_a(b) = \omega(\mu^{-1}(b))$, which is the required cross section. ▧

We will construct local cross sections for $\pi_T : G(H) \to S(T)$ which are restrictions of holomorphic maps defined on open subsets of $L(H)$, assuming the existence of them in the nilpotent case, in view of 1.6 and 2.1. Let $A \in L(H)$ be a nice Jordan operator,

$$A = \bigoplus_{i=1}^{r} (\lambda_i + Q_i),$$

where Q_i are nice Jordan nilpotents, $1 \leqslant i \leqslant r$.

Remarks 2.2. 1) We define $P_i(T) = \frac{1}{2\pi i} \int_{\partial B(\lambda_i, \varepsilon_i)} (\lambda - T)^{-1} \mathrm{d}\lambda$, $B(\lambda_i, \varepsilon_i) = \{z \in \mathbf{C}; |\lambda_i - z| < \varepsilon_i\}$, $1 \leqslant i \leqslant r$. $P_i(T)$ is a holomorphic map defined on a neighborhood of A in $L(H)$. It is clear that if $\sigma(T) = \{\lambda_1, \ldots, \lambda_r\}$, then $P_i(T)$ is the spectral projection associated to λ_i. We can obtain another holomorphic map Γ, also defined on an open neighborhood of A in $L(H)$, with values in $G(H)$, such that

$$(1) \qquad P_i(\Gamma(T)T\Gamma(T)^{-1}) = P_i(A), \quad 1 \leqslant i \leqslant r.$$

Indeed, $P_i(T)$, $1 \leqslant i \leqslant r$, define a system of idempotents. If T is sufficiently close to A so that

$$\sum_{i=1}^{r} P_i(A)P_i(T) \in G(H),$$

we can define

$$\Gamma(T) = \sum_{i=1}^{r} P_i(A)P_i(T).$$

Condition (1) holds because $P_i(\Gamma(T)\,T\Gamma(T)^{-1}) = \Gamma(T)P_i(T)\Gamma(T)^{-1}$ and $P_i(A)\Gamma(T) = \Gamma(T)P_i(T)$.

2) Suppose that $T \in S(A)$, T close to A, and moreover $P_i(T) = P_i(A)$, $1 \leqslant i \leqslant r$. Given $U \in G(H)$ such that $T = UAU^{-1}$ then U is "diagonal" with respect to $P_1(A), \ldots, P_r(A)$, that is, U commutes with $P_i(A)$, $1 \leqslant i \leqslant r$. Indeed

$$P_i(A) = P_i(T) = P_i(UAU^{-1}) = UP_i(A)U^{-1}.$$

With the previous notations, we have the following:

THEOREM 2.3. *$S(T)$ is a holomorphic submanifold of $L(H)$ if and only if T is similar to a nice Jordan operator.*

Proof. The necessary part is clear.

Now let T be similar to A,

$$A = \bigoplus_{i=1}^{r} (\lambda_i + Q_i), \quad Q_i = \bigoplus_{j=1}^{n_i} q_j^{(\alpha_{ij})},$$

$0 \leqslant \alpha_{ij} < \infty$ for every j but one index j_i, $1 \leqslant i \leqslant r$.

Set $P_i = P_i(A)$ and $H_i = \mathrm{R}(P_i)$. Let $B \in L(H)$, B in a neighborhood of A in $L(H)$ taken as in 2.2.1. Put

$$\psi(B) = \Gamma(B)B\Gamma(B)^{-1}.$$

Because of 1.6 and 2.1, for each Q_i, seen as an element of $L(H_i)$, there is a holomorphic map τ_i, $\tau_i : U_i \to G(H_i)$, defined on a open subset U_i of $L(H_i)$, such that τ_i restricted to $S(Q_i)$ is a local cross section for π_{Q_i}.

Consider the holomorphic map $\theta_i : V_i \to U_i$, from a neighborhood V_i of A in $L(H)$ into U_i, given by

$$\theta_i(B) = P_i\psi(B)P_i - \lambda_i P_i,$$

where we identify $L(H_i)$ with $P_iL(H)P_i$, and V_i is taken such that $\psi(T)$ is defined and $\theta_i(T) \in V_i$. Consider

$$\begin{array}{l} A \in V_i \xrightarrow{\theta_i} U_i \xrightarrow{\tau_i} G(H_i) \xhookrightarrow{l} L(H) \\ \quad\;\; \cap \qquad \cap \\ \;\; L(H)\, L(H_i) \end{array}$$

where l is an "inclusion" of $L(H_i)$ into $L(H)$.

Let $\bar{\omega}_i$ be the composition map. It is clear that $\bar{\omega}_i$ is holomorphic. Now put

$$V = \bigcap_{i=1}^{r} V_i \text{ and } \bar{\omega}: V \to L(H), \quad \bar{\omega}(B) = \sum_{i=1}^{r} \bar{\omega}_i(B);$$

then $\bar{\omega}$ is holomorphic and $\bar{\omega}(B)$ is a "diagonal" matrix with respect to $P_1, \ldots, P_r$. Also, it is clear that its entries are invertible in the corresponding $L(H_i)$; therefore $\bar{\omega}(B) \in G(H)$, $B \in V$.

Now take $C \in V \cap S(A)$. Then A, $\psi(C)$ and $\bar{\omega}(C)$ are diagonal with respect to $P_1, \ldots, P_r$, so operations between them can be done in each coordinate

$$\pi_A \circ \bar{\omega}(C) = \bar{\omega}(C)A\bar{\omega}(C)^{-1} = \sum_{i=1}^{r} \bar{\omega}_i(C)(Q_i + \lambda_i P_i)\bar{\omega}_i(C)^{-1} =$$

$$= \sum_{i=1}^{r} (\bar{\omega}_i(C)Q_i\bar{\omega}_i(C)^{-1} + \lambda_i P_i) = \sum_{i=1}^{r} (\theta_i(C) + \lambda_i P_i) = \psi(C).$$

Remember that $\psi(C) = \Gamma(C)C\Gamma(C)^{-1}$, so if we define $\omega(C) = \Gamma(C)^{-1} \mid \bar{\omega}(C)$ then ω is a holomorphic map defined on a neighborhood V of A in $L(H)$, and ω restricted to $S(A)$ is a local cross section for π_A. ▨

3. EXPLICIT CROSS SECTIONS

We will study now the differentiability of the map φ introduced in [1]. Let $T \in L(H)$, $T^n = 0$ and $T^{n-1} \neq 0$. Let $P_n^+(H) = \{(P_1, \ldots, P_n) \in L(H)^n : P_i^2 = P_i^* = P_i,\ P_iP_j = 0 \text{ if } i \neq j,\ \sum_{i=1}^{n} P_i = I \text{ and } \operatorname{rank}(P_{i+1}) \leqslant \operatorname{rank}(P_i)\}$, and define

$$\varphi(T) = (P_{\ker T},\ P_{\ker T^2} - P_{\ker T},\ \ldots,\ P_{\ker T^{n-1}} - P_{\ker T^{n-2}},\ I - P_{\ker T^{n-1}}).$$

In [1] we observed that $\varphi|S(T)$ is continuous when T is similar to a nice Jordan nilpotent. We know now that then $S(T)$ is a submanifold of $L(H)$.

On the other hand, $P_n^+(H)$ is a disjoint union of open connected components of $P_n(H)$, which has a rich structure studied in [4].

We will show that φ is a C^∞ map between these two manifolds. The reduced minimum modulus studied by Apostol in [2] plays an important role here.

DEFINITION 3.1. The *reduced minimum modulus* $\gamma(B)$ of $B \in L(H)$ is defined as follows:

$$\gamma(B) = \begin{cases} \inf\{\|Bx\| : \operatorname{dist}(x, \ker B) = 1\} & \text{if } B \neq 0 \\ 0 & \text{if } B = 0. \end{cases}$$

It is not difficult to see that

$$\gamma^2(B) = \inf\{\lambda \in \sigma(B^*B)\setminus\{0\}\}$$

and that $\gamma(B) > 0$ if $R(B)$ is closed.

We will use the following formula proved by Apostol in [2] (Proposition 1.1 (iii)). If $B,\ C \in L(H)$, then

(1) $$|\gamma(B) - \gamma(C)| \leqslant \|P_{\ker B} - P_{\ker C}\| \max\{\gamma(B), \gamma(C)\} + \|B - C\|.$$

PROPOSITION 3.2. *If T is similar to a nice Jordan nilpotent operator, then $\varphi|S(T) \to P_n^+(H)$ is a C^∞ map.*

Proof. Let $Q \in S(T)$; our purpose is to show that there exists an open subset U_Q of $L(H)$, and a map $\bar{\varphi}_Q : U_Q \to L(H)^n$, which is C^∞ and satisfies

$$\bar{\varphi}_Q|S(T) \cap U_Q = \varphi|S(T) \cap U_Q.$$

It is clear that $\gamma(Q) > 0$. Let $\rho(Q^*Q)$ be the spectral radius of Q^*Q, and put

$$U_{1,Q} = \{N \in L(H) : \sigma(N^*N) \subset \{z \in \mathbf{C} : |z| < \gamma^2(Q)/3\} \cup$$

$$\cup \{z \in \mathbf{C} : (2/3)\gamma^2(Q) < |z| < 2\rho(Q^*Q)\}\}.$$

$U_{1,Q}$ is open in $L(H)$ and $Q \in U_{1,Q}$.

Take $U_{,Q} = U_{1,Q^i}$, $1 \leqslant i \leqslant n-1$, and put

$$V_Q = \bigcap_{i=1}^{n-1} U_{i,Q}.$$

Using (1) and the continuity of $\varphi|S(T)$, we can see that there exists an open subset W_Q of $L(H)$ such that if $N \in W_Q \cap S(T)$, then $\gamma^2(N^i) > \gamma^2(Q^i)/3$, $1 \leqslant i \leqslant n-1$.

Put $U_Q = W_Q \cap V_Q$ and let

$$\eta_Q^i(A) = \frac{1}{2\pi i} \int_{\{z \in \mathbb{C}: |z| = \gamma^2(Q^i)/3\}} (\lambda - A^{*i}A^i)^{-1}\, d\lambda,$$

for $A \in U_Q$, $1 \leqslant i \leqslant n-1$.

η_Q^i is well defined and C^∞ on U_Q, and moreover, if $N \in S(T) \cap U_Q$, then

$$\eta_Q^i(N) = \chi_0(N^{*i}N^i) = P_{\ker N^i}, \quad 1 \leqslant i \leqslant n-1$$

because $\sigma(N^{*i}N^i) \cap \{z \in \mathbb{C}: |z| < \gamma^2(Q^i)/3\} = \{0\}$. We can now define

$$\bar{\varphi}_Q = (\eta_Q^1,\ \eta_Q^2 - \eta_Q^1,\ \ldots, \eta_Q^{n-1} - \eta_Q^{n-2},\ I - \eta_Q^{n-1}).$$

Clearly, $\bar{\varphi}_Q$ is C^∞ on U_Q and

$$\bar{\varphi}_Q|S(T) \cap U_Q = \varphi|S(T) \cap U_Q \text{ holds.}$$

∎

This result can be applied to the computation of an explicit cross section for $\pi_T: G(H) \to S(T)$, for the case when T is a nice Jordan nilpotent. Observing the way in which cross sections for general nice Jordan operators were obtained from cross sections for nilpotents in 2.3, it is possible to compute formulas in all the cases.

Let J be a Jordan nilpotent, $J^n = 0$, $J^{n-1} \neq 0$,

$$J = q_1^{(\alpha_1)} \oplus q_2^{(\alpha_2)} \oplus \ldots \oplus q_n^{(\alpha_n)}, \quad 0 \leqslant \alpha_i \leqslant \infty.$$

We will use the same notations as in 1.3. Then

$$J(H_{ij}) = H_{i,j-1}, \quad 1 < j \leqslant i.$$

Put $P_{ij} = P_{H_{ij}}$. Given $T \in L(H)$, let

$$U(T) = P_{11} + P_{22} + TP_{22}J^* + P_{33} + TP_{33}J^* + T^2P_{33}J^{*2} + \ldots$$

$$\ldots + P_{n-1,n-1} + \ldots + T^{n-2}P_{n-1,n-1}J^{*n-2} + P_{nn} + \ldots \tag{2}$$

$$\ldots + T^{n-1}P_{nn}J^{*n-1} = \sum_{i=1}^{n}\sum_{j=0}^{i-1} T^jP_{ii}J^{*j}.$$

Observe that $U(T)$ can be regarded as a polynomial mapping in T.

PROPOSITION 3.3. *If $T \in S(J)$ and $\varphi(T) = \varphi(J)$, then $U(T)J = TU(T)$. Moreover, $U(J) = I$; therefore if T is closed enough to J, then $U(T) \in G(H)$.*

Proof. Let us compute $TU(T)$:

$$TU(T) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} T^j P_{ii} J^{*j-1}. \tag{3}$$

Here the fact that $R(P_{ii}) \subset H_i$ and that $\ker J^i = \ker T^i$ imply that $T^i P_{ii} = 0$. On the other hand

$$U(T)J = \left[\sum_{i=1}^{n} \sum_{j=1}^{i-1} T^j P_{ii} J^{*j-1} \right] J^* J. \tag{4}$$

Indeed, the fact that $R(P_{ii}) \subset R(J)^\perp$ implies that $P_{ii}J = 0$. To prove that $R(P_{ii}) \subset R(J)^\perp$, observe that

$$R(P_{ii}) = H_i \ominus \ker J_i^{i-1} = H_i \ominus R(J_i) \subset R(J)^\perp.$$

Now

$$J^*J = I - P_{\ker J} = I - \sum_{i=1}^{n} P_{i1}.$$

Then

$$P_{ij} J^* J = P_{ij} \quad \text{if } j > 1. \tag{5}$$

It is straightforward to verify that $P_{ii}J^{*h} = P_{ii}J^{*h}P_{i,i-h}$ for $0 \leqslant h \leqslant i-1$. Indeed,

$$J^h(H_{ij}) = \begin{cases} 0 & \text{if } j \leqslant h \\ H_{i,j-h} & \text{if } h < j \leqslant i-1 \end{cases}$$

so that

$$J^{*h}(H_{ij}) = \begin{cases} H_{i,j+k} & \text{if } h \leqslant i-j \\ 0 & \text{if } h > i-j. \end{cases}$$

So $J^{*h}(H_{i,i-h}) = H_{ii}$ and $J^{*h}(H_k) \subseteq H_{ii}^\perp$ in any other case. Then it follows that

$$P_{ii}J^{*h} = P_{ii}J^{*h}\left(\sum_{k,l} P_{kl}\right) = P_{ii}J^{*h}P_{i,i-h} \quad \text{if } 0 \leqslant h \leqslant i-2,$$

or equivalently, if $i - h \geqslant 2$. Using (5) it is verified that

$$P_{ii}J^{*h} = P_{ii}J^{*h}P_{i,i-h} = P_{ii}J^{*h}P_{i,i-h}J^*J = P_{ii}J^{*h}J^*J.$$

Briefly, we have showed that if $0 \leqslant h \leqslant i-2$, then

$$P_{ii}J^{*h} = P_{ii}J^{*h}J^{*}J.$$

If we combine this identity with (4) and observe that the exponent of J^* in each summand is never greater than $i-2$, we see that expression (3) is equal to (4).

The fact that $U(J) = I$ is straightforward. ▨

REMARKS 3.4. (1) In the case when J is a very nice Jordan nilpotent, that is, $J = q_n^{(\infty)}$, then

$$U(T) = \sum_{j=0}^{n-1} T^j P_{nn} J^{*j}.$$

Looking through the proof of (3.3), it is clear that the condition $\varphi(T) = \varphi(J)$ is not necessary in this case. In this case U defines a local cross section for π_J, very similar to the one obtained by Fialkow in [6].

2) Also, in the case $J = q_n^{(\infty)}$, it is straightforward to verify that if $\varphi(T) = \varphi(J) = (P_1, \ldots, P_n) = P$, then $U(T) \in G(H)$. Put $G_P = \{V \in G(H) : V(\ker J^k) = \ker J^k,\ k = 1, \ldots, n-1\}$. Observe that $U(T) \in G_P$.

G_P is a Banach Lie subgroup of $G(H)$, and acts on $\varphi^{-1}(P)$, which is a C^∞ submaniold of $L(H)$. Note that φ is a submersion for it has C^∞ cross sections (see [1]). The action is simply the restriction of the usual action of $G(H)$ on $S(J)$.

So the bundle (and C^∞ homogeneous space)

$$\pi_J|G_P : G_P \to \varphi^{-1}(P)$$

has a global cross section which is C^∞.

Now we will go back to the case when J is nice Jordan. $\varphi: S(J) \to P_n^+$ is continuous, and we can define

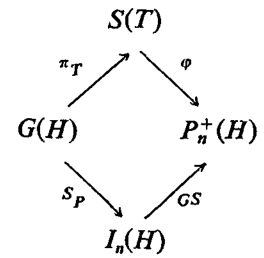
$$\alpha(T) = \sum_{i=1}^{n} \varphi_i(T)\varphi_i(J),$$

so that if T is close enough to J, then $\alpha(T) \in G(H)$.

LEMMA 3.5. *If $T \in S(J)$ and $\alpha(T) \in G(H)$, then*

$$\varphi(\alpha(T)^{-1}T\alpha(T)) = \varphi(J).$$

Proof. Consider the following commutative diagram introduced in [1]. Put $P = (P_1, \ldots, P_n) = \varphi(T)$,



where

$$I_n(H) = \left\{(Q_1, \ldots, Q_n) \in L(H)^n : Q_i^2 = Q_i, \sum_{i=1}^{n} Q_i = I \text{ and } Q_iQ_j = 0 \text{ if } i \neq j\right\},$$

$S_p(V) = (VP_1V^{-1}, \ldots, VP_nV^{-1})$ and GS is a process of "orthonormalization" of n-tuples of idempotents (see [1]). Using the commutativity of the diagram, we see that

$$\varphi(\alpha(T)^{-1}T\alpha(T)) = \varphi(\pi_T(\alpha(T)^{-1})) = GS(S_p(\alpha(T)^{-1})) =$$

$$= GS(\alpha(T)^{-1}\varphi_1(T)\alpha(T), \ldots, \alpha(T)^{-1}\varphi_n(T)\alpha(T)) =$$

$$= GS(\varphi_1(J), \ldots, \varphi_n(J)) = \varphi(J).$$ ▨

REMARK 3.7. Now we are able to compute cross sections for π_J when J is a nice Jordan nilpotent operator of order n. Let V be an open subset of $S(J)$, $J \in V$, such that $T \in V$ implies that $\alpha(T) \in G(H)$ and $U(\alpha(T)^{-1}T\alpha(T)) \in G(H)$ (U the same as in 3.3).

Put $\omega\colon V \to G(H)$,

$$\omega(T) = \left(\sum_{i=1}^{n} \varphi_i(T)\varphi_i(J)\right) U\left[\left(\sum_{i=1}^{n} \varphi_i(T)\varphi_i(J)\right)^{-1} T \left(\sum_{i=1}^{n} \varphi_i(T)\varphi_i(J)\right)\right].$$

It is straightforward to verify that ω is a C^∞ cross section for π_J.

REMARK 3.8. Let $A(H)$ be the Calkin algebra of H. It is well known that if $t \in A(H)$ and t is similar to $\pi(A)$ with A a nice Jordan, then $\pi_t\colon G(A(H)) \to S(t)$ has (polynomial) local cross sections (see [3], Chapter 16). Reasoning analogously as in $L(H)$, we obtain that $S(t)$ is a holomorphic submanifold of $A(H)$ if and only if t is similar to $\pi(A)$ with A a nice Jordan.

If t is also nilpotent, we can introduce $\tilde{\varphi}: S(T) \to P_n(A(H))$ and $\bar{S}_p: G(A(H)) \to$ $\to P_n(A(H))$ for $P = \tilde{\varphi}(t)$ (see [1]). Then we obtain the following commutative diagram of maps between homogeneous spaces:

$$\begin{array}{ccc} G(A(H)) & \xrightarrow{\pi_t} & S(t) \\ & \bar{S}_P \searrow \quad \swarrow \tilde{\varphi} & \\ & P_n(A(H)) & \end{array}$$

where $\tilde{\varphi}$ and $\bar{S}_p$ are C^∞ and π_t is holomorphic.

REMARK 3.9. If $J \in N_n(H)$ is Jordan but not nice Jordan, then $\varphi : S(J) \to$ $\to P_n^+(H)$ cannot be continuous at any point of $S(J)$. This is clear since for the construction of the cross section ω (3.3 to 3.5) we have only used that J is Jordan and that φ is continuous at $S(J)$. Therefore, if φ is continuous, then π_J has local cross sections and J must be then nice Jordan.

4. UNITARY ORBITS

Throughout this paragraph we will consider the real complemented subspaces of $L(H)$

$$L_h(H) = \{T \in L(H) : T^* = T\} \text{ and } L_{ah}(H) = \{A \in L(H) : A^* = -A\}.$$

Let $U(H)$ be the unitary group of H and for $B \in L(H)$, consider the map

$$\pi_B: U(H) \to U(B) = \{UBU^* : U \in U(H)\}, \quad \pi_B(U) = UBU^*.$$

It is widely known that $U(H)$ is a Lie group and a C^∞-submanifold of $L(H)$. A corollary of the Implicit Function Theorem (analogous to Proposition 1.1) assures the following conditions imply that $U(B)$ is a C^∞-submanifold of $L(H)$, and that the map π_B defines a C^∞ homogeneous space (see [10]):

i) $\pi_B: U(H) \to U(B)$ is an open map.

ii) $\{A \in L_{ah}(H) : AB = BA\} = \ker \delta_B \cap L_{ah}(H)$ is a real complemented subspace of $L_{ah}(H)$.

iii) $\{AB - BA : A \in L_{ah}(H)\} = \delta_B(L_{ah}(H))$ is a real complemented subspace of $L(H)$.

Observe that condition i) characterizes a class of operators studied by D. Deckard and L. A. Fialkow in [5] (Theorem 2.1). We transcribe the main theorem they proved:

THEOREM 4.1. *Let $T \in L(H)$. Then the following are equivalent:*

i) $\pi_T: U(H) \to U(T)$ *is an open map.*

ii) $U(T)$ *is norm closed in* $L(H)$.

iii) $C^*(T)$, *the C^*-algebra generated by T and J, is finite dimensional.*

iv) *T is unitarily equivalent to an operator of the form:* $A \oplus (B \oplus B \oplus \oplus B \oplus \ldots) = A \oplus (B \otimes I)$, *for A and B operators acting on finite dimensional Hilbert spaces.*

v) $\pi_T: U(H) \to U(T)$ *has local cross sections.*

We will prove that when T satisfies one of the above conditions, then $U(T)$ is a C^∞-submanifold of $L(H)$, and also that the converse is true.

PROPOSITION 4.2. *Let $T \in L(H)$. If $C^*(T)$ is finite dimensional, then* $\ker \delta_T \cap \cap L_{ah}(H)$ *is complemented in* $L_{ah}(H)$.

The proof is implicitly contained in the proof of Theorem 2.1 in [5].

LEMMA 4.3. *Let H' and H'' be complex separable Hilbert spaces, H' an n-fold copy of H_1, H'' an m-fold copy of H_2. Also let $A = a \otimes I_{H_1} \in L(H')$ for $a \in L(\mathbf{C}^n)$ and $B = b \otimes I_{H_2} \in L(H'')$ for $b \in L(\mathbf{C}^m)$. Then the linear subspace W of $L(H', H'') \times L(H', H'')$ given by $W = \{(C, D) \in L(H', H'')^2 : \exists\, X \in L(H', H'')$ such that $C = BX - XA$ and $D = B^*X - XA^*\}$ is closed and complemented.*

Proof. The matrices of the operators of $L(H', H'')$ relative to the decompositions $H' = H_1 \oplus \ldots \oplus H_1$ and $H'' = H_2 \oplus \ldots \oplus H_2$ determine an isomorphism between $L(H', H'')$ and $L(H_1, H_2)^{m \times n}$, so that $L(H', H'')^2$ is isomorphic to $L(H_1, H_2)^{m \times n \times 2}$.

We will show that the operator $\alpha \in L(L(H', H'')^2)$ defined by

$$\alpha(X, Y) = (BX - XA, B^*X - XA^*)$$

whose range is W, has a matrix relative to the mentioned decomposition of $L(H', H'')^2$ of the form

$$(1) \qquad (\alpha_{jk} I_{L(H_1, H_2)})_{1 \leq j,k < 2mn}$$

for certain $\alpha_{jk} \in \mathbf{C}$.

Indeed, as an example, we will show that the operator η, $\eta(X, Y) = (BX, 0)$ has a matrix of the described type. Let us index the entries of this matrix in the following fashion:

$$\eta = (\eta_{ijk,rst})_{\substack{1 \leq i,r \leq n \\ 1 \leq j,s \leq m \\ 1 \leq k,t \leq 2}}$$

based upon the decomposition $L(H', H'') \cong L(H_1, H_2)^{m\times n}$, where the indices k and t denote the first or the second coordinate in $L(H', H'')^2$.

It is straightforward to verify that with these notations,

$$\eta_{ijk,rst} = \begin{cases} 0, & \text{if } k = 2 \text{ or } t = 2 \\ \delta_{js} b_{ir} I_{L(H_1,H_2)}, & \text{if } t = k = 1, \end{cases}$$

where

$$(b_{ir}) = b \quad \text{and} \quad \delta_{js} = \begin{cases} 1, & \text{if } j = s \\ 0, & \text{if } j \neq s. \end{cases}$$

Proceeding analogously, "scalar" entries can be found for the other left or right multiplication operators in the first or second coordinate, defined by A, B^* and A^*.

So, it suffices to see that an operator $\alpha \in L(L(H',H'')^2)$ of the form (1), has closed and complemented range.

Indeed, let $\tilde{\alpha} \in L(\mathbf{C}^{2mn})$ given by $\tilde{\alpha}_{ij} = \alpha_{ij}$, in the canonical base. It is clear that there exists $\tilde{\gamma} \in L(\mathbf{C}^{2mn})$ satisfying

1) $\tilde{\alpha}\tilde{\gamma}\tilde{\alpha} = \tilde{\alpha}$ and

2) $\tilde{\gamma}\tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}$.

Then, if we define $\gamma \in L(L(H', H'')^2)$ by

$$\gamma = (\tilde{\gamma}_{ij}, I_{L(H_1,H_2)})_{1\leq i,j\leq 2mn},$$

α and γ satisfy $\alpha\gamma\alpha = \alpha$ and $\gamma\alpha\gamma = \gamma$. Therefore, $(\alpha\gamma)^2 = \alpha\gamma$ and $R(\alpha\gamma) = R(\alpha)$, so that $R(\alpha) = W$ is complemented. ▨

PROPOSITION 4.4. *Let $T \in L(H)$. If T is unitarily equivalent to an operator of the form $a \oplus (b \otimes I)$ for $a \in L(\mathbf{C}^n)$ and $b \in L(\mathbf{C}^m)$, then*

$$\delta_T(L_{ah}(H)) \text{ is complemented in } L(H).$$

Proof. We can suppose, without loss of generaility, that $T = a \oplus (b \otimes I_K)$, for $H = \mathbf{C}^n \oplus K^m$. Then $L(H)$ is isomorphic to

$$L(\mathbf{C}^n) \oplus L(K^m) \oplus L(\mathbf{C}^n, K^m) \oplus L(K^m, \mathbf{C}^n).$$

Let $A \in L_{ah}(H)$; its matrix relative to the decomposition of H is

$$\begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^* & A_{22} \end{pmatrix}, \qquad \text{with } A_{ii}^* = -A_{ii},\ i = 1, 2.$$

Then

$$\delta_T(A) = \begin{pmatrix} \delta_a(A_{11}) & A_{12}(b \otimes I) - aA_{12} \\ (b \otimes I)A_{12}^* - A_{12}^* a & \delta_{b\otimes I}(A_{22}) \end{pmatrix}.$$

To prove our statement it is enough to see that the following properties hold:

i) $\delta_a(L_{ah}(\mathbf{C}^n))$ is complemented in $L(\mathbf{C}^n)$.

ii) $\delta_{b\otimes I}(L_{ah}(K^m))$ is complemented in $L(K^m)$.

iii) $S = \{(C, D) \in L(K^m, \mathbf{C}^n) \times L(\mathbf{C}^n, K^m) : \exists\, X \in L(K^m, \mathbf{C}^n)$ such that $C = X(b \otimes I) - aX$ and $D = (b \otimes I)X^* - X^*a\}$ is complemented in $L(K^m, \mathbf{C}^n) \times L(\mathbf{C}^n, K^m)$.

Condition i) is clear.

Condition iii) follows from Lemma 4.3, setting $H' = K^m$, $H'' = \mathbf{C}^n$ and applying the real isomorphism of $L(K^m, \mathbf{C}^n)^2$ into $L(K^m, \mathbf{C}^n) \times L(\mathbf{C}^n, K^m)$ which maps (C, D) to $(-C, D^*)$.

To prove ii), let $\Delta : L(K^m) \to L(K^m)^2$, be given by

$$\Delta(X) = (\delta_{b\otimes I}(X), \delta_{(b\otimes I)^*}(X)).$$

Using 4.3, it is clear that $\mathbf{R}\Delta$ is complemented (put $H' = H'' = K^m$, and $a = b$). By 4.2, we can choose a closed real subspace V of $L_{ah}(K^m)$ such that $V \oplus [\ker \delta_{b\otimes I} \cap L_{ah}(K^m)] = L_{ah}(K^m)$. Then $V \oplus \mathrm{i}V$ is a supplement in $L(K^m)$ for

$$[\ker \delta_{b\otimes I} \cap L_{ah}(K^m)] \oplus [\ker \delta_{b\otimes I} \cap L_h(K^m)] =$$

$$= \ker \delta_{b\otimes I} \cap \ker \delta_{(b\otimes I)^*} = \ker \Delta.$$

Therefore, $\mathbf{R}\Delta = \Delta(V) \oplus \mathrm{i}\Delta(V)$. So $\Delta(V)$ is complemented in $L(K^m)^2$. Consider the real isomorphism $\rho : L(K^m)^2 \to L(K^m)^2$, $\rho(X, Y) = \left(\dfrac{X + Y^*}{2}, \dfrac{X - Y^*}{2}\right)$. It is clear that $\rho\Delta(V) = \delta_{b\otimes I}(V) \times \{0\}$.

Then $\delta_{b\otimes I}(V) = \delta_{b\otimes I}(L_{ah}(K^m))$ is complemented in $L(K^m)$. ▨

THEOREM 4.5. *Let $T \in L(H)$. Then $U(T)$ is a C^∞-submanifold of $L(H)$ if and only if T satisfies one of the conditions of Theorem* 4.1. *In that case, the natural projection $\pi_T \colon U(H) \to U(T)$ defines a C^∞ homogeneous space, and the tangent space $\mathrm{T}[U(T)]_T$ can be identified with $\delta_T(L_{ah}(H))$.*

Proof. Sufficiency is a consequence of 4.1, 4.2 and 4.4.

Suppose that $U(T)$ is a submanifold of $L(H)$. Then $U(T)$ is locally closed. There exists $\varepsilon > 0$ such that $B(T, \varepsilon) = \{A \in U(T) : \|T - A\| \leqslant \varepsilon\}$ is closed in $L(H)$. Then so is $UB(T, \varepsilon)U^* = B(UTU^*, \varepsilon)$, $U \in U(H)$.

Therefore it is easy to see that $U(T)$ is closed in $L(H)$. ∎

REMARK 4.6. Let $T \in L(H)$ such that $C^*(T)$ is finite dimensional. Then, using the existence of C^∞ local cross sections for π_T guaranteed by 4.5, it is easy to see that the cross section (φ, B), defined by Deckard and Fialkow in [5] (Theorem 2.1), is also C^∞.

Indeed, it suffices to observe in the proof of the mentioned theorem, that $\varphi \circ \pi_T$ has an expression that is analytic in U and U^*, for U in certain neighborhood of I in $U(H)$.

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E. ANDRUCHOW and D. STOJANOFF
Instituto Argentino de Matemática,
Viamonte 1636 1er. Piso,
1055 Buenos Aires,
Argentina.

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